HW 3

4.33) $B = B_0 \cos(\omega t)\hat{k}$

(a) $H = -\gamma \vec{B}\cdot\vec{S}$ for electron $\gamma = -e/m$

$H = -\gamma \frac{\hbar}{2} B_0 \cos(\omega t)\,\sigma_z$

$$\boxed{H = -\frac{1}{2}\gamma\hbar B_0 \cos(\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

(b) $\chi(0) = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ eigenspinor of S_x

get the eigenvalues of H : $\lambda_\pm = \mp\frac{1}{2}\gamma\hbar B_0 \cos(\omega t)$

get the eigenfunctions $\psi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ $\psi_- = \begin{pmatrix}0\\1\end{pmatrix}$ same as the eigenspinors for S_z

$$\chi(t) = a(t)\psi_+ + b(t)\psi_- = \begin{pmatrix}a(t)\\ b(t)\end{pmatrix}$$

$$i\hbar\frac{\partial\chi}{\partial t} = \vec{H}\chi = a(t)H\psi_+ + b(t)H\psi_-$$
$$= a(t)\lambda_+\psi_+ + b(t)\lambda_-\psi_-$$

$$i\hbar\begin{pmatrix}\dot{a}(t)\\ \dot{b}(t)\end{pmatrix} = \begin{pmatrix}a(t)\lambda_+\\ b(t)\lambda_-\end{pmatrix}$$

$$i\hbar\frac{da}{dt} = -\frac{1}{2}a\gamma\hbar B_0 \cos(\omega t)$$

$$a(t) = a(0)\exp\left[\frac{i}{2\omega}\gamma B_0 \sin(\omega t)\right]$$

$$i\hbar\frac{db}{dt} = \frac{1}{2}b\gamma\hbar B_0 \cos(\omega t)$$

$$b(t) = b(0)\exp\left[-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)\right]$$

from $\chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ we have $a(0) = b(0) = 1/\sqrt{2}$

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp\left(\frac{i}{2\omega}\gamma B_0 \sin(\omega t)\right)\\ \exp\left(-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)\right)\end{pmatrix}}$$

4.33) c) probability of getting $-\hbar/2$

This is the "down" state in the x-direction

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

probability $P = |\langle \chi_-^{(x)} | \chi(t)\rangle|^2 = \left| \frac{1}{2}(1 \ -1)\begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha}\end{pmatrix}\right|^2$

here I've defined $\alpha = \frac{1}{2\omega}\gamma B_0 \sin(\omega t)$

$$P = \left|\tfrac{1}{2}\left(e^{i\alpha} - e^{-i\alpha}\right)\right|^2 = \sin^2\alpha$$

$$\boxed{P = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)}$$

d) when $\chi = e^{i\theta}\chi_-^{(x)}$ then we have a complete flip

that is, when χ is equal to $\chi_-^{(x)}$ times some overall phase (which doesn't matter) we get

$$P = |\langle \chi_-^{(x)} | \chi\rangle|^2 = |e^{i\theta}|^2 = 1$$

so when $P = 1$ we have a complete flip

$\sin(\omega t)$ is some integer from -1 to 1
and the argument of $\sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)$
has to be $\frac{n\pi}{2}$, $n = 1, 3, 5, \ldots$

$$\frac{\gamma B_0}{2\omega}\sin(\omega t) = \frac{n\pi}{2}$$

$B_0 = \frac{\omega n \pi}{\gamma \sin(\omega t)}$ the smallest value of B_0 occurs at $n = 1$, and $\sin(\omega t) = 1$ (taking B_0 positive)

$$\boxed{B_0 = \frac{\omega \pi}{\gamma}}$$

6.38) $\mu_d = \frac{g_d e}{2m_d} S_d$

for hyperfine splitting $E' = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3} \langle S_d \cdot S_e \rangle$

note that S_p changed to S_d. Instead of a single proton of hydrogen we have the extra neutron for the deuteron, which changed our magnetic moment.

$S_d \cdot S_e = \frac{1}{2}(S^2 - S_d^2 - S_e^2)$ $\quad S_e^2 = \frac{3}{4}\hbar^2$ as before

$S_d^2 = 2\hbar^2$, since deuteron is a spin-1

total spin is then

$\frac{1}{2}+1, \frac{1}{2},$

$3/2, 1/2 \Rightarrow S^2 = \frac{15}{4}\hbar^2, \frac{3}{4}\hbar^2$

using [6.94], frequency $\nu = \frac{\Delta E}{\hbar}$, wavelength $\lambda = \frac{c}{\nu}$

$\langle S_d \cdot S_e \rangle = \begin{cases} 1/2\,\hbar^2 & \text{for } S = 3/2 \\ -\hbar^2 & \text{for } S = 1/2 \end{cases}$

$\Delta E = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\left(\frac{1}{2}\hbar^2 - (-\hbar^2)\right) = \frac{\mu_0 g_d e^2}{2\pi m_d m_e a^3}$

$\lambda = \frac{c\hbar}{\Delta E} = \boxed{92 \text{ cm}}$

3) $H' = 2^{3/2}\, b \sqrt{\dfrac{\mu^3 \omega^5}{\hbar}}\, x^3 = \alpha x^3$ where I've defined the whole constant factor as α

ⓐ $H|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle$

for harmonic oscillator: $\boxed{E_n^0 = \hbar\omega(n+\tfrac{1}{2})}$

ⓑ $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$

$$E_n^1 = \alpha\langle\psi_n^0|x^3|\psi_n^0\rangle$$

$$x^3 = \left(\frac{\hbar}{2m\omega}\right)^{3/2}(a_+ + a_-)^3$$

$$\begin{aligned}(a_+ + a_-)^3 &= (a_+a_- + a_-a_+ + a_+^2 + a_-^2)(a_+ + a_-)\\ &= a_+a_-a_+ + a_+a_-^2 + a_-a_+^2 + a_-a_+a_- + a_+^3\\ &\quad + a_+^2a_- + a_-^2a_+ + a_-^3\end{aligned}$$

there are no terms with equal powers of a_+ and a_-

$$\boxed{E_n^1 = 0}$$

ⓒ $$E_n^2 = \sum_{m\neq n} \frac{|\langle\psi_m|x^3|\psi_n\rangle|^2\alpha^2}{E_n^0 - E_m^0}$$

first of all: $E_n^0 - E_m^0 = \hbar\omega(n-m)$

$$E_n^2 = \cancel{2^3}\, b^2 \frac{\mu^3\omega^5}{\hbar}\left(\frac{\hbar^3}{\cancel{8}m^3\omega^3}\right)\frac{1}{\hbar\omega}\sum_{m\neq n}\frac{|\langle\psi_m|(a_+ + a_-)^3|\psi_n\rangle|^2}{n-m}$$

$$E_n^2 = \frac{b^2\mu^3\hbar\omega}{m^3}\sum\frac{|\langle\psi_m|(a_+ + a_-)^3|\psi_n\rangle|^2}{n-m}$$

3) c) we have the relations $a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$a_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a_+a_-|n\rangle = n|n\rangle \;,\; a_-a_+|n\rangle = (1+n)|n\rangle$$

looking at the terms in the expansion of x^3

$$\langle\psi_m^0|a_+a_-a_+|\psi_n^0\rangle = (n+1)\sqrt{n+1}\,\langle\psi_m^0|\psi_{n+1}^0\rangle = (n+1)\sqrt{n+1}\,\delta_{m,n+1}$$

$$\langle\psi_m^0|a_+a_-^2|\psi_n\rangle = (n-1)\sqrt{n}\,\langle\psi_m^0|\psi_{n-1}^0\rangle = (n-1)\sqrt{n}\,\delta_{m,n-1}$$

doing this for all 8 terms we get

$$E_n^2 = \frac{b^2\mu^3\hbar\omega}{m^3}\sum_{m\neq n}\left(\frac{(n+1)(n+2)(n+3)\delta_{m,n+3} + (n)(n-1)(n-2)\delta_{m,n-3}}{n-m} + \frac{3(n+1)(n+1)^2\delta_{m,n+1} + 3n^2 n\,\delta_{m,n-1}}{n-m}\right)$$

$$E_n^2 = \frac{b^2\mu^3\hbar\omega}{m^3}\left[\frac{(n)(n-1)(n-2) - (n+1)(n+2)(n+3)}{3} + 3n^3 - 3(n+1)^3\right]$$

d) we know $[H'] = \text{energy } (\varepsilon)$, $[\omega] = 1/s$, $[\hbar] = \varepsilon/[\omega]$

$[b] = \text{dimensionless}$, $[x^3] = m^3$

$$[H'] = [\mu]^{3/2}[\omega]^{5/2}[x]^3\frac{[\omega]^{1/2}}{\varepsilon^{1/2}} = \varepsilon$$

$$[\mu]^{3/2}[\omega]^3[x]^3 = \varepsilon^{3/2}$$

$$(\mu)^3(\omega)^6(x)^6 = \varepsilon^3 \qquad \varepsilon = kg\,m^2/s^2$$

$\boxed{[\mu] = kg}$ this can also be seen by looking at the constant factor in E_n^2

4) $V(r) = \begin{cases} 0 & r \le a \\ \infty & r > a \end{cases}$ $\quad \psi = R(r)\, Y_l^m(\theta,\phi)$

$$\frac{d^2 u}{dr^2} = \left[\frac{l(l+1)}{r^2} - k^2\right] u \qquad k = \frac{\sqrt{2mE}}{\hbar} \qquad u = \frac{R(r)}{r}$$

solutions are spherical bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \qquad n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$$

blows up at origin, no good

in this case $x = kr$

ⓐ $R_l = \frac{j_l(kr)}{r}$ note: R only depends on the quantum number l

use boundary conditions to determine the energies

$j_l(ka) = 0$ can look up the zeroes of the Bessel function in a table, but for now I'll call them β_{nl} (the n^{th} zero of $j_l(kr)$)

$ka = \beta_{nl}$

$$\boxed{E = \frac{\hbar^2 \beta_{nl}^2}{2ma^2}}$$

remember: $\psi = R_l(r)\, Y_l^m(\theta,\phi)$

so we now have 3 quantum numbers (n,l,m). Yet the energy only depends on n, l and there are $2l+1$ allowed values of m for a given n and l

$\boxed{\text{degeneracy} = 2l+1}$

ground state: $l=0, n=1$

1^{st} excited: $l=1, n=1$

2^{nd} excited: $l=2, n=1$

Can look at a table or figure (see example 4.1) to determine the l and n values of the 3 lowest zeroes of j_l

4) b) for $l=0$, $j_0 = \frac{\sin(kr)}{kr}$

at the boundary

$$j_0(ka) = \frac{\sin(ka)}{ka} = 0$$

$$ka = n\pi$$

at $n=1$ $\quad k = \pi/a$

$$\Rightarrow \beta_{10} = \pi$$

$$E_{10} = \frac{\hbar^2 \beta_{10}^2}{2ma^2} = \frac{\hbar^2 \pi^2}{2ma^2}$$

using $m = m_e \approx 9.12 \times 10^{-31}\,kg$

$a = 10^{-9}\,m$

$\hbar = 1.05 \times 10^{-34}\ J \cdot s$

$$E_{10} = 5.96 \times 10^{-20}\ J = 0.372\,eV$$

c) this energy is exactly the same as the ground state energy for the 1D infinite square well