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Quantum Mechanics B (Physics 130B) Fall 2014
Worksheet 4 – Solutions

Announcements

- The 130B web site is:

  http://physics.ucsd.edu/students/courses/fall2014/physics130b/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to discover why bra-ket notation is useful and do perturbation theory for a spin.

Problems

1. **Don't Give In**

   Suppose you're walking down the street and a man approaches you with well-prepared quantum state of the form:

$$\psi(\theta, \phi) = 2\sqrt{\frac{15}{16\pi}} \cos\theta \sin\theta \cos\phi \tag{1}$$

   He then asks you to predict average value of various angular momentum quantities. Snickering, he offers only one piece of advice:

$$Y_{2,\pm 1}(\theta, \phi) \equiv \mp\sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta e^{\pm \mathbf{i}\phi} \tag{2}$$

   Can you figure out the answers to these questions without doing any integrals?

   It will be helpful to split ψ as the following:

   $\psi(\theta, \phi) = \frac{1}{\sqrt{2}}(Y_{2,-1} - Y_{2,1}) \implies |\psi\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle - |2, 1\rangle)$

   (a) Calculate $\langle L_z \rangle$ for the state 1. If you need $L_z = -\mathbf{i}\partial_\phi$

   $\langle L_z \rangle = 0$ from several points of view.

   Most intuitively note that $\langle L_z \rangle = \int \psi^*(-\mathbf{i}\partial_\phi)\psi$ which since ψ is real $\langle L_z \rangle \propto \mathbf{i}$ which just isn't physical so the whole thing must vanish.

We can also do an explicit check with the form of $|\psi\rangle$ using $L_z|\ell, m\rangle = m|\ell, m\rangle$
$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle = \frac{1}{2}(\langle 2,-1| - \langle 2,1|)(-|2,1\rangle - |2,-1\rangle) = \frac{1}{2}(1-1) = 0$
where in the above I've used orthogonality. You could also do the integral but I wouldn't.

(b) Calculate $\langle L^2\rangle$ again for 1. If you need $L^2 = -\nabla^2$ restricted to the 2-sphere.
Here it's easiest to use the form of $|\psi\rangle$ and the fact $L^2|\ell, m\rangle = \ell(\ell+1)|\ell, m\rangle$
$\langle L^2\rangle = \frac{1}{2}(\langle 2,-1| - \langle 2,1|)(-2(2+1)|2,1\rangle + 2(2+1)|2,-1\rangle = \frac{1}{2}(6+6) = 6$
You can also do the integral. I've attached a Mathematica notebook where this is done and confirms my result.

2. **Sanity Check**

Consider a spin-$\frac{1}{2}$ particle in a magnetic field $\vec{B} = \{B_x, 0, B_z\}$

Generically the Hamiltonian to describe such a situation is:

$$\hat{H} = -\mu_B \vec{B} \cdot \boldsymbol{\vec{\sigma}} \tag{3}$$

where μ_B is the Bohr magneton and $\boldsymbol{\vec{\sigma}} = \{\sigma_x, \sigma_y, \sigma_z\}$ is a vector of Pauli matrices.

(a) Suppose $B_x = 0$, find the eigenstates and energies associated with 3
$E_{\uparrow,\downarrow} = \mp\mu_B B_z$ and for $\{|\uparrow\rangle, |\downarrow\rangle\}$ respectively

(b) Now suppose $B_z \gg B_x \neq 0$ and compute the first and second order corrections to the energy using perturbation theory.
$E_\uparrow^{(1)} = \langle\uparrow|(-\mu_B B_x\sigma_x)|\uparrow\rangle = 0$ and similar for $|\downarrow\rangle$ where we use $\sigma_x|\uparrow\rangle = |\downarrow\rangle$
The second order shifts are more interesting:
$E_\uparrow^{(2)} = \sum_{k\neq\uparrow} \frac{|\langle k|(-\mu_B B_x\sigma_x)|\uparrow\rangle|^2}{E_\uparrow^{(0)} - E_k^{(0)}} = (\mu_B B_x)^2 \frac{|\langle\downarrow|\sigma_x|\uparrow\rangle|^2}{E_\uparrow^{(0)} - E_\downarrow^{(0)}} = -\frac{\mu_B B_x^2}{2B_z}$
and similarly $E_\downarrow^{(2)} = -E_\uparrow^{(2)}$
Putting it all together the energy up to second order is:
$E_{\uparrow,\downarrow} = \mp\mu_B B_z(1 + \frac{1}{2}(\frac{B_x}{B_z})^2)$

(c) Now it turns out 3 is exactly solvable. Compute the energies of the exact eigenstates by direct diagonalization. Show by second order Taylor expansion this agrees with the above.
$\hat{H} = -\mu_B \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix} = \begin{pmatrix} E_\uparrow & 0 \\ 0 & E_\downarrow \end{pmatrix}$ where $E_{\uparrow,\downarrow} = \mp\mu_B\sqrt{B_x^2 + B_z^2} = \mp\mu_B B_z\sqrt{1 + (\frac{B_x}{B_z})^2}$
The Taylor expansion of $\sqrt{1+\epsilon^2} \approx 1 + \frac{1}{2}\epsilon^2$ which validates the above.