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Quantum Mechanics B (Physics 130B) Fall 2014
Midterm – Solutions

Problems

1. **Spin**

   (a) How many components does the wavefunction of a spin-$\frac{1}{2}$ particle have?

   The spin Hilbert space is $2D$ spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$ which are eigenstates of S_z. Therefore one needs at least two components; along $|\uparrow\rangle$ and another along $|\downarrow\rangle$

   (b) Give an example of a spin-$\frac{1}{2}$ particle besides an electron

   For the standard model particles any quark, neutrino, or other lepton work.

   Composite particle examples include the proton, neutron, and other baryons as well as various nuclei like He^3.

   (c) Apply the spin lowering operator S_- to the state $|\chi_+\rangle = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ What do you get?

   $S_- = S_x - \mathbf{i}S_y$ and $|\chi_+\rangle = -|\uparrow\rangle$
   $S_-|\uparrow\rangle = S_x|\uparrow\rangle - \mathbf{i}S_y|\uparrow\rangle = \frac{1}{2}(|\downarrow\rangle - \mathbf{i}^2|\downarrow\rangle) = |\downarrow\rangle$
   Therefore $S_-|\chi_+\rangle = -|\downarrow\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

2. **Fine Structure**

   (a) Write down the two components of the fine structure Hamiltonian for hydrogen and briefly explain what causes them. How big are they compared to the unperturbed ground state energy?

   One term comes relativistic corrections: $\Delta H_{rel} = -\frac{p^4}{8m^3c^2}$

   The other from spin-orbit coupling: $\Delta H_{SO} = \frac{e^2}{8\pi\epsilon_0 m^2c^2}\frac{\vec{L}\cdot\vec{S}}{r^3}$

   The unperturbed $E_{gs} = -\frac{1}{2}\alpha^2 mc^2$ where $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$ is the fine structure constant. Corrections from the terms above are suppressed by an additional α^2

   (b) What happens when a large magnetic field is applied to the hydrogen atom? Describe what happens to each of the two components of the fine structure Hamiltonian.

   A strong B-field is associated with Zeeman splitting/corrections to the hydrogen spectra. Corrections from ΔH_{Zeeman} are much larger than the above.

   Since the B-field couples directly to the spin $\vec{S}$ a sufficiently strong field can decouple $\vec{L}$ and $\vec{S}$ in ΔH_{SO}

(c) The fine structure effects depend on two quantum numbers: n and j. Evaluating corrections out to first order, how many (degenerate) ground states will there be?

$m_j = m_\ell + m_s = 0 \pm \frac{1}{2}$ is unaffected and so there is a two-fold ground state degeneracy due to the spin states.

(d) What does it mean to 'lift' or 'break' degeneracy in general? How could one, experimentally, lift the degeneracy arising in part (c)?

Splitting a degeneracy requires a perturbation to the original Hamiltonian which gives *differing* energy contributions to the otherwise degenerate states.

A fancy point: Often degeneracy arises from a *symmetry* of the system which is *spontaneously broken* by the ground state(s). For example in Hydrogen the operator which flips the spin of the electron S_x is a symmetry because $[H, S_x] = 0$; the energy doesn't depend on the spin orientation. However this operator acting on groundstates takes you to *different* states: $S_x|\uparrow\rangle = |\downarrow\rangle \neq |\uparrow\rangle$

The way to remove this degeneracy is to *explicitly* break the symmetry. For example, adding a term to H which depends on S_z. Then $[H', S_x] \neq 0$ because $[S_z, S_x] \neq 0$

Experimentally one can create such a term by introducing a magnetic field which will then energetically prefer one spin configuration over another.

3. **Linear Perturbation to SHO**

Consider a perturbation to the simple harmonic oscillator of the form $H' = b\hat{x}$

Calculate the first order correction to the ground state energy.

$\hat{x} \propto \hat{a} + \hat{a}^\dagger$ so $E_n^{(1)} = \langle n|H'|n\rangle \propto \langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle$

But $\hat{a}|n\rangle = \sqrt{n-1}|n-1\rangle$ and $\langle m|n\rangle = \delta_{mn}$ so $E_n^{(1)} \propto \langle n|n-1\rangle + \langle n-1|n\rangle = 0$

4. **Delta Square**

Consider a particle of mass M in an infinite square well of length a. Suppose at $t = 0$ we perturb the system by placing a delta function bump of the form $H' = \alpha\delta(x - \frac{a}{2})$ in the middle of the well.

Assume the bump is sufficiently small that first order time-independent perturbation theory is valid.

(a) What does sufficiently small mean, quantitatively?

$E_n^{(1)} \propto \frac{\alpha}{a} \ll E_n^{(0)}$

(b) What is the first order energy of the ground state to the perturbed system? What about excited states?

The unperturbed wavefunctions can be labelled by n as $\psi_n(x) \equiv \langle x|n\rangle = \sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$

$$\begin{aligned} E_n^{(1)} &= \langle n|H'|n\rangle = \textstyle\sum_x \langle n|x\rangle\langle x|H'|n\rangle \equiv \int \mathrm{d}x\ \psi_n(x)^* H' \psi_n(x) \\ &= \int \mathrm{d}x\ \psi_n(x)^* H' \psi_n(x) = \frac{2\alpha}{a}\int_0^a \mathrm{d}x\ \sin^2(\frac{n\pi}{a}x)\delta(x - \frac{a}{2}) = \frac{2\alpha}{a}\sin^2(\frac{n\pi}{2}) \end{aligned}$$

The total energy expression, to first order, is then $E_n = E_n^{(0)} + \frac{2\alpha}{a}\sin^2(\frac{n\pi}{2})$ where $E_n^{(0)} = \frac{\hbar^2}{2M}(\frac{n\pi}{a})^2$