Chapter 3

Symmetries

A famous theorem of Wigner shows that symmetries in a quantum theory must correspond to either unitary or anti-unitary operators. It seems fit to start with a review of what is meant by this. We will then proceed to study continuous symmetries, all represented by unitary operators. We will then turn our attention to discrete symmetries. It is then, in presenting time-reversal symmetry, that we will encounter anti-unitary operators.

3.1 Review of unitary and anti-unitary operators

The bra/ket notation is not quite suitable for anti-linear operators. So for this section we use the following notation:

- States are denoted by wave-functions: $\psi, \chi, \ldots$
- c-numbers are lowercase latin letters: $a, b, \ldots$
- Operators are uppercase: $A, B, \ldots, U, V, \ldots, \Omega, \ldots$
- Inner product and norm: (ψ, χ) and $\|\psi\|^2 = (\psi, \psi)$

An operator A is *linear* if $A(a\psi + b\chi) = aA\psi + bA\chi$. The *hermitian conjugate* $A^\dagger$ of an operator A, is such that for all χ, ψ

$$(\chi, A^\dagger\psi) = (A\chi, \psi)\,.$$

This is consistent only if A is linear:

$$\begin{aligned}
(A(a\psi + b\chi), \rho) &= (a\psi + b\chi, A^\dagger\rho) \\
&= a^*(\psi, A^\dagger\rho) + b^*(\chi, A^\dagger\rho) \\
&= a^*(A\psi, \rho) + b^*(A\chi, \rho) \\
&= (aA\psi + bA\chi, \rho)
\end{aligned}$$

for any ρ.

An invertible operator U is *unitary* if

$$(U\psi, U\chi) = (\psi, \chi) \qquad \text{(and therefore } \|U\psi\| = \|\psi\|\text{)}.$$

Unitary operators are linear. Proof:

$$\begin{aligned}
\|U(a\psi + b\chi) - aU\psi - bU\chi\|^2 &= \|U(a\psi + b\chi)\|^2 - 2\mathrm{Re}\,[a(U(a\psi + b\chi), U\psi)] \\
&\quad - 2\mathrm{Re}\,[b(U(a\psi + b\chi), U\chi)] + |a|^2\,\|U\psi\|^2 + |b|^2\,\|U\chi\|^2 \\
&= \|a\psi + b\chi\|^2 - 2\mathrm{Re}\,[a(a\psi + b\chi, \psi)] \\
&\quad - 2\mathrm{Re}\,[b(a\psi + b\chi, \chi)] + |a|^2\,\|\psi\|^2 + |b|^2\,\|\chi\|^2 \\
&= 0
\end{aligned}$$

where we have used unitarity in going form the first to the second line and we have expanded all terms in going from the second to the third line. Since only zero has zero norm we have $U(a\psi + b\chi) - aU\psi - bU = 0$, completing the proof. The inverse of a unitary operator is its hermitian conjugate:

$$(\psi, U^{-1}\chi) = (U\psi, U(U^{-1}\chi)) = (U\psi, \chi) = (\psi, U^{\dagger}\chi)$$

for any ψ, χ.

An invertible operator Ω is *anti-unitary* if for all ψ, χ

$$(\Omega\psi, \Omega\chi) = (\chi, \psi) \qquad \text{(notice inverted order)}.$$

An operator $\rotatebox{90}{A}$ is *anti-linear* if for all ψ, χ

$$\rotatebox{90}{A}(a\psi + b\chi) = a^*\rotatebox{90}{A}\psi + b^*\rotatebox{90}{A}\chi$$

Anti-unitary operators are anti-linear. The proof is the same as in linearity of unitary operators, $\|\Omega(a\psi + b\chi) - a^*\Omega\psi + b^*\Omega\chi\|^2 = 0$. Example: the complex conjugation operator, Ω_c. Clearly

$$\Omega_c(a\psi + b\chi) = a^*\Omega_c\psi + b^*\Omega_c\chi \qquad \text{and} \qquad (\Omega_c\psi, \Omega_c\chi) = (\chi, \psi)\,.$$

You can show that the products U_1U_2 and $\Omega_1\Omega_2$ of two unitary or two anti-unitary operators are unitary operators, while the products $U\Omega$ and ΩU of a unitary and an anti-unitary operators are anti-unitary.

Symmetries What properties are required of operator symmetries in QM? If A is an operator on the Hilbert space, $\mathcal{F} = \{\psi\}$, then it is a symmetry transformation if it preserves probabilities, $|(A\psi, A\chi)|^2 = |(\psi, \chi)|^2$. Wigner showed that the only two possibilities are A is unitary or anti-unitary.

Symmetry transformations as action on operators: for unitary operators, the symmetry transformation $\psi \to U\psi$, for all states, gives

$$(\psi, A\chi) \to (U\psi, AU\chi) = (\psi, U^\dagger AU\chi)$$

So we can transform instead operators, via $A \to U^\dagger AU$. For anti-unitary operators the hermitian conjugate is not defined, so we do not have "$\Omega^\dagger = \Omega^{-1}$". But

$$(\psi, A\chi) \to (\Omega\psi, A\Omega\chi) = (\Omega^{-1}A\Omega\chi, \Omega^{-1}\Omega\psi) = (\psi, (\Omega^{-1}A\Omega)^\dagger\chi)\,,$$

which is not very useful. For expectation values of observables, $A^\dagger = A$,

$$(\psi, A\psi) \to (\Omega\psi, A\Omega\psi) = (A\Omega\psi, \Omega\psi) = (\psi, \Omega^{-1}A\Omega\psi)$$

and in this limited sense, $A \to \Omega^{-1}A\Omega$.

3.2 Continuous symmetries, Generators

Consider a family of unitary transformations, $U(s)$, where s is a real number indexing the unitary operators. We assume $U(0) = \mathbb{1}$, the identity operator. Furthermore, assume $U(s)$ is continuous, differentiable. Then expanding about zero,

$$U(\epsilon) = \mathbb{1} + i\epsilon T + \mathcal{O}(\epsilon^2) \tag{3.1}$$

$$U(\epsilon)^\dagger U(\epsilon) = \mathbb{1} = (\mathbb{1} - i\epsilon T^\dagger)(\mathbb{1} + i\epsilon T) + \mathcal{O}(\epsilon^2) \quad \Rightarrow \quad T^\dagger = T \tag{3.2}$$

$$iT \equiv \left.\frac{dU(s)}{ds}\right|_{s=0} \tag{3.3}$$

The operator T is called a symmetry *generator*.

Now, the product of N transformations is a transformation, so consider

$$\left(\mathbb{1} + i\frac{s}{N}T\right)^N$$

In the limit $N \to \infty$, $\left(\mathbb{1} + i\frac{s}{N}T\right)$ is unitary , and therefore so is $\left(\mathbb{1} + i\frac{s}{N}T\right)^N$. This is a vulgarized version of the exponential map,

$$U(s) = \lim_{N\to\infty}\left(\mathbb{1} + i\frac{s}{N}T\right)^N = e^{isT}\,.$$

Note that

$$U(s)^\dagger U(s) = e^{-isT}e^{isT} = \mathbb{1}\,,$$

as it should. Also, $A \to U(s)^\dagger AU(s)$ becomes, for $s = \epsilon$ infinitesimal,

$$A \to A + i\epsilon[A, T] \qquad \text{or} \qquad \delta A = i\epsilon[A, T]\,.$$

A symmetry of the Hamiltonian has

$$U(s)^\dagger H U(s) = H \quad \Rightarrow \quad [H,T] = 0\,.$$

Since for any operator (in the Heisenberg picture)

$$i\frac{dA}{dt} = [H,A] + i\frac{\partial A}{\partial t}$$

we have that a symmetry generator T is a constant, $dT/dt = 0$ ($T(t)$ has no explicit time dependence). Conversely a constant hermitian operator T defines a symmetry:

$$dT/dt = 0 \quad \Rightarrow \quad [H,T] = 0 \quad \Rightarrow e^{-isT} H e^{isT} = H\,.$$

Let's connect this to symmetries of a Lagrangian density (or, more precisely, of the action integral).

3.3 Noether's Theorem

Consider a Lagrangian $L(t) = \int d^3x\, \mathcal{L}(\phi^a, \partial_\mu \phi^a)$, where the Lagrangian density is a function of N real fields ϕ^a, $a = 1, \ldots, N$. We investigate the effect of a putative symmetry transformation

$$\phi^a \to \phi'^a = \phi^a + \delta\phi^a, \quad \text{where } \delta\phi^a = \epsilon D^{ab}\phi^b,$$

with ϵ an infinitesimal parameter and $(D\phi)^a = D^{ab}\phi^b$ is a linear operator on the collection of fields (and may contain derivatives). Consider then $\mathcal{L}(\phi'^a, \partial_\mu \phi'^a)$. Suppose that by explicit computation we find

$$\delta\mathcal{L} = \mathcal{L}(\phi') - \mathcal{L}(\phi) = \epsilon \partial_\mu \mathcal{F}^\mu\,,$$

that is, that the variation of the Lagrangian density vanishes up to a derivative, the divergence of a four-vector. We do not require that the variation of $\mathcal{L}$ vanishes since we want invariance of the action integral, to which total derivatives contribute only vanishing surface terms. For any variation $\delta\phi$, not necessarily of the form displayed above,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a\,.$$

If ϕ^a are solutions to the equations of motion, the first term can be rewritten and the two terms combined,

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\right)\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\right)\,.$$

Using the variation that by assumption vanishes up to a total derivative we have

$$\partial_\mu \mathcal{F}^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} D^{ab}\phi^b \right)$$

so that

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} D^{ab}\phi^b - \mathcal{F}^\mu \quad \text{satisfies} \quad \partial_\mu J^\mu = 0\,.}$$

This is Noether's theorem. Note that this can be written in terms of a generalized momentum conjugate, $J_\mu = \pi^a_\mu D^{ab}\phi^b - \mathcal{F}_\mu$. Since J^μ is a conserved current, the spatial integral of the time component is a conserved "charge,"

$$T = \int d^3x\, J^0 \qquad \text{has} \quad \frac{dT}{dt} = 0\,.$$

The conserved charge is nothing but the generator of a continuous symmetry. We can show from its definition that it commutes with the Hamiltonian.

Let's look at some examples:

Translations The transformation is induced by $x^\mu \to x^\mu + \epsilon a^\mu$. Clearly

$$\delta\mathcal{L} = \epsilon a^\mu \partial_\mu \mathcal{L} \quad \Rightarrow \quad \mathcal{F}^\mu = a^\mu \mathcal{L}\,.$$

But $\mathcal{L} = \mathcal{L}(x)$, is a function of x^μ only through its dependence on $\phi^a(x)$, so

$$\delta\phi^a = \epsilon a^\mu \partial_\mu \phi^a \quad \Rightarrow \quad J^\mu_{(a)} = \pi^{a\mu} a^\nu \partial_\nu \phi^a - a^\mu \mathcal{L} = a_\nu(\pi^{a\mu}\partial^\nu \phi^a - \eta^{\mu\nu}\mathcal{L})\,.$$

Since a^ν is arbitrary, we can choose it to be alternatively along the direction of any of the four independent directions of space-time, and we then have in fact four independently conserved currents:

$$\boxed{T^{\mu\nu} = \pi^{a\mu}\partial^\nu\phi^a - \eta^{\mu\nu}\mathcal{L}}$$

This is the *energy and momentum tensor*, also known as the *stress-energy tensor*. The first index refers to the conserved current and the second labels which of the four currents.

The conserved "charges" are

$$P^\mu = \int d^3x\, T^{0\mu}\,.$$

They are associated with the transformation $x \to x + a$ and hence they are momenta. For example,

$$P^0 = \int d^3x(\pi^a\phi^a - \mathcal{L}) = \int d^3x\, \mathcal{H} = H$$

as expected.

Example: In our Klein-Gordon field theory

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 \tag{3.4}$$
$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \tag{3.5}$$
$$\Rightarrow \quad T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} \tag{3.6}$$

We can verify this is conserved, by using equations of motion,

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial^2\phi\partial^\nu\phi + \partial^\mu\phi\partial^\nu\partial_\mu\phi - \partial^\nu(\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2) \\
&= -m^2\phi\partial^\nu\phi + \partial^\mu\phi\partial^\nu\partial_\mu\phi - \partial^\mu\phi\partial^\nu\partial_\mu\phi + m^2\phi\partial^\nu\phi \\
&= 0
\end{aligned}$$

Compute now the conserved 4-momentum operator in terms of creation and annihilation operators. For the temporal component, $P^0 = H$, the computation was done in the previous chapter. For the spatial components we have

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, (\pi\partial^i\phi - \eta^{0i}\mathcal{L}) = \int d^3x\, \pi\partial^i\phi$$

so that

$$\begin{aligned}
P^i &= \int d^3x\, \partial_t\phi\partial^i\phi \\
&= \int d^3x \int (dk')(dk)\, [(-iE_{\vec{k}'})(\alpha_{\vec{k}'}e^{-ik'\cdot x} - \alpha^\dagger_{\vec{k}'}e^{ik'\cdot x})][(-ik^i)(\alpha_{\vec{k}}e^{-ik\cdot x} - \alpha^\dagger_{\vec{k}}e^{ik\cdot x})] \\
&= \int (dk')(dk)\,(-iE_{\vec{k}'})(-ik^i)\Big[\Big((2\pi)^3\delta^{(3)}(\vec{k}' + \vec{k})\alpha_{\vec{k}'}\alpha_{\vec{k}}e^{-i(E_{\vec{k}'}+E_{\vec{k}})t} + \text{h.c.}\Big) \\
&\qquad\qquad - \Big((2\pi)^3\delta^{(3)}(\vec{k}' - \vec{k})\alpha_{\vec{k}'}\alpha^\dagger_{\vec{k}}e^{-i(E_{\vec{k}'}-E_{\vec{k}})t} + \text{h.c.}\Big)\Big] \\
&= \frac{1}{2}\int (dk)k^i\left[\alpha_{\vec{k}}\alpha_{-\vec{k}}e^{-2iE_{\vec{k}}t} + \alpha^\dagger_{\vec{k}}\alpha^\dagger_{-\vec{k}}e^{2iE_{\vec{k}}t} + \alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} + \alpha_{\vec{k}}\alpha^\dagger_{\vec{k}}\right]
\end{aligned}$$

In the last line the first two terms are odd under $\vec{k} \to -\vec{k}$ so they vanish upon integration and we finally have

$$P^i = \int (dk)\, \tfrac{1}{2}k^i\left(\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} + \alpha_{\vec{k}}\alpha^\dagger_{\vec{k}}\right) = \int (dk)k^i\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}}\,.$$

It follows that

$$[\vec{P}, \alpha^\dagger_{\vec{k}}] = \vec{k}\alpha^\dagger_{\vec{k}} \quad \text{so that} \quad P^\mu|\vec{k}\rangle = k^\mu|\vec{k}\rangle\,.$$

Lorentz Transformations The transformation of states was introduced earlier, $U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle$. Assuming the vacuum state is invariant under Lorentz transformations we then may take $U(\Lambda)\alpha^\dagger_{\vec{p}}U(\Lambda)^\dagger = \alpha^\dagger_{\Lambda\vec{p}}$ and $U(\Lambda)\alpha_{\vec{p}}U(\Lambda)^\dagger = \alpha_{\Lambda\vec{p}}$. Therefore

$$U(\Lambda)\phi(x)U(\Lambda)^\dagger = \phi(\Lambda x) \quad \text{or} \quad U(\Lambda)^\dagger\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Let $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\omega^\mu{}_\nu$ with ϵ infinitesimal. Then $\omega^{\mu\nu} = -\omega^{\nu\mu}$, and therefore has $4\times 3/2 = 6$ independent components, three for rotations (ω^{ij}) and three for boosts (ω^{0i}). We assume $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ is Lorentz invariant. What does this mean? For scalar fields $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(x')$ and by the chain rule $\partial_\mu\phi(x') = \partial_\mu x'^\lambda\partial'_\lambda\phi(x') = (\Lambda^{-1})^\lambda{}_\mu\partial'_\lambda\phi(x') = \Lambda_\mu{}^\lambda\partial'_\lambda\phi(x')$, which gives

$$\mathcal{L}(\phi, \partial_\mu\phi) \to \mathcal{L}' = \mathcal{L}(\phi'(x), \partial_\mu\phi'(x)) = \mathcal{L}(\phi(x'), \partial_\mu\phi(x')) = \mathcal{L}(\phi(x'), \Lambda_\mu{}^\lambda\partial'_\lambda\phi(x'))\,.$$

Invariance means $\mathcal{L}'$ has the same functional dependence as $\mathcal{L}$ but in terms of x', that is, $\mathcal{L}$ is a scalar, $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x')$. This gives

$$\mathcal{L}(\phi(x'), \Lambda_\mu{}^\lambda\partial'_\lambda\phi(x')) = \mathcal{L}(\phi(x'), \partial'_\mu\phi(x'))\,.$$

For example, $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to \eta^{\mu\nu}\Lambda_\mu{}^\lambda\Lambda_\nu{}^\sigma\partial'_\lambda\phi\partial'_\sigma\phi = \eta^{\lambda\sigma}\partial'_\lambda\phi\partial'_\sigma\phi$ works, but not so $a^\mu\partial_\mu\phi$ for constant vector a^μ since a^μ does not transform (it is a fixed constant).

We assume $\mathcal{L}$ is Lorentz invariant and compute:

$$\begin{aligned}
\delta\phi &= \phi(x^\mu - \epsilon\omega^\mu{}_\nu x^\nu) - \phi(x^\mu) = -\epsilon\omega^\mu{}_\nu x^\nu\partial_\mu\phi = -\epsilon\omega^{\mu\nu}x_\nu\partial_\mu\phi \\
\delta\mathcal{L} &= -\epsilon\omega^{\mu\nu}x_\nu\partial_\mu\mathcal{L} = -\left[\partial_\mu(\epsilon\omega^{\mu\nu}x_\nu\mathcal{L}) - \epsilon\omega^{\mu\nu}\eta_{\mu\nu}\mathcal{L}\right] = -\partial_\mu(\epsilon\omega^{\mu\nu}x_\nu\mathcal{L})
\end{aligned}$$

and then the conserved currents are

$$\begin{aligned}
J^\mu_{(\omega)} &= -\pi^\mu\omega^{\lambda\sigma}x_\sigma\partial_\lambda\phi + \omega^{\mu\nu}x_\nu\mathcal{L} \\
&= -\omega^{\lambda\sigma}(\pi^\mu x_\sigma\partial_\lambda\phi - \delta^\mu{}_\lambda x_\sigma\mathcal{L})
\end{aligned}$$

or, since $\omega^{\mu\nu}$ is arbitrary, we have six conserved currents,

$$\begin{aligned}
M^{\mu\nu\lambda} &= (\pi^\mu x^\nu\partial^\lambda - \eta^{\mu\lambda}x^\nu\mathcal{L}) - \nu \leftrightarrow \lambda \\
&= x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}\,.
\end{aligned}$$

This has

$$M^{\nu\lambda} = \int d^3x\, M^{0\nu\lambda}$$

as generator of rotations (M^{ij}) and boosts (M^{0i}).

Comments:

(i) The expression for $M^{\mu\nu\lambda}$ is specific for scalar fields since we have used $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$. This *defines* scalar fields. We expect, for example, that a vector field, $A^\mu(x)$ will transform like $\partial^\mu\phi(x)$,

$$U^\dagger(\Lambda)A_\mu(x)U(\Lambda) = \Lambda_\mu{}^\lambda A_\lambda(\Lambda^{-1}x) \quad \text{or} \quad U(\Lambda)^\dagger A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\lambda A^\lambda(\Lambda^{-1}x)$$

Then

$$\begin{aligned}\delta A^\mu &= (\delta^\mu{}_\nu + \epsilon\omega^\mu{}_\nu)A(x^\lambda - \epsilon\omega^\lambda{}_\sigma x^\sigma) - A^\mu(x)\\ &= -\epsilon\big[\underbrace{\omega^\lambda{}_\sigma x^\sigma \partial_\lambda A^\mu}_{\text{as before}} - \underbrace{\omega^\mu{}_\nu A^\nu}_{\text{new term}}\big]\end{aligned}$$

This then gives a new term in $J^\mu_{(\omega)}$ of the form $\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}(\omega^{\rho\nu}A_\nu)$ leading to an additional term in $M^{\mu\rho\nu}$,

$$\begin{aligned}\Delta M^{\mu\rho\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}A^\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}A^\rho\\ &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}A^\sigma(\delta^\rho_\lambda\delta^\nu_\sigma - \delta^\rho_\sigma\delta^\nu_\lambda)\end{aligned} \tag{3.7}$$

There is a generalization of the matrices $(\mathcal{I}^{\rho\nu})_{\lambda\sigma} = \delta^\rho_\lambda\delta^\nu_\sigma - \delta^\rho_\sigma\delta^\nu_\lambda$ to the case of fields other than vectors, that is, fields that have other Lorentz transformations, like spin$\frac{1}{2}$ fields. The generalization has $(\mathcal{I}^{\rho\nu})_{\lambda\sigma} \to (\mathcal{I}^{\rho\nu})_{ab}$, where $\rho\nu$ labels the matrices and ab give the specific matrix elements with a, b running over the number of components of the new type of field. The matrices $\mathcal{I}^{\rho\nu}$ satisfy the same commutation relations as $M^{\rho\nu}$. We will explore this in more detail later. For now, the important point is that physically we have a clear interpretation:

$$M^{\mu ij} = \underbrace{x^iT^{\mu j} - x^jT^{\mu i}}_{\text{orbital ang mom } \epsilon^{ijk}L^k} + \underbrace{\Delta M^{\mu ij}}_{\text{intrinsic ang mom (spin) } \epsilon^{ijk}S^k}$$

(ii) The conserved quantities M^{ij} are angular momentum. What are M^{0i}? To get some understanding consider a classical "field" for point particles, with

$$T^{0i} = \sum_n p^i_n\delta^{(3)}(\vec{x} - \vec{x}_m(t)) \quad \text{so that} \quad P^i = \sum_n p^i_n\,.$$

Then we can compute

$$M^{ij} = \int d^3x\,(x^jT^{0i} - x^iT^{0j}) = \sum_n(x^i_np^j_n - x^j_np^i_n) = \epsilon^{ijk}\sum_n L^k_n$$

as expected. Turning to the mysterious components,

$$M^{0i} = \int d^3x \, (x^0 T^{0i} - x^i T^{00}) = x^0 P^i - \int d^3x \, x^i T^{00}$$

We can now see what conservation of M^{0i}, namely $\frac{dM^{0i}}{dt} = 0$, gives:

$$0 = P^i - \frac{d}{dt} \int d^3x \, x^i T^{00}$$

The quantity $\int d^3x \, x^i T^{00}$ is a relativistic generalization of center of mass, say, a "center of energy." This can be seen from T^{00} being an energy density, so that when all particles are at rest it corresponds to the mass density. So conservation of M^{0i} means that the center of energy motion is given by the total momentum. This is the relativistic analogue of $\vec{P} = M\vec{V}$ where $\vec{P}$ is the total momentum of the system, M its total mass and $\vec{V}$ the velocity of the center of mass. In the relativistic case we have that

$$\frac{P^i}{P^0} = \text{velocity of "C.M."} = \frac{\frac{d}{dt} \int d^3x \, x^i T^{00}}{P^0}$$

That the "C.M." moves with a constant speed is a relativistic conservation law.

3.4 Internal Symmetries

We have discussed symmetries that have to be present in any of the models we will care about: invariance under translations and under Lorentz transformations, or *Poincare invariance*. These symmetries transform fields at one space-time point to fields at other space-time points. In this sense they are geometrical. But there may be, in addition, symmetries that transform fields at the same space-time point. These are not generic, they are specific to each model. They are called *internal symmetries*. The name comes from the associated conserved quantities giving "internal" characteristics fo the particles. For example, baryon number or isotopic spin are symmetries and they are associated with the baryon number or the isotopic spin of particles.

Let's study a simple example. Consider a model with two real scalar fields,

$$\mathcal{L} = \sum_{n=1}^{2} \tfrac{1}{2} \partial^\mu \phi_n \partial_\mu \phi_n - V(\sum_{n=1}^{2} \phi_n^2)$$

This satisfies $\mathcal{L}(\phi'_n) = \mathcal{L}(\phi_n)$ where

$$\begin{aligned} \phi'_1(x) &= \cos\theta \phi_1(x) - \sin\theta \phi_2(x) \\ \phi'_2(x) &= \sin\theta \phi_1(x) + \cos\theta \phi_2(x) \end{aligned}$$

or more concisely

$$\vec{\phi}' = R\vec{\phi}, \quad \text{where} \quad \vec{\phi} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \text{ and } R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.8}$$

so that invariance is just the statement that the length of a two dimensional vector is invariant under rotations, $(R\vec{r}) \cdot (R\vec{r}) = \vec{r} \cdot \vec{r}$. Note that $R^T R = \mathbb{1}$. The set of symmetry transformations form a group, $O(2)$; the rotations given explicitly in (3.8) have determinant $+1$ and tehy form the group $SO(2)$. For this discussion we focus on transformations that can be reached from $\mathbb{1}$ continuously, so we restrict our attention to $SO(2)$. Setting $\theta = \epsilon$ infinitesimal in (3.8),

$$R = \mathbb{1} + \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbb{1} + \epsilon D \qquad \text{and} \qquad \delta\phi_n = \epsilon D_{nm}\phi_m$$

Under this transformation $\delta CL = 0$. By Noether's theorem

$$J^\mu = \pi^\mu_n D_{nm}\phi_m = \pi^\mu_2\phi_1 - \pi^\mu_1\phi_2 = \phi_1\partial^\mu\phi_2 - \phi_1\overleftarrow{\partial}^\mu\phi_2 \equiv \phi_1\overleftrightarrow{\partial}^\mu\phi_2$$

is a conserved current.

To check that the current is conserved we need equations of motion,

$$\partial^2\phi_n + 2V'\phi_n = 0$$

Then

$$\partial^\mu J_\mu = \phi_1\partial^2\phi_2 - (\partial^2\phi_1)\phi_2 = \phi_1(2V'\phi_2) - (2V'\phi_1)\phi_2 = 0$$

The conserved charge is

$$Q = \int d^3x\,(\phi_1\partial_t\phi_2 - \phi_2\partial_t\phi_1) = \int d^3x\,(\phi_1\pi_2 - \phi_2\pi_1) \tag{3.9}$$

For the QFT at a fixed time, say $t = 0$, we have (equal-time) commutation relations

$$i[\pi_n(\vec{x}), \phi_m(\vec{x}\,')] = \delta_{nm}\delta^{(3)}(\vec{a} - \vec{x}\,')$$

and the others (ϕ-ϕ and π-π) vanish. It follows that

$$[Q, \phi_1(\vec{x}\,)] = \int d^3x'\,[\phi_1(\vec{x}\,')\pi_2(\vec{x}\,') - \phi_2(\vec{x}\,')\pi_1(\vec{x}\,'), \phi_1(\vec{x}\,)] = i\phi_2(\vec{x}\,)$$

and similarly

$$[Q, \phi_2(\vec{x}\,)] = -i\phi_1(\vec{x}\,)\,.$$

Together

$$[Q, \phi_n(\vec{x}\,)] = -iD_{nm}\phi_m(\vec{x}\,) = -i\frac{1}{\epsilon}\delta\phi_n(\vec{x}\,) \qquad \text{or} \qquad \delta\phi_n(\vec{x}\,) = i\epsilon[Q, \phi_n(\vec{x}\,)]\,.$$

Since Q is time independent (commutes with H), $\delta\phi_n(\vec{x},t) = i\epsilon[Q,\phi_n(\vec{x},t)]$.

This is in fact a very general result: if $\delta\phi_n = D_{nm}\phi_m$ is a symmetry of $\mathcal{L}$ then the Noether charge Q associated with it has, in the QFT, $\delta\phi_n(\vec{x},t) = i[Q,\phi_n(\vec{x},t)]$.

In order to better understand the physical content of this conserved charge, let's solve the eigensystem for D:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi = \lambda\chi \qquad \Rightarrow \qquad \chi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

Therefore we define the field

$$\psi = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

We do not define a separate field for $\frac{\phi_1+i\phi_2}{\sqrt{2}}$ since this is just $\psi^\dagger$ (classically this would be ψ^*, but recall we are dealing with operators in the quantum theory). Then,

$$[Q,\psi] = \tfrac{1}{\sqrt{2}}(i\phi_2 - i(-i\phi_1)) = -\psi \quad \text{and} \quad [Q,\psi^\dagger] = \tfrac{1}{\sqrt{2}}(i\phi 2 + i(-i\phi_1)) = \psi^\dagger$$

So ψ has charge -1 while $\psi^\dagger$ has charge $+1$. This is better seen from the rotation by a finite amount,

$$\begin{aligned}\psi \to &\ \tfrac{1}{\sqrt{2}}\left[(\cos\theta\phi_1 - \sin\theta\phi_2) - i\left(\sin\theta\phi_1 + \cos\theta\phi_2\right)\right] \\ &= \cos\theta\left(\frac{\phi_1 - i\phi_2}{\sqrt{2}}\right) - i\sin\theta\left(\frac{\phi_1 - i\phi_2}{\sqrt{2}}\right)\end{aligned}$$

or simply

$$\psi \to e^{-i\theta}\psi \qquad \text{and} \qquad \psi^\dagger \to e^{i\theta}\psi^\dagger \tag{3.10}$$

displaying again that $\psi(\psi^\dagger)$ has charge $+1(-1)$. The set of transformations by unitary $n\times n$ matrices form a group, $U(n)$. The transformations in (3.10) are by 1×1 unitary matrices (the phases $e^i\theta$). So the symmetry group is $U(1)$. Since we already knew the symmetry group is $SO(2)$ we see that these groups are really the same (isomorphic). In terms of ψ the classical Lagrangian density is

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - V(\psi^*\psi)$$

exhibiting the symmetry under (3.10) quite explicitly.

We can get some further insights by inspecting the action of Q on states. For this we need to expand the fields in terms of creation and annihilation operators but we do not know how to do that for interacting theories (that is, for general "potential" V), nor do we know how to do that for complex fields ψ. So let's take $V = \frac{1}{2}m^2(\phi_1^2 + \phi_2^2)$ and analyze in terms of real fields. We have

$$\phi_n(x) = \int (dk)\left(e^{-ik\cdot x}\alpha_{\vec{k},n} + e^{ik\cdot x}\alpha^\dagger_{\vec{k},n}\right)$$

Plugging this in (3.9) and computing we get

$$Q = i\int(dk)\left(\alpha^\dagger_{\vec{k},2}\alpha_{\vec{k},1} - \alpha^\dagger_{\vec{k},1}\alpha_{\vec{k},2}\right)$$

If we label the particles by an "internal" quantum number, $|\vec{k},1\rangle = \alpha^\dagger_{\vec{k},1}|0\rangle$ and $|\vec{k},2\rangle = \alpha^\dagger_{\vec{k},2}|0\rangle$, then

$$Q|\vec{k},1\rangle = i|\vec{k},2\rangle, \quad Q|\vec{k},2\rangle = i|\vec{k},1\rangle$$

just like the transformation of the fields. This gives relations between probability amplitudes. For this it is important that $[Q,H]=0$ (which we know holds, but you can verify explicitly for Q and H in terms of creation/annihilation operators). The relation obtained by an infinitesimal transformation is of the form

$$\langle\psi_f|e^{iHt}\,(Q|\psi_i\rangle) = (\langle\psi_f|Q)\,e^{iHt}|\psi_i\rangle$$

relating the amplitude for $Q|\psi_i\rangle$ to evolve into $|\psi_f\rangle$ in time t, to the amplitude for $|\psi_i\rangle$ to evolve into $Q|\psi_f\rangle$ in the same time. The finite rotation version of this is

$$\langle\psi_f|e^{iHt}|\psi_i\rangle = \langle\psi_f|e^{iHt}e^{-i\theta Q}e^{i\theta Q}|\psi_i\rangle = \langle\psi_f|e^{-i\theta Q}e^{iHt}e^{i\theta Q}|\psi_i\rangle = \langle\psi'_f|e^{iHt}|\psi'_i\rangle$$

where $|\psi'\rangle = e^{i\theta Q}|\psi\rangle$.

Going back to complex fields we have

$$\begin{aligned}
\psi(x) &= \int(dk)\left[\left(\frac{\alpha_{\vec{k},1} - i\alpha_{\vec{k},2}}{\sqrt{2}}\right)e^{-ik\cdot x} + \underbrace{\left(\frac{\alpha^\dagger_{\vec{k},1} - i\alpha^\dagger_{\vec{k},2}}{\sqrt{2}}\right)}_{\neq \text{ h.c. of the first term}} e^{ik\cdot x}\right] \\
&= \int(dk)\left(b_{\vec{k}}e^{-ik\cdot x} + c^\dagger_{\vec{k}}e^{ik\cdot x}\right) \\
\psi^\dagger(x) &= \int(dk)\left(c_{\vec{k}}e^{-ik\cdot x} + b^\dagger_{\vec{k}}e^{ik\cdot x}\right)
\end{aligned}$$

Notice that

$$[b_{\vec{k}}, b^\dagger_{\vec{k}'}] = [c_{\vec{k}}, c^\dagger_{\vec{k}'}] = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k}) \qquad [b_{\vec{k}}, b_{\vec{k}'}] = [c_{\vec{k}}, c_{\vec{k}'}] = 0$$

So these are also creation and annihilation operators, but the one particle states they create are not $|\vec{k},1\rangle$ or $|\vec{k},2\rangle$, but rather superpositions,

$$|\vec{k},+\rangle = b^\dagger_{\vec{k}}|0\rangle = \frac{1}{\sqrt{2}}(|\vec{k},1\rangle + i|\vec{k},2\rangle), \quad \text{and} \quad |\vec{k},-\rangle = c^\dagger_{\vec{k}}|0\rangle = \frac{1}{\sqrt{2}}(|\vec{k},1\rangle - i|\vec{k},2\rangle),$$

It is straightforward to get Q in terms of these operators,

$$\alpha_2^\dagger\alpha_1 - \alpha_1^\dagger\alpha_2 = \tfrac{i}{2}(b^\dagger + c^\dagger)(-b+c) - \left(-\tfrac{i}{2}\right)(-b^\dagger + c^\dagger)(b+c) = -i(b^\dagger b - c^\dagger c)$$

so that

$$Q = \int (dk)\,(b_{\vec{k}}^\dagger b_{\vec{k}} - c_{\vec{k}}^\dagger c_{\vec{k}}) = N_+ - N_-$$

where $N_+(N_-)$ counts the number of particles of charge $+(-)$, so that the total charge $Q = N_+ - N_-$ is conserved.

While complex fields can always be recast in terms of pairs of real fields, they can be very useful! So let's discuss briefly the formulation of a field theory directly in terms of ψ and $\psi^\dagger$. Given $\mathcal{L}(\psi, \psi^*, \partial_\mu\psi, \partial_\mu\psi^*)$ (say, as above), how do we obtain the equations of motion? We can do this by varying ϕ_1 and ϕ_2 independently, but how do we make a variation with respect to a complex field? The seemingly dumbest thing to do is to forget that ψ^* is not independent and vary with respect to both ψ and ψ^* (as if tehy were independent). Surprisingly this works. The general argument is this. Suppose you want to find the extremum of a *real* function $F(z, z^*)$,

$$\delta F = f\delta z + f^*\delta z^*\,,$$

for some f. If we naively treat δz and δz^* as independent we obtain the conditions $f = 0 = f^*$. To do this correctly we write $z = x + iy$. For fixed y, $\delta z^* = \delta z$ so that $f + f^* = 0$; for fixed x, $\delta z^* = -\delta z$ so that $f - f^* = 0$. Combining these conditions we obtain $f = f^* = 0$. This is true for any number, even a continuum, of complex variables. So the equations of motion read

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} - \frac{\partial\mathcal{L}}{\partial\psi^*}$$

Example:

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi$$

We have

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} = \partial^\mu\psi, \qquad \frac{\partial\mathcal{L}}{\partial\psi^*} = -m^2\psi$$

so that

$$(\partial^2 + m^2)\psi = 0\,.$$

This is in accord with the above expansion in terms of plane waves. The real and imaginary parts of ψ satisfy the Klein-Gordon equation. For the Poisson brackets we need the momentum conjugate to ψ,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \partial_t\psi^*\,.$$

Then the equal-time commutation relation in the QFT is

$$i[\pi(\vec{x}), \psi(\vec{x}')] = \delta^{(3)}(\vec{x} - \vec{x}') \quad \Rightarrow \quad i[\partial_t \psi^\dagger(\vec{x}), \psi(\vec{x}')] = \delta^{(3)}(\vec{x} - \vec{x}') \,.$$

The commutation relation for $\psi^\dagger$ and its conjugate momentum is just the hermitian conjugate of this relation.

Internal symmetries: a non-abelian symmetry example Take now a generalization of the previous example, with N real scalar fields, $\phi_n(x)$, with $n = 1, \ldots, N$, and assume $\mathcal{L}(\phi') = \mathcal{L}(\phi)$ where $\phi'_n = R_{nm}\phi_m$ with R_{nm} real and $R_{nl}R_{ml} = \delta_{nm}$, or, in matrix notation, $\phi' = R\phi$ with $R^T R = RR^T = \mathbb{1}$. For example,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_n\partial^\mu\phi_n - V(\phi_n\phi_n) = \tfrac{1}{2}\partial_\mu\phi^T\partial^\mu\phi - V(\phi^T\phi)$$

where in the second step we have rewritten the first expression in matrix form. Note that the set of matrices R that are continuously connected to $\mathbb{1}$ form the group of special orthogonal transformations, $SO(N)$. With $R = \mathbb{1} + \epsilon T$, ϵ infinitesimal, the condition $R^T R = \mathbb{1}$ gives $T^T + T = 0$. That is T is a real antisymmetric and real: there are $\frac{1}{2}N(N-1)$ independent such matrices. For example, for $N = 2$, there is only one such matrix,

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as in our first example. For $N = 3$ there are three independent matrices which we can take to be

$$T^{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Here "12" is a label for the matrix T^{12}, etc. We could as well label the matrices T^a with $a = 1, 2, 3$, with $(T^a)_{mn} = \epsilon_{amn}$. In the general case take

$$(T^{ij})_{kl} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k$$

Then $J^\mu = (\pi^\mu)^T D\phi$ is a set of $\frac{1}{2}N(N-1)$ conserved currents

$$J^{mn}_\mu = \partial_\mu\phi^T T^{mn}\phi = \partial_\mu\phi_k(T^{mn})_{kl}\phi_l = \partial_\mu\phi_m\phi_n - \partial_\mu\phi_n\phi_m = \phi_n \overleftrightarrow{\partial_\mu} \phi_m$$

The matrices T^{mn} don't all commute with each other, so we cannot find simultaneous eigenstates to all of them. We will have more to say about this later and in homework.

Notice the similarity between $J^{\mu mn}$ and $\Delta M^{\mu\nu\lambda}$. In both cases we have the derivative of a field times the field. Notice also the similarity between the numerical coefficients, the matrices $(T^{mn})_{kl}$ and $(\mathcal{I}^{\rho\nu})_{\lambda\sigma}$ encountered in (3.7). This is not an accident. The currents $J^{\mu mn}$ generate $SO(N)$ rotations among the fields ϕ_n while $\Delta M^{\mu\nu\lambda}$ generate $SO(1,3)$ "rotations" among the fields A^μ.

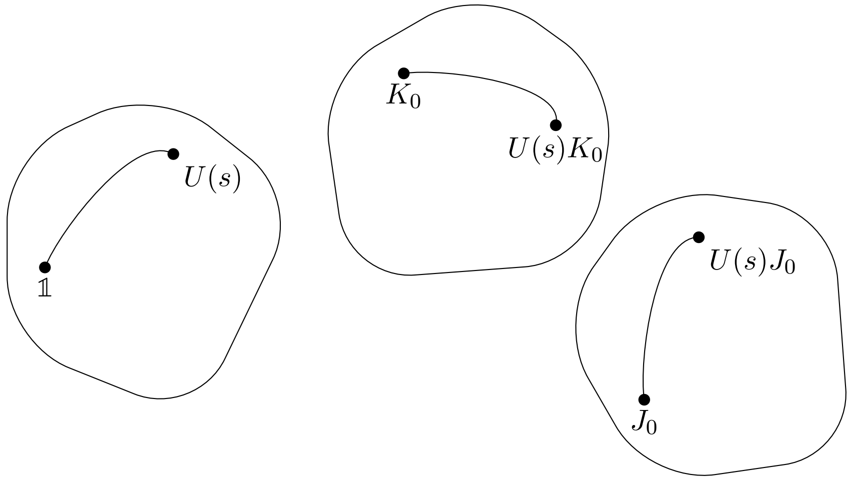


Figure 3.1: Disconnected components of a group of symmetry transformations

3.5 Discrete Symmetries

Not every symmetry transformation is continuously connected to the identity. Example,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - V(\phi^2) \quad \text{is invariant under } \phi(x) \to \phi'(x) = -\phi(x) \tag{3.11}$$

More genrally we can have both transformations that are continuously connected to the identity and transformations that are not. The situation is depicted in Fig. 3.1. The three regions shown form together this example of a group G. One of the disconnected components contains a transformation, K_0 that cannot be reached from $\mathbb{1}$ by a continuous transformation. By definition the connected component of $\mathbb{1}$ contains transformations $U(s)$ that are continuous functions of s and satisfy $U(0) = \mathbb{1}$. We can get to all the elements of G that are in the connected component of K_0 by $U(s)K_0$ (stated without proof). Likewise, the other disconnected component contains a reference element J_0 and all elements are obtained by taking $U((s)J_0$. Since we already understand the physical content of $U(s)$ it suffices to look at a discrete set of symmetry transformations, one per disconnected component, in order to understand this type of group of transformations. These are the *discrete symmetries* we consider now.

Going back to the example in 3.11, we can introduce a unitary operator K such that $K^\dagger \phi(x) K = -\phi(x)$, $K^\dagger K = K K^\dagger = \mathbb{1}$. Note that

$$K^\dagger (K^\dagger \phi(x) K) K = \phi(x) \quad \Rightarrow \quad K^2 = 1 \Rightarrow K^{-1} = K^\dagger = K$$

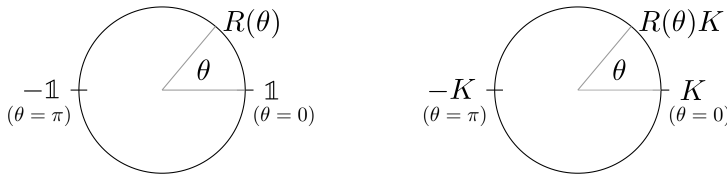


Figure 3.2: Disconnected components of $SO(2) \approx U(1)$

that is K is hermitian. Strictly speaking we did not show that $K^2 = 1$, since we could as well have $K^2 = e^{i\alpha}$; but we are free to choose the transformation and we make the most convenient choice. Clearly $K^\dagger a_{\vec{k}} K = -a_{\vec{k}}$, $K^\dagger a^\dagger_{\vec{k}} K = -a^\dagger_{\vec{k}}$, so assuming the vacuum is symmetric, $K|0\rangle = |0\rangle$, we have $K|\vec{p}_1, \ldots, \vec{p}_n\rangle = (-1)^n|\vec{p}_1, \ldots, \vec{p}_n\rangle$. Since n is just the number of particles in the state we can write a representation of K in terms of the number operator, $K = (-1)^N$. Note that we may not have an explicit representation of N in terms of creation and annihilation operators, as in $N = \int (dk)\, a^\dagger_{\vec{k}} a_{\vec{k}}$, because the potential $V(\phi^2)$ generically produces non-linearities ("interactions") in the equation of motion (the case $V(\phi^2) = \frac{1}{2}m^2\phi^2$ is special). Still, it is still true that $K = (-1)^N$, but the number of particles is not conserved. Since $K^\dagger H K = H$, K is conserved, number of particles, N, is conserved mod 2. That means that evolution can change particle number by even numbers; for example, if we call the particle the "chion", χ, then we can have a reaction $\chi + \chi \to \chi + \chi + \chi + \chi$ but not $\chi + \chi \to \chi + \chi + \chi$. Likewise $2\chi \to 84\chi$ and $3\chi \to 11\chi$ are allowed, but not $7\chi \to 16\chi$. The transformations K and $K^2 = \mathbb{1}$ form a group, $G = \{\mathbb{1}, K\}$, isomorphic to $\mathbb{Z}_2$.

3.5.1 Charge Conjugation (C)

Above we saw the example of a Lagrangian with two real fields,

$$\mathcal{L} = \sum_{n=1}^{2} \tfrac{1}{2}\partial^\mu\phi_n\partial_\mu\phi_n - V\Big(\sum_{n=1}^{2}\phi_n^2\Big) = \partial_\mu\psi^*\partial^\mu\psi - V(\psi^*\psi)$$

which is invariant under $\phi_n(x) \to -\phi_n(x)$. But this is not new, it is an $SO(2)$ transformation, a rotation by angle π:

$$\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \to \left.\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\right|_{\theta=\pi}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}$$

In terms of the complex field $\psi \to e^{i\pi}\psi = -\psi$. The more general form of a transformation by a matrix R that preserves the form of the Lagrangian requires only that R be a real orthogonal matrix, that is, that it satisfies $R^TR = RR^T = \mathbb{1}$. This implies $\det(R) = \pm 1$; the matrices $R(\theta)$ with $\det(R(\theta)) = +1$ are the rotations, elements of $SO(2)$. For each of them $R(\theta)K$ with $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a matrix with negative determinant, an element of $O(2)$ that is not in $SO(2)$. This is shown in Fig. 3.2, where the left disconnected component is $SO(2)$ and the two components together form $O(2)$.

So let's study K. It is just like above, with $K = (-1)^{N_2}$, and for the linear Klein-Gordon theory $N_2 = \int (dk) a^\dagger_{\vec{k},2} a_{\vec{k},2}$. In terms of the complex field, $K^\dagger(\phi_i - i\phi_2)K = \phi_i + i\phi_2$, so that $\psi \to \psi^\dagger$, that is, K acts as hermitian conjugation.

Since ψ carries charge, $\psi \to \psi^\dagger$ conjugates charge. Hence the name,

$$K^\dagger Q K = -Q \qquad \textit{charge conjugation}$$

Recall also that $b = (a_1 - ia_2)/\sqrt{2}$ and $c = (a_1 + ia_2)/\sqrt{2}$; hence $K^\dagger b_{\vec{k}} K = c_{\vec{k}}$ and $K^\dagger c_{\vec{k}} K = b_{\vec{k}}$. Since $b^\dagger$ ($c^\dagger$) creates particles with charge $+1(-1)$ the action of K is to exchange one particle states of charge $+1$ with one particle states of charge -1. We refer to the $Q = +1$ states as particles and to the $Q = -1$ states as anti-particles, and what we have is that charge conjugation exchanges particles with antiparticles.

In the generic case charge conjugation C has $C|\text{particle}\rangle = |\text{antiparticle}\rangle$ and one still has $C^2 = 1$ and $C^{-1} = C^\dagger = C$.

3.5.2 Parity (P)

A familiar discrete symmetry in particle mechanics is space inversion,

$$\vec{x} \to -\vec{x}$$

It's QFT version is called *parity*. This is different than the above in that it acts on x^μ. It is part of the Lorentz group, an orthochronous Lorentz transformation (${\Lambda^0}_0 > 0$) with $\det\Lambda = -1$. As we saw earlier, there are four disconnected components of the Lorentz group. And we want representatives from each component:

(i) $\Lambda = \mathrm{diag}(1,1,1,1)$, ${\Lambda^0}_0 > 0, \det\Lambda = +1$ Identity

(ii) $\Lambda = \mathrm{diag}(-1,1,1,1)$, ${\Lambda^0}_0 < 0, \det\Lambda = -1$ Time Reversal (T, $t \to -t$)

(iii) $\Lambda = \mathrm{diag}(1,-1,-1,-1)$, ${\Lambda^0}_0 > 0, \det\Lambda = -1$ Parity (P, $\vec{x} \to -\vec{x}$)

(iv) $\Lambda = \mathrm{diag}(-1,-1,-1,-1)$, ${\Lambda^0}_0 < 0, \det\Lambda = +1$ PT ($x^\mu \to -x^\mu$)

We'll consider T in the next section.

Consider again the Lagrangian in (3.11). Notice that it satisfies $\mathcal{L}(\phi') = \mathcal{L}(\phi)$ with $\phi'(\vec{x},t) = \phi(-\vec{x},t)$. But we could just as well take $\phi'(\vec{x},t) = -\phi(-\vec{x},t)$. If $\mathcal{L}$ has an internal symmetry under $\phi \to U^\dagger \phi U$ and under a parity transformation $\phi \to U_P^\dagger \phi U_P$, then $\widetilde{U}_P = UU_P$ (and also $U_P U$) defines an equally good parity symmetry transformation, $\phi \to \widetilde{U}_P^\dagger \phi \widetilde{U}_P$.

Terminology: if $\phi(\vec{x},t) \to \phi(-\vec{x},t)$ is a symmetry we say that ϕ is a scalar field, as opposed to if $\phi(\vec{x},t) \to -\phi(-\vec{x},t)$ a symmetry, in which case we say it is a *pseudo-scalar* field. If both are symmetries the distinction is immaterial .

The action of parity on states easily understood in terms of creation and annihilation operators (applicable to Klein-Gordon theory, but the result applies more generally):

$$U_P^\dagger \phi(\vec{x},0) U_P = \phi(-\vec{x},0) \quad \Rightarrow \quad \int (dk) \left(U_P^\dagger \alpha_{\vec{k}} U_P e^{i\vec{k}\cdot\vec{x}} + \text{h.c.} \right) = \int (dk) \left(\alpha_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \text{h.c.} \right)$$
$$\Rightarrow \quad U_P^\dagger \alpha_{\vec{k}}^\dagger U_P = \alpha_{-\vec{k}}^\dagger \quad \Rightarrow \quad U_P |\vec{k}_1, \dots, \vec{k}_n\rangle = |-\vec{k}_1, \dots, -\vec{k}_n\rangle$$

More generally, $\mathcal{L}$ is symmetric under parity if it is invariant under a transformation of the form

$$\phi_n(\vec{x},t) \to \phi'_n(\vec{x},t) = R_{nm} \phi_m(-\vec{x},t) \quad n,m = 1,\dots,N$$

for some real matrix R.

Examples:

(i) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi^2)$ is P and $\mathbb{Z}_2$ invariant.

(ii) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - g\phi^3$ is P invariant with ϕ a scalar (but not a pseudo-scalar).

(iii) Pions are known to be pseudo-scalars. Here is an example related to $\pi^0 \to \gamma\gamma$. It involves the 4-vector potential A^λ for the electro-magnetic field ($\vec{E} = -\partial_0 \vec{A} - \vec{\nabla} A_0$, $\vec{B} = \vec{\nabla} \times \vec{A}$):

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 + g\phi \epsilon^{\mu\nu\lambda\sigma} \partial_\mu A_\nu \partial_\lambda A_\sigma$$

where $\epsilon^{0123} = +1$ is the totally antisymmetric 4-index symbol. You may recall that under parity $\vec{A}(\vec{x},t) \to -\vec{A}(-\vec{x},t)$ and $A_0(\vec{x},t) \to A_0(-\vec{x},t)$. If you do not recall you can either look at Maxwell's equations or you can take a shortcut: from minimal substitution, $\vec{\nabla} \to \vec{\nabla} - e\vec{A}$ one must have $\vec{A}$ transform as $\vec{\nabla}$, and by Lorentz invariance also A_0 as ∂_0. Then the interaction term contributes one of ∂_0 or A_0 and thee of ∂_i or A_i, so $\epsilon^{\mu\nu\lambda\sigma} \partial_\mu A_\nu \partial_\lambda A_\sigma$ is odd under P. Hence $\mathcal{L}$ is P symmetric only if ϕ is a pseudo-scalar, $\phi(\vec{x},t) \to \phi(-\vec{x},t)$. Note that the conclusion is unchanged if A^λ is a *pseudo-vector*, that is, if $\vec{A}(\vec{x},t) \to \vec{A}(-\vec{x},t)$ and $A_0(\vec{x},t) \to -A_0(-\vec{x},t)$ under P.

(iv) Add $h\phi^3$ to the previous example. Now there is no possibility of defining parity that leaves $\mathcal{L}$ invariant.

(v) Consider

$$\mathcal{L} = \sum_{1,2} \left(\tfrac{1}{2}(\partial_\mu \phi_n)^2 - \tfrac{1}{2}m^2\phi_n^2\right) + g\epsilon^{\mu\nu\lambda\sigma}\partial_\mu A_\nu \partial_\lambda A_\sigma(\phi_1^2 - \phi_2^2) + h(\phi_1^3\phi_2 - \phi_1\phi_2^3)\,.$$

This is symmetric under the parity transformation

$$\begin{aligned}\phi_1(\vec{x}, t) &\to -\phi_2(-\vec{x}, t)\\ \phi_2(\vec{x}, t) &\to \phi_1(-\vec{x}, t)\end{aligned}$$

In this example $(U_P^\dagger)^2 \phi_n(\vec{x}, t) U_P^2 = -\phi_n(\vec{x}, t)$. It is not true that $U_P^2 = \mathbb{1}$ in general (although for common applications it is).

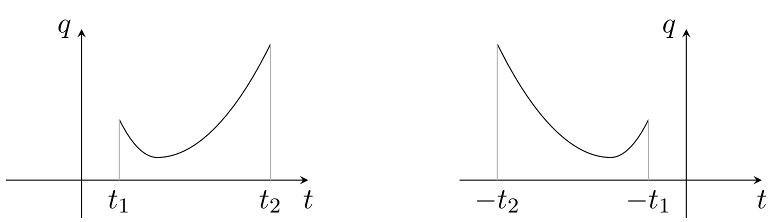
Remark: in the examples above you may feel cheated by the introduction of 4-vector fields, since we have not discussed them in any length. You may instead replace $\epsilon^{\mu\nu\lambda\sigma}\partial_\mu A_\nu \partial_\lambda A_\sigma \to \epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\lambda\phi_3\partial_\sigma\phi_4$, for which you need four additional fields. By itself this term in the Lagrangian does not give rise to any dynamics since it is a total derivative. But when multiplied by a function of yet another field it is no longer a total derivative and gives a non-trivial contribution to equations of motion. So consider

$$\mathcal{L} = \sum_{n=1}^{5} \tfrac{1}{2}(\partial_\mu \phi_n)^2 - g\phi_5 \epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\lambda\phi_3\partial_\sigma\phi_4\,.$$

Under P, $\phi_n(\vec{x}, t) \to (-1)^{\pi_n}\phi_n(-\vec{x}, t)$ and the interaction term transforms by a factor of $-(-1)^{\sum_n \pi_n}$.

3.5.3 Time Reversal (T)

In classical mechanics $t \to -t$ is *not* a symmetry. If $L = L(t)$ only though its dependence on $q = q(t)$ and $\dot{q} = \dot{q}(t)$, then while $L(t) \to L(-t)$ is form invariant, so that $q(-t)$ is a solution of equations of motion, the boundary conditions $q(t_1) = q_1$ and $q(t_2) = q_2$ break the symmetry. This simply means that if L has no explicit time dependence and the motion of $q(t)$ from t_1 to t_2 with $q(t_1) = q_1$ and $q(t_2) = q_2$ is allowed, then so is $q(-t)$ from $q(-t_2)$ to $q(-t_1)$:



We'd like to aim for something analogous in QM: an operator that takes a solution of the equation of motion to another,

$$U_T^{-1} q(t) U_T \stackrel{?}{=} q(-t)$$

However, if U_T is a unitary operator we encounter contradictions:

(i) Since we want $U_T^{-1}\dot{q}(t)U_T \stackrel{?}{=} -\dot{q}(-t)$, then $U_T^{-1}p(t)U_T \stackrel{?}{=} -p(-t)$. Then

$$U_T^{-1}[q(t),p(t)]U_T \stackrel{?}{=} \begin{cases} U_T^{-1} i U_T = i & \text{if commutator is computed first} \\ [q(-t),-p(-t)] = -i & \text{if } U_T \text{ applied to operators first} \end{cases}$$

(ii) For any operator, $A(t) = e^{iHt}A(0)e^{-iHt} \equiv e^{iHt}Ae^{-iHt}$ implies

$$A(-t) \stackrel{?}{=} U_T^{-1}A(t)U_T = U_T^{-1}e^{iHt}U_T U_T^{-1}A(0)U_T U_T^{-1}e^{-iHt}U_T = \left(U_T^{-1}e^{iHt}U_T\right) A \left(U_T^{-1}e^{-iHt}U_T\right)$$

but this is also

$$A(-t) = e^{-iHt}Ae^{iHt}\,.$$

Since this is to hold for any A, we must have,

$$U_T^{-1}e^{iHt}U_T \stackrel{?}{=} e^{-iHt}\,.$$

For $t = \epsilon$, infinitesimal, we expand to get $U_T^{-1}(\mathbb{1} + i\epsilon H)U_T \stackrel{?}{=} \mathbb{1} - i\epsilon H$ or $U_T^{-1}HU_T \stackrel{?}{=} -H$. This means the spectrum of H is the same of $-H$. If H is not bounded from above then it is not bounded from below. This does not seem right, that time reflection symmetry requires a negative energy catastrophe!

The solution to these difficulties is to replace an anti-unitary transformation Ω_T for the unitary U_T. Then in (i) and (ii) above $\Omega_T^{-1} i \Omega_T = -i$, and problem solved! While the problem with (ii) above is resolved, it still leads to the requirement

$$\Omega_T^{-1} H \Omega_T = H \qquad \textit{T-invariance}$$

When this is satisfied

$$\begin{aligned}
(\psi_f, e^{-iH\Delta t}\psi_i) &= (\psi_f, \Omega_T^{-1}e^{iH\Delta t}\Omega_T\psi_i) \qquad (\Delta t = t_f - t_i)\\
&= (\Omega_T\Omega_T^{-1}e^{iH\Delta t}\Omega_T\psi_i, \Omega_T\psi_f)\\
&= (e^{iH\Delta t}\Omega_T\psi_i, \Omega_T\psi_f)\\
&= (\Omega_T\psi_i, e^{-iH\Delta t}\Omega_T\psi_f)
\end{aligned}$$

so the amplitude for ψ_i at $t = t_i$ to evolve to ψ_f at $t = t_f$ is the same as the amplitude for $\Omega_T\psi_f$ at $t = -t_f$ to evolve to $\Omega_T\psi_i$ at $t = -t_i$.

For a single free scalar field,

$$\Omega_T^{-1}\phi(\vec{x},t)\Omega_T = \eta_T\phi(\vec{x},-t)\,,$$

where $\eta_T = \pm 1$ (same ambiguity by $\mathbb{Z}_2$ as in case of P). Then, if ϕ satisfies the Klein-Gordon equation we can expand in terms of creation and annihilation operators and

$$\Omega_T^{-1}\alpha_{\vec{p}}\Omega_T = \alpha_{-\vec{p}},\quad \Omega_T^{-1}\alpha^\dagger_{\vec{p}}\Omega_T = \alpha^\dagger_{-\vec{p}},\quad \Rightarrow \Omega_T|\vec{k}_1,\ldots,\vec{k}_n\rangle = (\eta_T)^n|-\vec{k}_1,\ldots,-\vec{k}_n\rangle\,.$$

Notice that this is much like P. We can define a PT anti-unitary operator $\Omega_{PT}^{-1}\phi(x^\mu)\Omega_{PT} = \eta_{PT}\phi(-x^\mu)$, which is simpler in that it leaves the states $|\vec{k}_1,\ldots,\vec{k}_n\rangle$ unchanged (save for a factor of $(\eta_{PT})^n = (\eta_P\eta_T)^n$).

3.5.4 CPT Theorem (baby version)

Since Ω_{PT} does not seem to do much, maybe any theory is invariant under PT? Answer: no. Example, a complex scalar field with

$$\mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - (h\psi^3 + h^*\psi^{\dagger 3} + g\psi^4 + g^*\psi^{\dagger 4})\,.$$

Then $\Omega_{PT}^{-1}(h\psi^3(x) + g\psi^4(x))\Omega_{PT} = h^*\psi^3(-x) + g^*\psi^4(-x)$ and the interaction part of the Hamiltonian, $H_{\text{int}} = \int d^3x\,(h\psi^3 + h^*\psi^{\dagger 3} + g\psi^4 + g^*\psi^{\dagger 4})$ has $\Omega_{PT}^{-1}H_{\text{int}}\Omega_{PT} = \int d^3x\,(h^*\psi^3 + h\psi^{\dagger 3} + g^*\psi^4 + g\psi^{\dagger 4})$ which is invariant only if $h^* = h$ and $g^* = g$. (Actually the condition is only $(h^*/h)^4 = (g^*/g)^3$, because one can redefine $\psi \to e^{i\alpha}\psi$ and choose to make g or h real, and when $(h^*/h)^4 = (g^*/g)^3$ both can be made simultaneously real). But note that since $U_C^{-1}\psi(x)U_C = \psi^\dagger(x)$, if we combine C with PT we obtain

$$\Omega_{CPT}^{-1}H\Omega_{CPT} = H\,.$$

For this to work it was crucial that $H^\dagger = H$, as well as that $\mathcal{L}$ is Lorentz invariant. More generally, if we have Lorentz invariance but not hermiticity of the Hamiltonian, we would have $\Omega_{CPT}^{-1}H\Omega_{CPT} = H^\dagger$. For example, if $H_{\text{int}} = \int d^3x\,(g\psi^4 + h\psi^{\dagger 4})$ then $\Omega_{CPT}^{-1}H_{\text{int}}\Omega_{CPT} = \int d^3x\,(h^*\psi^4 + g^*\psi^{\dagger 4})$.

Notice that we took $\Omega_{PT}^{-1}\psi(x)\Omega_{PT} = \psi(-x)$, which is natural for a complex field: it leaves $b_{\vec{k}}$ and $c_{\vec{k}}$ unchanged, as was the case of $\alpha_{\vec{k}}$ for real fields. The operation that takes $\psi(x)$ into $\psi^\dagger(-x)$ is CPT. If $\Omega_{CPT}^{-1}\phi_n(x)\Omega_{CPT} = \phi_n(-x)$, then

$$\Omega_{CPT}^{-1}\left(\frac{\phi_1(x) - i\phi_2(x)}{\sqrt{2}}\right)\Omega_{CPT} = \frac{\phi_1(-x) + i\phi_2(-x)}{\sqrt{2}} = \psi^\dagger(-x)\,.$$

To obtain a PT transformation $\psi(x) \to \psi(-x)$ from the transformation of the real fields one must have $\Omega_{PT}^{-1}\phi_1(x)\Omega_{PT} = \phi_1(-x)$ and $\Omega_{PT}^{-1}\phi_2(x)\Omega_{PT} = -\phi_2(-x)$. Of course we are free to define anti-unitary operations from the product of a putative Ω_T and any unitary ones, say, U_C or U_P, and investigate then which of these may be a symmetry of the system under consideration. But we want Ω_T to stand for what we physically interpret as time-reversal, which does not involve exchanging anti-particles for particles.

More generally we arrive at the following *CPT theorem*. Consider $\mathcal{L}$ to be a Lorentz invariant function of real scalar fields $\phi_n(x)$ and complex scalar fields $\psi_i(x)$. If $H^\dagger = H$ then $\Omega_{CPT}^{-1}H\Omega_{CPT} = H$. The proof should be obvious by now. Roughly, $H = H(g, \phi_n, \psi_i, \psi_i^\dagger)$; $H^\dagger = H$ implies $H = H(g^*, \phi_n, \psi_i^\dagger, \psi_i)$, and

$$\begin{aligned}
\Omega_{CPT}^{-1}H(g, \phi_n, \psi_i, \psi_i^\dagger)\Omega_{CPT} &= H(\Omega_{CPT}^{-1}g\Omega_{CPT}, \Omega_{CPT}^{-1}\phi_n\Omega_{CPT}, \Omega_{CPT}^{-1}\psi_i\Omega_{CPT}, \Omega_{CPT}^{-1}\psi_i^\dagger\Omega_{CPT}) \\
&= H(g^*, \phi_n, \psi_i^\dagger, \psi_i) \\
&= H(g, \phi_n, \psi_i, \psi_i^\dagger)\,.
\end{aligned}$$

There is an implicit analysis of the monomials that sum up to H, that shows that Lorentz invariance is sufficient to make the change $x^\mu \to -x^\mu$ a formal invariance. Note that for this it is important that the combination PT is a Lorentz transformation with $\det\Lambda = +1$.

The CPT theorem is a surprising consequence of relativistic invariance in consistent (hermitian Hamiltonian) quantum field theory. It implies, for example, that if a theory is CP invariant (which involves unitary transformations) it automatically is T invariant (an anti-unitary transformation). It also gives equality of properties, like mass, of particles and anti-particles. The latter may seem trivial, but it is not once you consider particles that are complex bound states due to srong forces (like the proton, which by CPT has the same mass, magnitude of charge and magnetic moment, as the anti-proton).