

# Quantum Mechanics B (Physics 130B) Fall 2014 Worksheet 5

## Announcements

- The 130B web site is:

<http://physics.ucsd.edu/students/courses/fall2014/physics130b/> .

Please check it regularly! It contains relevant course information!

- Greetings everyone! This week we're going to add angular momentum.

## Problems

### 1. Combine?

Consider a system of two particles, one of spin-1 and another of spin-2. Let  $\{s_1, m_1; s_2, m_2\}$  denote their spins and  $\mathcal{H}_1$  and  $\mathcal{H}_2$  their Hilbert spaces respectively.

Suppose they interact with a Hamiltonian of the form:

$$H = -\epsilon \vec{S}_1 \cdot \vec{S}_2 \quad (1)$$

Let's understand the space of states for these particles

- (a) How many different spin states are allowed for particle 1? Equivalently, what is the dimension of  $\mathcal{H}_1$ ? Particle 2?

What's the dimension of  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  ?

One possible basis for  $\mathcal{H}$  is the tensor product of the bases for  $\mathcal{H}_1$  and  $\mathcal{H}_2$

Denote this as:

$$|m_1; m_2\rangle \equiv |s_1 = 1, m_1\rangle \otimes |s_2 = 2, m_2\rangle \quad (2)$$

Another possible basis is that of a combined angular momentum operator:

$$\vec{S} \equiv \vec{S}_1 + \vec{S}_2 \quad (3)$$

This operator allows us to analyze the Hamiltonian **1** in terms of better quantum numbers. It also makes physical sense as the spin of the composite system.

A basis  $|S, M\rangle$  associated with  $S^2$  and  $S_z$  of the combined pair is:

$$S^2|S, M\rangle = S(S+1)|S, M\rangle \quad S_z|S, M\rangle = M|S, M\rangle \quad (4)$$

The values of  $S$  are not independent of  $s_1$  and  $s_2$ ; they can be thought of as the lengths allowed by adding independent  $S_i$  vectors.

The allowed range is thus<sup>1</sup>:

$$|s_1 - s_2| \leq S \leq s_1 + s_2 \quad (5)$$

The  $M$  quantum number is also directly determinable from the  $m_i$  of the tensor product states as we'll see.

- (b) Determine the number of independent  $|S, M\rangle$  states. Does this match the value for  $\dim \mathcal{H}$  obtained previously?
- (c) Rewrite the Hamiltonian **1** in terms of  $S^2$ . What are the energies associated with the  $|S, M\rangle$  states?

Now let's derive explicit relations between the two bases we've constructed.

Recall that we define  $S_{\pm} \equiv S_x \pm iS_y = S_{1,\pm} + S_{2,\pm}$  such that:

$$S_{\pm}|S, M\rangle = \sqrt{S(S+1) - M(M \pm 1)}|S, M \pm 1\rangle \quad (6)$$

The *highest weight* state is  $|3, 3\rangle \equiv |m_1 = 1; m_2 = 2\rangle$  such that  $S_+|3, 3\rangle = 0$

- (d) Using the  $S_-$  operator and normalization/orthogonality constraints determine the values  $a, b$  for which:

$$|3, 2\rangle = a|1; 1\rangle + b|0; 2\rangle \quad (7)$$

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<sup>1</sup>This is the same fact as  $1 \otimes 2 = 1 \otimes 2 \otimes 3$ ; we're multiplying different  $SU(2)$  representations