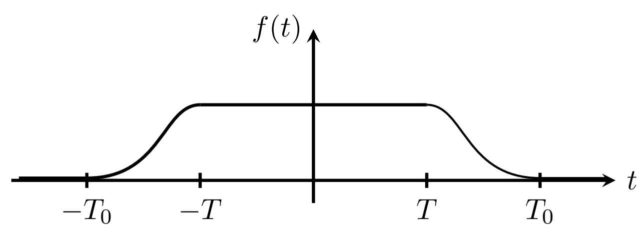
Chapter 5

Elementary Processes

We want to extend the previous discussion to the case where fields interact among themselves, rather than with an external source. The aim is to give an expression for the S-matrix in terms of "in" fields, as before. We will see that uit is convenient to express the matrix elements of S in terms of Green's functions, that is, vacuum expectation values of time-ordered products of elementary fields, as in $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$.

As before the S matrix connects $|\text{in}\rangle$ states to $|\text{out}\rangle$ states, $|\text{out}\rangle = S^{-1}|\text{in}\rangle$, and correspondingly ϕ_{in} fields to ϕ_{out} fields. But we can no longer say that $\phi(x) \to \phi_{\text{in}}(x)$ as $t \to -\infty$ (nor $\phi(x) \to \phi_{\text{out}}(x)$ as $t \to +\infty$) with $\phi_{\text{in,out}}(x)$ free fields (a *free field* is one without interactions, e.g., it satisfies the KG equation).

Before we explain that in more detail let us better understand the role of $|\text{in}\rangle$ states (and ϕ_{in} fields). In a collision process we start with particles that are wildly separated, so interactions between them can be initially neglected. As particles approach each other the interactions can no longer be neglected, they "turn on." So one could think of the situation by replacing H' (the interaction part of the Hamiltonian) by $f(t)H'$, where $f(t)$ is a smooth function that turns on slowly (adiabatically), then stays on for some long period over which the collision takes place, say, $f(t) = 1$ for $-T < t < T$, and then turns off slowly again, $f(t) = 0$ $|t| > T_0$ and $f(t)$ smoothly decreasing (increasing) in $T < t < T_0$ $(-T_0 < t < -T)$:



We want both T and T_0 to be arbitrarily large, and we want $T_0 \gg T$ in the process to avoid sudden changes that can introduce extraneous effects, e.g., pair production.

We will use this model, but it is not quite general enough. The reason is that if the interaction is turned off we may not be able to describe $|\text{in}\rangle$ and $|\text{out}\rangle$ states of interest, namely, bound states that arise because of the interaction. For example, we may want to study collisions of an electron with an H atom due to electromagnetic interactions. But it is the electromagnetic interactions that binds a p and an e into an H atom. More poignant is the case of collisions of protons by the strong interactions when it is these interactions that keeps quarks bound in protons. The idea of collision theory is that oen can set up a muck theory of free particles that happen to have the same mass (and other quantum numbers, e.g., spin) as the bound states. These are the $|\text{in}\rangle$ and $|\text{out}\rangle$ states. For very early (or late) times these states describe the evolution of the particles that later (earlier) participate in the collision. And the S-matrix uses information in the interacting theory to connect the pre- and post-collision states. For theories without bound states we can use the simpler approximation of turning on and off the interaction via $f(t)H'$. Remarkably the expression for the S-matrix obtained via this simplified treatment is the same as in a more complete and rigorous analysis that does not employ it.

If we adiabatically turn on and off the interactions then we can use our previous approach:

$$\phi(x) = \phi_{\text{in}}(x) + \int d^4y\, G_{\text{ret}}(x-y)J(y) \tag{5.1}$$

But now

$$(\partial^2 + m^2)\phi = \frac{\mathcal{L}'}{\partial\phi}$$

(e.g., if $\mathcal{L}' = g\phi^3 + \lambda\phi^4$ then $\frac{\mathcal{L}'}{\partial\phi} = 3g\phi^2 + 4\lambda\phi^4$). So for $J(x)$ use $f(t)\frac{\mathcal{L}'}{\partial\phi}$. Note that in the absence of $f(t)$ this would not work, the "source" would not be localized in time. Now, we had two ways of obtaining the S-matrix from this. One was

$$S^\dagger\phi_{\text{in}}(x)S = \phi_{\text{in}}(x) + \int d^4y\,[\phi_{\text{in}}(x), \phi_{\text{in}}(y)]J(y)\,.$$

But now $J(y)$ depends on $\phi(x)$ so it does not commute with $\phi_{\text{in}}(x)$. This makes it harder to solve for S using this method.

The second method used $\phi = U^\dagger(t)\phi_{\text{in}}(x)U(t)$ and constructed $U(t)$ in terms of $\mathcal{H}'$. This will work. The result was, and still is,

$$S = T\left[\exp\left(i\int d^4x\,\mathcal{L}'_{\text{in}}\right)\right]$$

which we can use, but now with, say, $\mathcal{L}'_{\text{in}} = g\phi^3_{\text{in}} + \lambda\phi^4_{\text{in}}$.

However, the above discussion has to be modified, as we will see shortly, because in general one cannot take $\phi(x) \to \phi_{\text{in}}(x)$ as $t \to -\infty$.

5.1 Källen-Lehmann Spectral Representation

Here we will see that we cannot take $\phi(x) \to \phi_{\rm in}(x)$ as $t \to -\infty$. We study $[\phi(x), \phi(y)]$ and compare with $[\phi_{\rm in}(x), \phi_{\rm in}(y)]$. In particular,

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \sum\!\!\!\!\!\!\int_n \left(\langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle - x \leftrightarrow y\right)$$

Now

$$\langle 0|\phi(x)|n\rangle = \langle 0|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|n\rangle = e^{-ip_n\cdot x}\langle 0|\phi(0)|n\rangle$$

where $\hat{P}^\mu|n\rangle = p_n^\mu|n\rangle$, $\hat{P}^\mu|0\rangle = 0$, and

$$\begin{aligned}\langle 0|[\phi(x), \phi(y)]|0\rangle &= \sum\!\!\!\!\!\!\int_n \left(e^{-ip_n\cdot(x-y)}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle - x \leftrightarrow y\right)\\ &= \sum\!\!\!\!\!\!\int_n \int d^4k\, \delta^{(4)}(p_n - k)\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)|\langle 0|\phi(0)|n\rangle|^2\\ &= \int \frac{d^4k}{(2\pi)^3}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)\rho(k)\end{aligned}$$

where in going from the first to the second line we introduce a factor of $1 = \int d^4k\, \delta^{(4)}(p_n - k)$ and in the last line we defined

$$\rho(k) \equiv \sum\!\!\!\!\!\!\int_n (2\pi)^3\delta^{(4)}(p_n - k)|\langle 0|\phi(0)|n\rangle|^2 = \sigma(k^2)\theta(k^0)\,.$$

The last equality follows from (i) Lorentz invariance and (ii) $p_n^0 > 0$.

Now compare this with the case of free fields. We have computed this, but it is easy to derive from above: $|n\rangle$ is only the one particle states $\vec{p}$, $\sum\!\!\!\!\!\int_n$ is $\int(dp)$, and

$$\langle 0|\phi_{\rm in}(0)|\vec{p}\rangle = \int (dk)\langle 0|\alpha_{\vec{k}}\alpha^\dagger_{\vec{p}}|0\rangle = 1\,.$$

Then

$$\rho(k) = \int (dp)(2\pi)^3\delta^{(4)}(p-k) = \int d^4p\,\theta(p^0)\delta(p^2-m^2)\delta^{(4)}(p-k) = \theta(k^0)\delta(k^2-m^2)\,,$$

and

$$\begin{aligned}\langle 0|[\phi_{\rm in}(x), \phi_{\rm in}(y)]|0\rangle &= \int \frac{d^4k}{(2\pi)^3}\,\theta(k^0)\delta(k^2 - m^2)\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)\\ &= \int \frac{d^4k}{(2\pi)^3}\,\varepsilon(k^0)\delta(k^2 - m^2)e^{-ik\cdot(x-y)} \equiv i\Delta(x-y; m)\end{aligned}$$

Hence

$$\begin{aligned}\langle 0|[\phi(x),\phi(y)]|0\rangle &= \int \frac{d^4k}{(2\pi)^3}\,\sigma(k^2)\varepsilon(k^0)e^{-ik\cdot(x-y)}\\ &= \int \frac{d^4k}{(2\pi)^3}\int_0^\infty d\bar{m}^2\,\delta(k^2-\bar{m}^2)\sigma(k^2)\varepsilon(k^0)e^{-ik\cdot(x-y)}\\ &= \int_0^\infty d\bar{m}^2\sigma(\bar{m}^2)\int \frac{d^4k}{(2\pi)^3}\varepsilon(k^0)\delta(k^2-\bar{m}^2)e^{-ik\cdot(x-y)}\end{aligned}$$

or

$$\boxed{\langle 0|[\phi(x),\phi(y)]|0\rangle = i\int_0^\infty d\bar{m}^2\sigma(m^2)\Delta(x-y;\bar{m})} \tag{5.2}$$

This is the *Källen-Lehmann representation.*

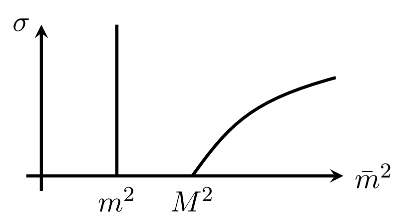
Now we separate the contributions to σ of 1-particle states from $\geq$ 2-particles states. We will assume that $p^0 \geq M > m$ for $\geq$ 2-particle states. For two free particles $p^0 \geq 2m$. If interacting we expect $p^0 \geq 2m-\varepsilon$ where ε is some interaction energy; we are assuming $\varepsilon < m$. If we had $\varepsilon > m$ then the 2-particle energy would be smaller than m and the 1-particle "state" is not a state because it can decay into a lower energy state. Then, if in fact we could demand $\phi(x) \to \phi_{\text{in}}(x)$ as $t \to -\infty$ we should have

$$\langle 0|\phi(0)|\vec{p}\rangle \stackrel{?}{=} \langle 0|\phi_{\text{in}}(0)|\vec{p}\rangle = 1\,,$$

and therefore

$$\sigma(\bar{m}^2) \stackrel{?}{=} \delta(m^2-\bar{m}^2) + \sigma(\bar{m}^2)\theta(\bar{m}-M)\,. \tag{5.3}$$

This is illustrated in the following figure:



Inserting (5.3) in (5.2) gives

$$\langle 0|[\phi(x),\phi(y)]|0\rangle \stackrel{?}{=} i\Delta(x-y;m) + i\int_{M^2}^\infty d\bar{m}^2\sigma(m^2)\Delta(x-y;\bar{m})\,.$$

Taking $\partial/\partial x^0$ of this, and then the limit $y^0 \to x^0$ we obtain on the left hand side the equal time commutator $[\pi(x),\phi(y)] = -i\delta^{(3)}(\vec{x}-\vec{y})$. Then note that on the right hand we can do this again since $i\Delta(x-y;m) = \langle 0|[\phi_{\text{in}}(x),\phi_{\text{in}}(y)]|0\rangle$. This gives

$$1 \stackrel{?}{=} 1 + \int_{M^2}^\infty d\bar{m}^2\,\sigma(\bar{m}^2)$$

If this equation holds then $\sigma(\bar{m}^2) = 0$ for $m > M$. That is $\sigma(\bar{m}^2) = \delta(\bar{m}^2 - m^2)$, which means $\langle 0|\phi(0)|n\rangle = 0$ for any state $|n\rangle$ which has ≥ 2 particles. This then gives $\phi(x) = \phi_{\rm in}(x)$ for all times which makes $\phi(x)$ a free field. We conclude that we cannot demand $\phi(x) \to \phi_{\rm in}(x)$ as $t \to -\infty$. Assume instead

$$\phi(x) \to Z^{\frac{1}{2}}\phi_{\rm in}(x) \qquad \text{as } t \to -\infty$$

Then, repeating the steps above,

$$1 = Z + \int_{M^2}^{\infty} d\bar{m}^2\, \sigma(\bar{m}^2)$$

so that $\sigma > 0$ requires $0 \leq Z < 1$ (that $Z \geq 0$ is from it being $(Z^{\frac{1}{2}})^2$). Similarly, we assume $\phi(x) \to Z^{\frac{1}{2}}\phi_{\rm out}(x) \qquad$ as $t \to +\infty$.

We conclude the section with some useful observations. Uniqueness of the vacuum state gives $|0\rangle_{\rm out} = |0\rangle_{\rm in} = |0\rangle$. (In principle one can have a relative phase, $|0\rangle_{\rm out} = e^{i\alpha}|0\rangle_{\rm in}$ but we conventionally set $\alpha = 0$). Since we are assuming the 1-particle states are stable, they are eigenstates of the Hamiltonian, so they evolve simply, by a phase, e^{-iEt}. Hence $|\vec{k}\rangle_{\rm out} = |\vec{k}\rangle_{\rm in}$ (up to a constant phase that we conventionally set to zero). Now $\langle 0|\phi(x)|\vec{k}\rangle = \langle 0|\phi(0)|\vec{k}\rangle e^{-ik\cdot x}$ so that the prescription to evaluate at $t \to -\infty$ in order to compare with the corresponding expectation value of $\phi_{\rm in}(x)$ is superfluous, and similarly for $t \to \infty$ and expectation values of $\phi_{\rm out}(x)$. So we have

$$\langle 0|\phi(x)|\vec{k}\rangle = Z^{\frac{1}{2}}\langle 0|\phi_{\rm in}(x)|\vec{k}\rangle = Z^{\frac{1}{2}}\langle 0|\phi_{\rm out}(x)|\vec{k}\rangle\,.$$

We collect some basic results for the S-matrix:

$$\phi_{\rm in}(x) = S\phi_{\rm out}(x)S^{-1}\,, \qquad |\psi\rangle_{\rm in} = S|\psi\rangle_{\rm out}\,,$$
$${}_{\rm out}\langle\chi|\psi\rangle_{\rm in} = {}_{\rm out}\langle\chi|S|\psi\rangle_{\rm out} = {}_{\rm in}\langle\chi|S|\psi\rangle_{\rm in}\,.$$

For ψ, χ the vacuum or 1-particle states

$$\langle 0|S|0\rangle = \langle 0|0\rangle = 1\,,$$
$$\langle\vec{k}\,|S|\vec{k}'\rangle = \langle\vec{k}\,|\vec{k}'\rangle = (2\pi)^3 2E_{\vec{k}}\,\delta^{(3)}(\vec{k}' - \vec{k}\,)\,.$$

Finally, if $U = U(a^\mu, \Lambda)$ is a Poincare transformation, covariance means

$$USU^{-1} = S\,.$$

It is often stated in textbooks that $\phi(x) \to Z^{\frac{1}{2}}\phi_{\rm in}(x)$ cannot hold in the strong sense. That is, that it can only hold for separate matrix elements. Else we'd have, the argument goes, for example, $[\phi(x), \phi(y)] \to Z[\phi_{\rm in}(x), \phi_{\rm in}(y)]$ for non-equal, early times, and since for "in" fields this is a c-number, one would be able to argue that $\phi(x)$ is a free field. I think this is overkill. Obviously $\phi(x0$ and $\phi_{\rm in}$ are different, one is an interacting field and one is not. One can produce multiple particle states out of the vacuum —that's the statement that $\sigma > 0$— the other cannot. The statement that $\phi(x) \to Z^{\frac{1}{2}}\phi_{\rm in}(x)$ at $t \to -\infty$ is useful because it gives us the correct way of relating wildly separated initial state (single)-particles created by ϕ to those created by $\phi_{\rm in}$. To make sense of this we need particles that are truly separated, which means we have to consider wave-packets rather than plane waves. We will comment on this when we discuss the S-matrix for multi-particle states.

5.2 LSZ reduction formula: stated

LSZ stands for Lehmann, Symanzik and Zimmermann. The LSZ formula gives the probability amplitude for scattering any number of particles into any number of particles:

$$
{}_{\rm out}\langle \vec{p}_1, \dots, \vec{p}_l | \vec{k}_1, \dots, \vec{k}_n \rangle_{\rm in} = (iZ^{-\frac{1}{2}})^{n+l} \int \prod_{i=1}^{l} d^4y_i \int \prod_{j=1}^{n} d^4x_j \, e^{i\sum_{i=1}^{l} p_i\cdot y_i - i\sum_{j=1}^{n} k_j\cdot x_j}
$$

$$
\times \prod_{i=1}^{l} (\partial^2_{y_i} + m^2) \prod_{j=1}^{n} (\partial^2_{x_j} + m^2) \langle 0| T\left(\phi(y_1)\cdots\phi(y_l)\phi(x_1)\cdots\phi(x_n)\right)|0\rangle \,. \quad (5.4)
$$

Comments:

(i) Computing S matrix elements reduced to computing Green's functions,

$$
G^{(n)}(x_1, \dots, x_n) = \langle 0| T\left(\phi(x_1)\cdots\phi(x_n)\right)|0\rangle \,.
$$

(ii) One can do a more general treatment in term of 1-particle wave-packets. Since the LSZ formula is multilinear in the plane waves for the in and out states, the result amounts to replacing $e^{-ik_i\cdot x_i)} \to f_i(x_i)$ and $e^{ip_i\cdot y_i)} \to f_i^*(y_i)$.

(iii) Integrating by parts $(\partial_x^2 + m^2)e^{\pm ip\cdot x)} = -(p^2 - m^2)e^{\pm ip\cdot x)} = 0$. The result has to be interpreted with care. Let

$$
\int \prod_{i=1}^{n} d^4x_i \, e^{-i\sum_{i=1}^{n} k_i\cdot x_i} G^{(n)}(x_1, \dots, x_n) = (2\pi)^4 \delta^{(4)}(\sum_i k_i) \widetilde{G}^{(n)}(k_1, \dots, k_n) \,. \quad (5.5)
$$

That we always have a $\delta^{(4)}(\sum_i k_i)$ follows form translation invariance. We can change variables to the differences $x_{i+1} - x_i$ together with the center of mass $X = \sum_i x_i$. Then since $G^{(n)}$ does not depend on X we will have $\int d^4X e^{-iR\cdot\sum k_i}$ times the rest. Now, $\widetilde{G}^{(n)}(k_1,\ldots,k_n)$ is defined for arbitrary four vectors, $k_1,\ldots,k_n$, not necessarily satisfying the *on-shell* condition $k_i^2 = m^2$; we say that k_i is *off-shell* if $k_i^2 \neq m^2$, or alternatively, that the "energies," k_i^0, are arbitrary, not given by $\pm E_{\vec{k}_i}$. Incidentally, the on/off-shell language is simply short for the momentum being on/off the *mass-shell*. It may be that $\widetilde{G}^{(n)}(k_1,\ldots,k_n)$ has simple poles as $k^0 \to \pm E_{\vec{k}}$. In fact, by Lorentz invariance the poles must be paired, appearing as poles in $k^2 - m^2$. These poles cancel the zeroes from $\prod(\partial^2 + m^2)$ and the S-matrix element is just the residue:

$$
{}_{\rm out}\langle \vec{p}_1,\ldots,\vec{p}_l|\vec{k}_1,\ldots,\vec{k}_n\rangle_{\rm in} = (iZ^{-\frac{1}{2}})^{n+l}\int\prod_{i=1}^{l} d^4y_i \int\prod_{j=1}^{n} d^4x_j\, e^{i\sum_{i=1}^{l} p_i\cdot y_i - i\sum_{j=1}^{n} k_j\cdot x_j}
$$

$$
\times\int\prod_{k=1}^{l+n} d^4q_k e^{i\sum_{i=1}^{l} q_i\cdot y_i + i\sum_{j=1}^{n} q_j\cdot x_j}\prod_{i=1}^{l+n}(-q_k^2+m^2)(2\pi)^4\delta^{(4)}(\sum_i k_i)\widetilde{G}^{(n)}(q_1,\ldots,q_{n+l})
$$

$$
= (-iZ^{-\frac{1}{2}})^{n+l}(2\pi)^4\delta^{(4)}(\sum_j k_j - \sum_i p_i)
$$

$$
\lim_{\substack{p_i^2\to m^2\\ k_j^2\to m^2}}\prod_{i=1}^{l}(p_i^2 - m^2)\prod_{j=1}^{n}(k_j^2 - m^2)\widetilde{G}^{(n)}(k_1,\ldots,k_n,-p_1,\ldots,-p_l)\,.
$$

and note that each factor of $p^2 - m^2$ comes with a $1/(iZ^{\frac{1}{2}})$.

(iv) It is therefore useful to summarize this in terms of a *scattering amplitude*,

$$
\begin{aligned}
i\mathcal{A} &= i\mathcal{A}(k_1,\ldots,k_n;p_1,\ldots,p_l)\\
&= \lim_{\substack{p_i^2\to m^2\\ k_j^2\to m^2}}\prod_{i=1}^{l}\frac{(p_i^2 - m^2)}{iZ^{\frac{1}{2}}}\prod_{j=1}^{n}\frac{(k_j^2 - m^2)}{iZ^{\frac{1}{2}}}\widetilde{G}^{(n)}(k_1,\ldots,k_n,-p_1,\ldots,-p_l)
\end{aligned}
$$

so that

$$
{}_{\rm out}\langle \vec{p}_1,\ldots,\vec{p}_l|\vec{k}_1,\ldots,\vec{k}_n\rangle_{\rm in} = (2\pi)^4\delta^{(4)}(\sum_j k_j - \sum_i p_i)\, i\mathcal{A}\,.
$$

5.3 S-matrix: perturbation theory

Continuing to present results without justification (which will be given later) we now give the Green's functions (or *correlators* or *n-point functions* in terms of "in" fields:

$$\begin{aligned} G^{(n)}(x_1,\ldots,x_n) &= \langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle \\ &= \langle 0|T(\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)e^{-i\int d^4x\mathcal{H}'_{\rm in}})|0\rangle \end{aligned} \tag{5.6}$$

Expanding the exponential, writing $\epsilon\mathcal{H}'_{\rm in}$ for $\mathcal{H}'_{\rm in}$, with $\epsilon = 1$, just a counting device, and retaining up to some power in ϵ, say ϵ^N, we are approximating $G^{(n)}$ as a perturbative expansion. If $\mathcal{H}'_{\rm in}$ is written explicitly in terms of some parameters, and these can be considered as small, we are then approximating $G^{(n)}$ as an expansion in powers of these small parameters. For example, with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ and $\mathcal{L}' = -\mathcal{H}' = -(\lambda/4!)\phi^4$, then we are expanding $G^{(n)}$ in powers of λ, the *coupling constant*.

Let's see this explicitly in this example. Compute $G^{(4)}(x_1,\ldots,x_4)$:

0th order We take the 1 in the expansion fo the exponential:

$$G_0^{(4)}(x_1,\ldots,x_4) = \langle 0|T(\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_4))|0\rangle$$

To make the notation more compact we drop the label "in" for now and use ϕ_1 for $\phi(x_1$, etc . Using Wick's theorem,

$$\begin{aligned} G_0^{(4)}(x_1,\ldots,x_4) &= \langle 0|T(\phi_1\cdots\phi_4)|0\rangle \\ &= \langle 0|\, :\phi_1\cdots\phi_4: + :\wick{\c1\phi_1\c1\phi_2}\phi_3\phi_4: + :\wick{\c1\phi_1\phi_2\c1\phi_3}\phi_4: + :\wick{\c1\phi_1\phi_2\phi_3\c1\phi_4}: \\ &\quad + :\phi_1\wick{\c1\phi_2\c1\phi_3}\phi_4: + :\phi_1\wick{\c1\phi_2\phi_3\c1\phi_4}: + :\phi_1\phi_2\wick{\c1\phi_3\c1\phi_4}: \\ &\quad + :\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}:\,|0\rangle \end{aligned}$$

Only the last line is non-vanishing, the previous two have normal ordered operators acting on the vacuum. The last line gives

$$G_0^{(4)}(x_1,\ldots,x_4) = \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}$$

Now, we compute $\widetilde{G}^{(4)}$ and then $i\mathcal{A}$:

$$\begin{aligned} \int\prod_{n=1}^{4} d^4x_n\, e^{-i\sum k_n\cdot x_n} G_0^{(4)}(x_1,\ldots,x_4) &= \int d^4x_1\, d^4x_2 e^{-ik_1\cdot x_1 - ik_2\cdot x_2}\wick{\c1\phi_1\c1\phi_2} \\ &\quad\times\int d^4x_3\, d^4x_4 e^{-ik_3\cdot x_3 - ik_4\cdot x_4}\wick{\c1\phi_3\c1\phi_4} + \text{permutations.} \end{aligned}$$

We need

$$\begin{aligned}\int d^4x_1\, d^4x_2 e^{-ik_1\cdot x_1 - ik_2\cdot x_2} \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x_1-x_2)} \frac{i}{q^2-m^2+i\epsilon} \\ = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon} \int d^4x_1\, e^{-ix_1\cdot(k_1+q)} \int d^4x_2\, e^{-ix_2\cdot(k_2-q)} \\ = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon} (2\pi)^4\delta^{(4)}(k_1+q)\,(2\pi)^4\delta^{(4)}(k_2-q) \\ = (2\pi)^4\delta^{(4)}(k_1+k_2)\frac{i}{k_1^2-m^2}\end{aligned}$$

There is no $i\epsilon$ in the last step since we have performed the integral over q^0. Using this above we have

$$\begin{aligned}\widetilde{G}^{(4)}(k_1,\ldots,k_4) &= \frac{i}{k_1^2-m^2}\frac{i}{k_3^2-m^2}(2\pi)^4\delta^{(4)}(k_1+k_2)(2\pi)^4\delta^{(4)}(k_3+k_4) + \text{perms} \\ &= (2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\left[\frac{i}{k_1^2-m^2}\frac{i}{k_3^2-m^2}(2\pi)^4\delta^{(4)}(k_1+k_2) + \text{perms}\right]\end{aligned}$$

Moreover, the scattering amplitude is

$$\begin{aligned}&i\mathcal{A}(k_1,k_2;p_1,p_2) = \\ &\lim_{\substack{p_i^2\to m^2 \\ k_j^2\to m^2}} \left(\frac{p_1^2-m^2}{i}\right)\left(\frac{p_2^2-m^2}{i}\right)\left(\frac{k_1^2-m^2}{i}\right)\left(\frac{k_2^2-m^2}{i}\right)\widetilde{G}^{(4)}(k_1,\ldots,k_4) = 0\,,\end{aligned}$$

which makes sense since at this order in the expansion there is no interaction so there is no scattering.

1st order We now expand the exponential to linear order so that

$$G_1^{(4)}(x_1,\ldots,x_4) = \langle 0|T(\phi_1\cdots\phi_4(-i)\int d^4x\,\tfrac{\lambda}{4!}\phi^4(x))|0\rangle$$

To calculate this we use Wick's theorem. We know from experience gained above that we need the terms with all fields contracted. Let's distinguish terms like

$$\overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4}\,(-i)\int d^4x\,\frac{\lambda}{4!}\overline{\phi(x)\phi(x)}\,\overline{\phi(x)\phi(x)} \quad \text{or} \quad \overline{\phi_1\phi_2}\,(-i)\int d^4x\frac{\lambda}{4!}\overline{\phi_3\overline{\phi_4\phi(x)}\phi(x)}\phi(x)\phi(x)$$

where at least two of the $\phi_1,\ldots,\phi_4$ are contracted among themselves, from terms like

$$-i\frac{\lambda}{4!}\int d^4x\,\overline{\phi_1\overline{\phi_2\overline{\phi_3\overline{\phi_4\phi(x)}\phi(x)}\phi(x)}\phi(x)} \tag{5.7}$$

We call the first kind *disconnected*, the second *connected*. The reason for the terminology will become clear when we introduce a graphical representation of these contractons. We say the fields $\phi_1, \ldots, \phi_4$ are *external* while the ones that appear from inserting powers of the Hamiltonian into the time ordered product are *internal*.

For disconnected terms each contraction $\phi_i \phi_j$ of a pair of external fields will give a single factor of $(2\pi)^4\delta^{(4)}(k_i + k_j)i(k_i^2 - m^2)^{-1}$, and the rest of the factors in that term will be independent of k_i and k_j. Then, when computing the amplitude $i\mathcal{A}$ we'll have

$$\lim_{k_{i,j}^2 \to m^2} (k_i^2 - m^2)(k_j^2 - m^2)\frac{i}{k_i^2 - m^2}(2\pi)^4\delta^{(4)}(k_i + k_j) \times (k_{i,j}\text{-independent}) = 0$$

To get a non-vanishing amplitude we look in connected terms. Note that the one in (5.7) is but one of 4! contractions of this type, and they all give the same result. So we have

$$\begin{aligned} G^{(4)}_{1,\text{conn}}(x_1, \ldots, x_4) &= -i\lambda \int d^4x\, \phi(x_1)\phi(x)\phi(x_2)\phi(x)\phi(x_3)\phi(x)\phi(x_4)\phi(x) \\ &= -i\lambda \int d^4x \prod_{n=1}^{4}\left(\int \frac{d^4q_n}{(2\pi)^4} e^{-iq_n\cdot(x_n - x)}\frac{i}{q_n^2 - m^2 + i\epsilon}\right) \\ &= -i\lambda \int \prod_{n=1}^{4} \frac{d^4q_n}{(2\pi)^4} e^{-iq_n\cdot(x_n - x)}(2\pi)^4\delta^{(4)}(\sum_{n=1}^{4} q_n)\prod_{n=1}^{4}\frac{i}{q_n^2 - m^2 + i\epsilon} \end{aligned}$$

from which we read off

$$\widetilde{G}^{(4)}_{1,\text{conn}}(k_1, \ldots, k_4) = -i\lambda \prod_{n=1}^{4}\frac{i}{q_n^2 - m^2 + i\epsilon}$$

We now use the LSZ formula to compute the scattering amplitude. This entails removing the four propagators and multiplying by Z^{-2}. However, Z itself has a perturbative expansion, with $Z = 1 + \mathcal{O}(\lambda)$, so to the order we are working we can set $Z = 1$. We obtain

$$i\mathcal{A}(k_1, k_2; p_1, p_2) = -i\lambda\,, \tag{5.8}$$

that is

$$_{\text{out}}\langle \vec{p}_1\vec{p}_2|\vec{k}_1\vec{k}_2\rangle_{\text{in}} = -i\lambda(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)$$

5.3.1 Graphical Representation

A graphical representation gives an effective way of communicating and organizing these calculations. Consider the Green's function that we would need to compute the scattering amplitude at p-th order in λ in the perturbative expansion

$$(-i\lambda)^p \langle 0|T\left(\phi(x_1)\cdots\phi(x_4)\,\frac{\phi^4(y_1)}{4!}\cdots\frac{\phi^4(y_p)}{4!}\right)|0\rangle \tag{5.9}$$

In using Wick's theorem to compute this is expanded into a sum of many terms. Each term in the sum is represented by a diagram, and the set of all diagrams, is constructed by drawing:

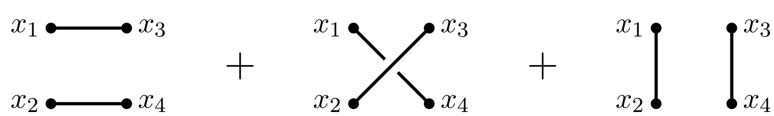
1. An endpoint of a line for each $x_1,\ldots,x_4$. We call these lines *external*.

2. A point, or *vertex*, from which four lines originate for each $y_1,\ldots,y_p$.

3. Connections between the loose ends of the lines, so that all points $x_1,\ldots,y_p$ are connected with lines (with one line emerging from the x's and four from the y's).

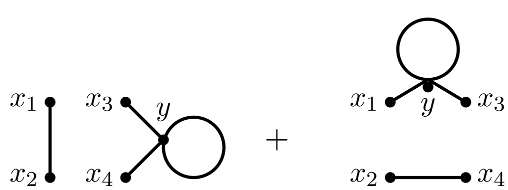
We associate with each line a factor of $\overbracket{\phi(z_a)\phi(z_b)}$, where z_a and z_b are the two points connected by the line. To each vertex we associate a $-i\Lambda$. Finally there is a combinatorial factor arising from equivalent contractions, to compensate for less than 4! possible equivalent contractions; see example and fuller explanation below.

Let's recover the results of our 0-th and 1st order calculations. At lowest order, $p = 0$, so there are not vertices, only four endpoints of external lines:



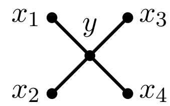
$$= \overbracket{\phi_1\phi_3}\,\overbracket{\phi_2\phi_4} + \overbracket{\phi_1\phi_4}\,\overbracket{\phi_2\phi_3} + \overbracket{\phi_1\phi_2}\,\overbracket{\phi_3\phi_4}$$

These are disconnected terms, and the diagrams are *disconnected diagrams*. As such they give $\mathcal{A} = 0$. Now at first order in perturbation theory, the $p = 1$ term, we have disconnected diagrams,



$$+\ \cdots = -i\lambda\tfrac{1}{2}\,\overbracket{\phi_1\phi_y}\,\overbracket{\phi_3\phi_y}\,\overbracket{\phi_y\phi_y}\,\overbracket{\phi_2\phi_4} + \cdots$$

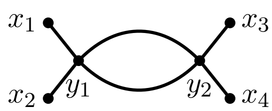
and one connected diagram,



$$= -i\lambda\, \overline{\phi_1\phi_y}\; \overline{\phi_2\phi_y}\; \overline{\phi_3\phi_y}\; \overline{\phi_4\phi_y}$$

Note that the first disconnected diagram has a combinatorial factor of $\frac{1}{2}$, as indicated. This can be seen as follows. Starting from (5.9) with $p = 1$, and insisting that ϕ_1 is contracted with ϕ_2 and both ϕ_3 and ϕ_4 are contracted with ϕ_y's, we see that there is only one possible contraction of ϕ_1 with ϕ_2, then we have 4 ways of contracting ϕ_3 with ϕ_y^4, which ϕ_y^3 un-contracted, and finally we have 3 ways of contracting ϕ_4 with ϕ_y^3. The last step leaves ϕ_y^2 which can give a single contraction of ϕ_y with ϕ_y. That is, there are 4×3 contractions. This times the pre-factor of $\frac{1}{4!}$ gives the symmetry factor of $\frac{1}{2}$.

Here is another example of a combinatorial factor, now for a connected diagram, from $p = 2$:



$$= \frac{1}{2}(-i\lambda)^2\, \overline{\phi_1\phi_{y_1}}\; \overline{\phi_2\phi_{y_1}}\; \overline{\phi_3\phi_{y_2}}\; \left(\overline{\phi_{y_1}\phi_{y_2}}\right)^2$$

The combinatorial factor is obtained as follows. There are 4 ways of contracting ϕ_1 with $\phi_{y_1}^4$, which leaves 3 ways of contracting ϕ_2 with $\phi_{y_1}^3$. Similarly, here are 4 ways of contracting ϕ_3 with $\phi_{y_2}^4$, which leaves 3 ways of contracting ϕ_4 with $\phi_{y_2}^3$. Finally we have to contract $\phi_{y_1}^2$ with $\phi_{y_2}^2$, and there are 2 ways of doing this. We have

$$\left(\frac{1}{4!}\right)^2 \times 4 \cdot 3 \cdot 4 \cdot 3 \cdot 2 = \frac{1}{2}$$

As an exercise you should verify there are five other connected diagrams for the $p = 2$ case and they all have the same combinatorial factor of $\frac{1}{2}$.

5.4 Feynman Graphs

The diagrammatic language above is very useful in computing Green's functions in perturbation theory. But often we are interested in scattering amplitudes which are obtained from the Fourier transform $\widetilde{G}^{(n)}(k_1, \ldots, k_N)$ by the LSZ reduction formula. So it is convenient to replace the rules for the diagrammatic analysis above so that one obtains directly the Fourier transforms $\widetilde{G}^{(n)}$ or even the corresponding scattering amplitude $i\mathcal{A}$.

To this effect, in computing (5.9) we use

$$\overline{\phi_a\phi_b} = \int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot(x_a - x_b)} \frac{i}{q^2 - m^2 + i\epsilon}\,. \tag{5.10}$$

For perturbation theory the interaction term, $-i\lambda \int d^4y\, \phi^4(y)$, is as in (5.9) but integrated over space-time. Each interaction term gives a factor of

$$-i\lambda \int d^4y\, e^{-iy\cdot\sum_n q_n} = -i\lambda(2\pi)^4\delta^{(4)}(\sum_n q_n)\,,$$

where the q_n are from the four contractions $\prod_{i=1}^4 \overline{\phi(y)\phi}(x_i)$. To keep track of the signs $\pm q\cdot x$ in the arguments of the exponential, it is convenient to think of q as an arrow: in (5.10) it is directed from x_a to x_b.

So we have new rules:

1. Draw diagrams with n external "legs" (all topologically distinct diagrams).

2. For each topology assign momenta q_i to each line, including external legs. The assignment is directional: draw an arrow to indicate the direction of q_i, arbitrarily.

3. Every external line carries a factor

$$\int \frac{d^4q_n}{(2\pi)^4} e^{\pm iq_n\cdot x_n} \frac{i}{q_n^2 - m^2 + i\epsilon}\,.$$

   with the plus sign if the arrow for q_n is drawn pointing into the diagram, minus if it points out.

4. Every internal line carries a factor

$$\int \frac{d^4q_i}{(2\pi)^4} \frac{i}{q_i^2 - m^2 + i\epsilon}\,.$$

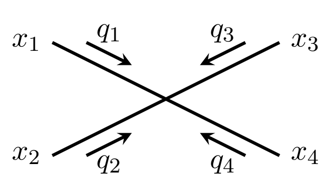
5. Each vertex carries a factor

$$-i\lambda(2\pi)^4\delta^{(4)}(\sum_n (\pm)q_n)$$

   where q_n are the momenta of the lines at the vertex, with the sign assignment $+1$ if q_n is directed into the vertex and -1 if directed out of the vertex.

6. Introduce a correction symmetry factor, as before.

For example, the connected diagram,



gives

$$\prod_{i=1}^{4}\int\frac{d^4q_n}{(2\pi)^4}e^{iq_n\cdot x_n}\frac{i}{q_n^2-m^2+i\epsilon}(-i\lambda)(2\pi)^4\delta^{(4)}(\sum_{n=1}^{4}q_n)$$

If the contraction involves an external leg, when taking the Fourier transform the corresponding coordinate x_i is integrated, $\int d^4x_i e^{-ik_i\cdot x_i}$. This gives

$$\int d^4x_i e^{-ik_i\cdot x_i}\int\frac{d^4q}{(2\pi)^4},e^{iq\cdot(x_i-y)}\frac{i}{q^2-m^2+i\epsilon}=e^{-ik_i\cdot y}\frac{i}{k_i^2-m^2+i\epsilon}$$

Now recall, Eq. (5.5), that $\widetilde{G}^{(n)}(k_1,\ldots,k_n)$ is not really the Fourier transform of $G^{(n)}(x_1,\ldots,x_n)$, but rather

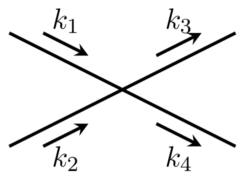
$$\int\prod_{i=1}^{n}d^4x_i\, e^{-i\sum_{i=1}^{n}k_i\cdot x_i}G^{(n)}(x_1,\ldots,x_n)=(2\pi)^4\delta^{(4)}(\sum_i k_i)\widetilde{G}^{(n)}(k_1,\ldots,k_n)\,.$$

The *Feynman rules* tell us how to compute for $(2\pi)^4\delta^{(4)}(\sum_i k_i)\widetilde{G}^{(n)}(k_1,\ldots,k_n)$ in perturbation theory:

1. Draw diagrams with n external "legs" (all topologically distinct diagrams).
2. For each topology find the inequivalent ways of assigning momenta k_i to each external leg. The assignment is directional: k_i goes into the diagram, "out" to "in" if $k_i^0>0$ (draw an arrow to indicate this).
3. Assign a momentum q_n, $n=1,\ldots,I$ to each internal line. Draw an arrow to indicate this momentum direction, arbitrarily.
4. Every external line carries a factor
$$\frac{i}{k_i^2-m^2+i\epsilon}\,.$$
5. Every internal line carries a factor
$$\int\frac{d^4q_i}{(2\pi)^4}\frac{i}{q_i^2-m^2+i\epsilon}\,.$$
6. Each vertex carries a factor
$$-i\lambda(2\pi)^4\delta^{(4)}(\sum_n(\pm)p_n)$$
where p_n are the momenta of the lines at the vertex, with the sign assignment $+1$ if p_n is directed into the vertex and -1 if directed out of the vertex.

7. Introduce a correction symmetry factor, as before.

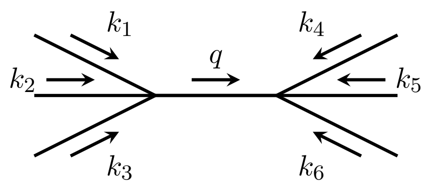
So, for example, to order λ in the perturbative expansion, the connected diagram for the 4-point function is



corresponding to

$$\widetilde{G}_1^{(4)}(k_1,\ldots,k_4) = \prod_{n=1}^{4} \frac{i}{k_n^2 - m^2 + i\epsilon} (-i\lambda)(2\pi)^4\delta^{(4)}(\sum_{n=1}^{4} k_n)$$

Here is another example, a contribution to order λ^2 to $\widetilde{G}^{(6)}$:



corresponding to

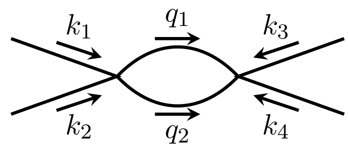
$$\prod_{n=1}^{6}\left(\frac{i}{k_n^2 - m^2 + i\epsilon}\right)\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\epsilon}$$
$$\times\left[(-i\lambda)(2\pi)^4\delta^{(4)}(\sum_{n=1}^{3} k_n - q)\right]\left[(-i\lambda)(2\pi)^4\delta^{(4)}(\sum_{n=3}^{6} k_n + q)\right]$$
$$= (2\pi)^4\delta^{(4)}(\sum_{n=1}^{6} k_n)\left[-\lambda^2\prod_{n=1}^{6}\left(\frac{i}{k_n^2 - m^2}\right)\frac{i}{(k_1+k_2+k_3)^2 - m^2}\right]$$

leading to

$$\widetilde{G}_{\rm conn}^{(6)}(k_1,\ldots,k_6) = -\lambda^2\prod_{n=1}^{6}\left(\frac{i}{k_n^2 - m^2}\right)\frac{i}{(k_1+k_2+k_3)^2 - m^2} + \cdots$$

where the ellipses stand for other connected graphs at order λ^2 (can you display them?) plus terms of higher order in λ. We have removed the $i\epsilon$ from the propagators in the last line since all integrals have been performed.

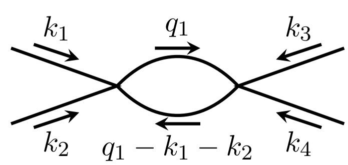
One more example:



gives

$$\frac{1}{2}\prod_{n=1}^{4}\left(\frac{i}{k_n^2-m^2}\right)\prod_{i=1}^{2}\left(\int\frac{d^4q_i}{(2\pi)^4}\,\frac{i}{q_i^2-m^2+i\epsilon}\right)$$
$$\times\left[(-i\lambda)(2\pi)^4\delta^{(4)}(k_1+k_2-q_1-q_2)\right]\left[(-i\lambda)(2\pi)^4\delta^{(4)}(k_3+k_4+q_1+q_1)\right]$$
$$=(2\pi)^4\delta^{(4)}\big(\sum_{n=1}^{4}k_n\big)\frac{1}{2}(-i\lambda)^2\prod_{n=1}^{4}\left(\frac{i}{k_n^2-m^2}\right)\int\frac{d^4q_1}{(2\pi)^4}\,\frac{i}{q_1^2-m^2+i\epsilon}\,\frac{i}{(k_1+k_2-q_1)^2-m^2+i\epsilon}$$

Notice that this result involves a non-trivial integration. This occurs in any diagram for which there is a closed circuit of internal lines: momentum conservation at each vertex, enforced by δ-functions, does not completely fix the momentum of the internal lines. In this example the momentum q_1 appears in two propagators, tracing a closed trajectory, a *loop*. The situation is depicted in a new type of diagram in which the delta functions of momentum conservation have been explicitly accounted for (except an single factor that gives momentum conservation of the external momentum):

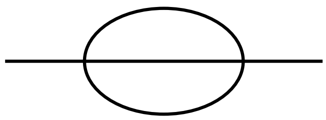


There is no longer an integration for each internal line. Instead there as in integral only over the undetermined momentum q_1. We call q_1 a *loop momentum* and $\int d^4q$ a *loop integral*.

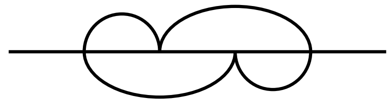
This generalizes. It is always the case that that the δ-functions at each vertex impose momentum conservation and therefore one can always recast the product of *delta*-functions as one $\delta^{(4)}(\sum_n k_n)$ of external momentum times the remaining delta functions. The number L of loop integrals we are left to do in any given diagram is the number of internal lines I minus the number of *delta*-functions, taking away the one for overall momentum conservation. If there are V vertices, we then have $V-1$ δ-functions and therefore

$$L = I - V + 1$$

In our example above, $I = 2$, $V = 2$ and we had $L = 2 - 2 + 1 = 1$ loops. Here are few more examples:

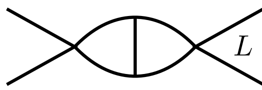


$L = 3 - 2 + 1 = 2$



$L = 7 - 4 + 1 = 4$

We can also have a theory with 3-point and 4-point vertices, as in $\frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4$; here is an example:



$L = 5 - 4 + 2 = 2$

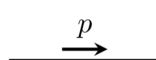
This suggest a more compact set of *Feynman rules* to compute $\widetilde{G}^{(n)}(k_1, \ldots, k_n)$:

1. Draw diagrams with n external "legs" (all topologically distinct diagrams).

2. For each topology find the inequivalent ways of assigning momenta k_i to each external leg. The assignment is directional: k_i goes into the diagram. Draw an arrow to indicate this.

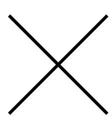
3. Assign q_i, $i = 1, \ldots, L$ momenta to internal lines; draw arrow indicating direction. Assign momenta to remaining $I - L = V - 1$ internal lines by enforcing momentum conservation: at each vertex $\sum p_{\text{in}} = \sum p_{\text{out}}$.

4. For each line



$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

5. For each vertex,



$$= -i\lambda$$

6. Integrate:

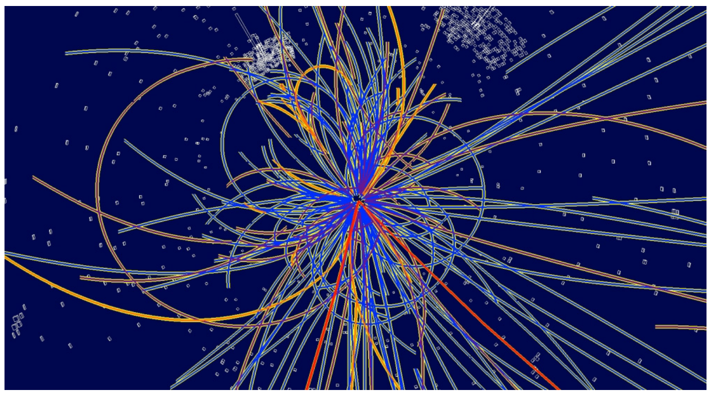
$$\prod_{n=1}^{L} \int \frac{d^4 q_n}{(2\pi)^4}$$

.

7. Symmetry factor $1/S$ as needed.

5.5 Cross Section

In a common experimental setup two beams of elementary particles are accelerated in opposite directions and brought into face to face encounter. Some fraction of the particles in the beams collide. The collision results in a spray of elementary particles emanating form the collision point and an array of detectors surrounding the area register these outgoing particles. Here as a computer generated image of the tracks made by charged particles that are sprayed out of the head-to-head collision of two protons, projected onto a plane transverse to the direction of the protons:



In another common setup a beam of particle impinges on a collection of stationary targets. This second set-up is, of course, just the first one as seen by an observer at rest with the second "beam." We say this observer is in the "lab frame."

We are after a measure of how likely are these collisions to occur. The cross section, σ, for scattering is defined through

$$\frac{\text{number of collisions}}{\text{unit time}} = (\text{flux}) \times \sigma\,.$$

To calculate, rather than computing the number of collisions per unit time from the actual flux, we use unit flux (that of one-on-one particles) and therefore

$$\frac{\text{collision probability}}{\text{unit time}} = (\text{unit flux}) \times \sigma\,.$$

We have an initial state $|i\rangle$ that consists of two particles, a final state $|f\rangle$ that consists of n-particles ($n/ge2$). The probability that $|i\rangle$ evolves into $|f\rangle$ is

$$P_{i\to f} = |\langle f\,\text{out}|i\,\text{in}\rangle|^2 = |\langle f\,\text{in}|S|i\,\text{in}\rangle|^2 = |\langle f|S-1|i\rangle|^2$$

where in the last step we ignore the $f = i$ case (no collision, hence subtract 1 from S) and suppressed the "in" label (we will get tired of carrying it around).

We have to be careful to ask the right question: since we have continuum normalization of states, if we are overly selective in what we want for $|f\rangle$, the probability of finding it in $|i\rangle$ will vanish. Recall if you drop a pin on a piece of paper the probability of hitting a given point on the paper, say, (x_0, y_0), is zero, since a point is a set of measure zero in the set of points that comprise the area of the paper. Likewise, if we set $|f\rangle = |\vec{k}_1, \ldots, \vec{k}_n\rangle$ we'll find $P_{i\to f} = 0$. Instead we project out a subspace of $\mathcal{F}$, rather than a single state. Instead of

$$|\langle f|S-1|i\rangle|^2 = \langle i|(S-1)^\dagger|f\rangle\langle f|S-1|i\rangle$$

we take

$$\langle i|(S-1)^\dagger\Big(\sum_{\substack{f \\ \text{some states}}}|f\rangle\langle f|\Big)(S-1)|i\rangle$$

In particular, for n particles in the final state we have

$$\sum_f |f\rangle\langle f| \to \int (dk_1)\cdots(dk_n)|\vec{k}_1, \ldots, \vec{k}_n\rangle\langle\vec{k}_1, \ldots, \vec{k}_n|$$

where

- we may not want to sum over all possible momenta, so the integrals can be restricted
- must avoid double counting from indistinguishable particles

Suppose particles 1 and 2 are indistinguishable (but the rest are not). Then to avoid double counting one should write

$$\sum_f |f\rangle\langle f| \to \frac{1}{2}\int (dk_1)\cdots(dk_n)|\vec{k}_1, \ldots, \vec{k}_n\rangle\langle\vec{k}_1, \ldots, \vec{k}_n|$$

If 1,2,3 are indistinguishable then the pre-factor becomes 1/3! since the order of $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ in the label of the state is immaterial. More generally,

$$\sum_f |f\rangle\langle f| \to \frac{1}{S}\prod_{i=1}^{n}(dk_i)\,|\vec{k}_1, \ldots, \vec{k}_n\rangle\langle\vec{k}_1, \ldots, \vec{k}_n|$$

where $S = m_1!m_2!\cdots$ where m_i is the number of identical particles of type i $(\sum_i m_i = n)$.

We are ready to give a probability:

$$P_{i\to f} = \frac{\langle i|(S-1)^\dagger \sum |f\rangle\langle f|S-1|i\rangle}{\langle i|i\rangle}$$

We have divided by $\langle i|i\rangle$ because states must be normalized for proper interpretation. Did not divide by normalization of $|f\rangle$ because it is included properly in relativistic measure in the sum over states. We next recast this in terms of the scattering amplitude,

$$\langle f|S-1|i\rangle = (2\pi)^4\delta^{(4)}(P_f - P_i)\, i\mathcal{A}(i\to f).$$

At this point we choose an initial state of plane waves with definite momentum, $|i\rangle = |\vec{p}_1, \vec{p}_2\rangle$. We have

$$\langle i|i\rangle = \langle\vec{p}_1,\vec{p}_2|\vec{p}_1,\vec{p}_2\rangle = \langle\vec{p}_1|\vec{p}_1\rangle\langle\vec{p}_2|\vec{p}_2\rangle$$

but since in general $\langle\vec{p}|\vec{k}\rangle = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})$, we have, $\langle\vec{p}_1|\vec{p}_1\rangle = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(0)$. This is embarrassing, but not disastrous. There are two ways of dealing with this problem. It is not very hard to use wave-packets, which can be properly normalized, instead of plane waves, but won't do here; check it out in some of the textbooks in our bibliography. We have an alternative means of dealing with this problem, which is by putting the system in a finite box of volume V. We have already done so in computing phase space. We discovered there that the correct interpretation of this infinity is $(2\pi)^3\delta^{(3)}(0)\to V$. In fact, we will also use, more generally, $(2\pi)^4\delta^{(4)}(0)\to VT$, where $T = t_{\text{final}} - t_{initial}$. So we can write $\langle i|i\rangle = 4E_1E_2V^2$. Similarly

$$\begin{aligned}
|\langle f|S-1|i\rangle|^2 &= \Big((2\pi)^4\delta^{(4)}(P_f-P_i)\Big)^2 |\mathcal{A}(i\to f)|^2\\
&= \Big((2\pi)^4\delta^{(4)}(0)\Big)(2\pi)^4\delta^{(4)}(P_f-P_i)|\mathcal{A}(i\to f)|^2\\
&= VT(2\pi)^4\delta^{(4)}(P_f-P_i)|\mathcal{A}(i\to f)|^2
\end{aligned}$$

Putting it all together:

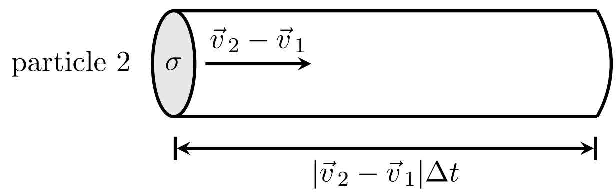
$$\begin{aligned}
\frac{\text{probability}}{\text{time}} &= \frac{1}{T}\frac{VT\frac{1}{S}\int\prod_i(dk_i)\,(2\pi)^4\delta^{(4)}(P_f-P_i)|\mathcal{A}(i\to f)|^2}{4E_1E_2V^2}\\
&= \frac{1}{V}\frac{1}{4E_1E_2}\int|\mathcal{A}(i\to f)|^2 \mathrm{d}\Phi_n
\end{aligned}$$

where

$$\mathrm{d}\Phi_n = \frac{1}{S}\,(2\pi)^4\delta^{(4)}(P_f-P_i)\,(dk_1)\cdots(dk_n)$$

is the Lorentz-invariant n-particle phase space.

Finally, in order to determine the cross section σ we need to divide the above by the unit flux. Assume particle 1 is uniformly distributed in a box of volume V. The probability of finding it is a sub-volume v is v/V. We want v to be the interaction volume, so project a volume forward of particle 2, in the direction of the relative motion $\vec{v}_2-\vec{v}_1$, with cross sectional area σ perpendicular to that direction, when particle 2 moves over a time Δt, as in the following figure:



This has

$$\frac{\text{volume}}{V} = \frac{(|\vec{v}_2 - \vec{v}_1|\Delta t)\sigma}{V} \quad \Rightarrow \quad \frac{\text{probability}}{\text{unit time}} = \frac{|\vec{v}_2 - \vec{v}_1|\sigma}{V}$$

Comparing with the above probability per unit time we have

$$\boxed{\mathrm{d}\sigma = \frac{1}{4E_1E_2|\vec{v}_2 - \vec{v}_1|}\,|\mathcal{A}|^2\mathrm{d}\Phi}$$

where we have written $\mathrm{d}\sigma$ rather than σ to remind us that $\mathrm{d}\Phi$ will be integrated over: $\sigma = \int \mathrm{d}\sigma = \frac{1}{4E_1E_2|\vec{v}_2-\vec{v}_1|}\int |\mathcal{A}|^2\mathrm{d}\Phi$.

The factor $E_1E_2|\vec{v}_2-\vec{v}_1|$ is invariant under boosts along the direction of $\vec{v}_2-\vec{v}_1$. This is most easily seen in a frame where $\vec{v}_1$ and $\vec{v}_2$ are along the z-axis. Then

$$E_1E_2|\vec{v}_2 - \vec{v}_1| = E_1E_2|\frac{p_2}{E_2} - \frac{p_1}{E_1} = |p_2E_1 - p_1E_2| = |\epsilon_{12\mu\nu}p_1^\mu p_2^\nu|$$

is invariant to boosts in the 3-direction. It is useful to compute this factor in the two most common frames, once and for all. In the *Lab frame*: $\vec{p}_2 = 0$, $E_2 = m_2$ so $E_1E_2|\vec{v}_2 - \vec{v}_1| = |p_2E_1 - p_1E_2| = m_2p_1$. In the center of mass, or *CM frame*, $\vec{p}_2 + \vec{p}_1 = 0$, so that $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $E_1E_2|\vec{v}_2 - \vec{v}_1| = |p_2E_1 - p_1E_2| = |\vec{p}|\sqrt{(p_1+p_2)^2}$ where $p_{1,2}$ are 4-vectors. Let

$$s \equiv (p_1 + p_2)^2$$

so that $s = m_1^2 + m_2^2 + 2E_1E_2 + 2|\vec{p}|^2$. Since $E_i = \sqrt{|\vec{p}|^2 + m_i^2}$ we have an equation relating $|\vec{p}|^2$ to s, which we solve:

$$|\vec{p}|^2 = \frac{(s - m_1^2 - m_2^2)^2 - 4m_2^2m_2^2}{4s}$$

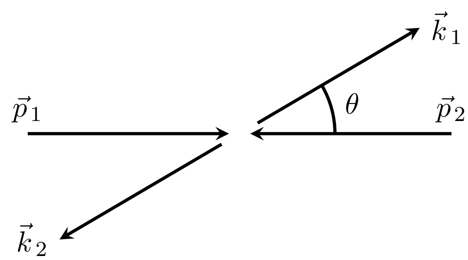
This gives

$$4E_1E_2|\vec{v}_2 - \vec{v}_1| = |\vec{p}|\sqrt{s} = 2\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2}\,.$$

The last expression is valid in any frame boosted along $\vec{v}_2 - \vec{v}_1$, and we can write

$$\mathrm{d}\sigma = \frac{1}{2\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2}}\,|\mathcal{A}|^2\mathrm{d}\Phi$$

Example: $2 \to 2$ scattering, identical particles



We have for identical particles $m_1 = m_2 = m$

$$\mathrm{d}\Phi_2 = \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(P - k_1 k_2)$$
$$= \frac{d^4k_1\, d^4k_2}{(2\pi)^2}\theta(k_1^0)\theta(k_2^0)\delta(k_1^2 - m^2)\delta(k_2^2 - m^2)\delta^{(4)}(P - k_1 k_2)$$

Change variables:

$$\begin{array}{lcl} p = k_1 + k_2 & \Leftrightarrow & k_1 = \frac{1}{2}p + q \\ q = \frac{1}{2}(k_1 - k_2) & & k_2 = \frac{1}{2}p - q \end{array}$$

Note that the Jacobian of the transformation, $\left|\frac{\partial(k_1,k_2)}{\partial(p,q)}\right| = 1$. Then,

$$\mathrm{d}\Phi_2 = \frac{1}{(2\pi)^2}d^4p\delta^{(4)}(P-p)d^4q\theta(\tfrac{1}{2}p^0 + q^0)\theta(\tfrac{1}{2}p^0 - q^0)\delta((\tfrac{1}{2}p + q)^2 - m^2)\delta((\tfrac{1}{2}p - q)^2 - m^2)$$
$$= \frac{1}{(2\pi)^2}d^4q\,\theta(\tfrac{1}{2}P^0 - |q^0|)\delta(\tfrac{1}{4}P^2 + P\cdot q + q^2 - m^2)\delta(2P\cdot q)$$

In the CM frame, $\vec{P} = 0$, this is simple:

$$\mathrm{d}\Phi_2 = \frac{1}{(2\pi)^2}dq^0|\vec{q}|^2 d|\vec{q}|d\cos\theta d\phi\,\theta(\tfrac{1}{2}P^0 - |q^0|)\delta(\tfrac{1}{4}P^2 + (q^0)^2 - |\vec{q}|^2 - m^2)\tfrac{1}{2P^0}\delta(q^0)$$
$$= \frac{1}{(2\pi)^2}d\cos\theta d\phi\,\frac{1}{2P^0}\frac{1}{2}\sqrt{\tfrac{1}{4}(P^0)^2 - m^2}$$
$$= \frac{1}{8(2\pi)^2}d\cos\theta d\phi\,\sqrt{1 - 4m^2/s}$$

where $s = P^2 = (p_1 + p_2)^2$ as before. Therefore

$$\frac{\mathrm{d}\sigma}{d\cos\theta d\phi} = \frac{1}{2}\frac{1}{8(2\pi)^2}\sqrt{1 - 4m^2/s}\frac{1}{2\sqrt{(s - 2m^2)^2 - 4m^2}}|\mathcal{A}|^2$$
$$= \frac{1}{32}\frac{1}{(2\pi)^2}\frac{1}{s}|\mathcal{A}|^2$$

Often $\mathcal{A}$ is independent of ϕ, so

$$\frac{\mathrm{d}\sigma}{d\cos\theta} = \frac{1}{64\pi}\frac{1}{s}|\mathcal{A}|^2$$

5.6 LSZ reduction, again

We want to establish the LSZ reduction formula. We don;t pretend to give a complete proof. The objective is to express the S-matrix in terms of Green' s functions, vacuum expectation values of time ordered products, of in fields. Consider

$$_{\mathrm{out}}\langle\psi|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle\psi|\vec{k}_1\chi\rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle\psi|\alpha^\dagger_{\vec{k}_1\mathrm{in}}\chi\rangle_{\mathrm{in}}$$

Recall

$$\begin{aligned}
\phi_{\mathrm{in}}(x) &= \int (dq)\left(\alpha_{\vec{q}\mathrm{in}}e^{-iq\cdot x} + \alpha^\dagger_{\vec{q}\mathrm{in}}e^{iq\cdot x}\right)\\
\partial_t\phi_{\mathrm{in}}(x) &= \int (dq)\left(-iE_{\vec{q}}\alpha_{\vec{q}\mathrm{in}}e^{-iq\cdot x} + iE_{\vec{q}}\alpha^\dagger_{\vec{q}\mathrm{in}}e^{iq\cdot x}\right)
\end{aligned}$$

Inverting these,

$$\alpha^\dagger_{\vec{q}\mathrm{in}} = -i\int d^3x\, e^{-iq\cdot x}\overleftrightarrow{\partial_t}\,\phi_{\mathrm{in}}(x)$$

So we have

$$\begin{aligned}
{}_{\mathrm{out}}\langle\psi|\vec{k}_1\chi\rangle_{\mathrm{in}} &= -i\int d^3x\, e^{-ik_1\cdot x}\overleftrightarrow{\partial_t}\;{}_{\mathrm{out}}\langle\psi|\phi_{\mathrm{in}}(x)|\chi\rangle_{\mathrm{in}}\\
&= -iZ^{-\frac{1}{2}}\int d^3x\, e^{-ik_1\cdot x}\overleftrightarrow{\partial_t}\;{}_{\mathrm{out}}\langle\psi|\phi(x)|\chi\rangle_{\mathrm{in}} \quad \text{as } t\to-\infty.
\end{aligned}$$

Now we use the fundamental theorem of calculus, $g(t_2) = g(t_1) + \int_{t_1}^{t_2} dt\frac{dg}{dt}$, with $t_2\to\infty$ and $t_1\to-\infty$ so that

$$\begin{aligned}
\lim_{t\to-\infty}\int d^3x\, e^{-ik_1\cdot x}\overleftrightarrow{\partial_t}\;{}_{\mathrm{out}}\langle\psi|\phi(x)|\chi\rangle_{\mathrm{in}} &= \lim_{t\to\infty}\int d^3x\, e^{-ik_1\cdot x}\overleftrightarrow{\partial_t}\;{}_{\mathrm{out}}\langle\psi|\phi(x)|\chi\rangle_{\mathrm{in}}\\
&\quad - \int d^4x\,\partial_t\left(e^{-ik_1\cdot x}\overleftrightarrow{\partial_t}\;{}_{\mathrm{out}}\langle\psi|\phi(x)|\chi\rangle_{\mathrm{in}}\right)
\end{aligned}$$

The first term on the right hand side times $-iZ^{-\frac{1}{2}}$ is ${}_{\mathrm{out}}\langle\psi|\alpha^\dagger_{\vec{k}_1\mathrm{out}}|\chi\rangle_{\mathrm{in}}$, as can be seen by reversing the steps. If $|\psi\rangle_{\mathrm{out}} = |\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_{n'}\rangle_{\mathrm{out}}$ then $\alpha^\dagger_{\vec{k}_1\mathrm{out}}|\psi\rangle_{\mathrm{out}} = (2\pi)^3 2E_{\vec{k}_1}\delta^{(3)}(\vec{k}_1-\vec{p}_1)|\vec{p}_2, \ldots, \vec{p}_{n'}\rangle_{\mathrm{out}} + \cdots + (2\pi)^3 2E_{\vec{k}_1}\delta^{(3)}(\vec{k}_1-\vec{p}_{n'})|\vec{p}_1, \ldots, \vec{p}_{n'-1}\rangle_{\mathrm{out}}$ corresponds to a particle not participating in the scattering. We have no use for this. For the second term on the right hand side we use $\partial_t(f(t)\overleftrightarrow{\partial_t}\, g(t)) = f\partial_t^2 g - (\partial_t^2 f)g$ so it is

$$-\int d^4x\, e^{-iq\cdot x}(\partial_t^2 + E_{\vec{q}}^2)\,{}_{\text{out}}\langle\psi|\phi(x)|\chi\rangle_{\text{in}}$$

Using $E_{\vec{q}}^2 e^{-iq\cdot x} = (|\vec{q}|^2 + m^2)e^{-iq\cdot x} = (-\vec{\nabla}^2 + m^2)e^{-iq\cdot x}$ and integrating by parts we have finally

$${}_{\text{out}}\langle\psi|\vec{k}_1\chi\rangle_{\text{in}} = {}_{\text{out}}\langle\psi|\alpha^\dagger_{\vec{k}_1\text{out}}|\chi\rangle_{\text{in}} + iZ^{-\frac{1}{2}}\int d^4x\, e^{-ik_1\cdot x}(\partial^2 + m^2)\,{}_{\text{out}}\langle\psi|\phi(x)|\chi\rangle_{\text{in}}$$

Similarly

$${}_{\text{out}}\langle\psi\vec{p}|\chi\rangle_{\text{in}} = {}_{\text{out}}\langle\psi|\alpha^\dagger_{\vec{p}\text{in}}|\chi\rangle_{\text{in}} + iZ^{-\frac{1}{2}}\int d^4x\, e^{ip\cdot x}(\partial^2 + m^2)\,{}_{\text{out}}\langle\psi|\phi(x)|\chi\rangle_{\text{in}} \quad (5.11)$$

We woud like to repeat the process until we remove all particles from $|\psi\rangle_{\text{out}}$ and $|\chi\rangle_{\text{in}}$. To see how this goes move a particle off from $|\chi\rangle_{\text{in}}$ from what we already had:

$$\begin{aligned}
{}_{\text{out}}\langle\psi|\phi(0)|\vec{k}\,\chi'\rangle_{\text{in}} &= {}_{\text{out}}\langle\psi|\phi(0)\alpha^\dagger_{\vec{k}\text{in}}|\chi'\rangle_{\text{in}} = -iZ^{-\frac{1}{2}}\lim_{x^0\to-\infty}\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|\phi(0)\phi(x)|\chi'\rangle_{\text{in}} \\
&= iZ^{-\frac{1}{2}}\int d^4x\,\partial_{x^0}\left(e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|\phi(0)\phi(x)|\chi'\rangle_{\text{in}}\right) \\
&\quad - iZ^{-\frac{1}{2}}\lim_{x^0\to\infty}\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|\phi(0)\phi(x)|\chi'\rangle_{\text{in}} \qquad (5.12)
\end{aligned}$$

In the last expression we would like to move $\phi(x)$ to the left of $\phi(0)$ so that we may turn $\phi(x)$ as $x^0 \to \infty$ into $\alpha^\dagger_{\vec{k}\text{out}}$ acting on $\langle\text{out}|$. To this end we rewrite the first term in the last expression in (5.12) using

$$\begin{aligned}
\phi(0)\phi(x) &= (\theta(-x^0) + \theta(x^0))\phi(0)\phi(x) + \theta(x^0)(\phi(x)\phi(0) - \phi(x)\phi(0)) \\
&= T(\phi(x)\phi(0)) + \theta(x^0)[\phi(0), \phi(x)]
\end{aligned}$$

Then in

$$\int d^4x\,\partial_{x^0}\left(e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|\theta(x^0)[\phi(0),\phi(x)]|\chi'\rangle_{\text{in}}\right) \qquad (5.13)$$

when $\overleftrightarrow{\partial}_{x^0}$ hits $\theta(x^0)$ we get the equal time commutator $[\phi(0), \phi(x)] = 0$. So we have (5.13) is

$$\begin{aligned}
&= \int d^4x\,\partial_{x^0}\left[\theta(x^0)\left(e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|[\phi(0),\phi(x)]|\chi'\rangle_{\text{in}}\right)\right] \\
&\qquad\qquad = \lim_{x^0\to\infty}\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial}_{x^0}\,{}_{\text{out}}\langle\psi|[\phi(0),\phi(x)]|\chi'\rangle_{\text{in}}
\end{aligned}$$

Combining this with the last term in (5.12) gives precisely what we want: it reverses the order of $\phi(0)\phi(x)$ so that

$$_{\rm out}\langle\psi|\phi(0)|\vec{k}\,\chi'\rangle_{\rm in} = iZ^{-\frac{1}{2}}\int d^4x\,\left(e^{-ik\cdot x}(\partial^2+m^2)\,_{\rm out}\langle\psi|T(\phi(0)\phi(x))|\chi'\rangle_{\rm in}\right) + {}_{\rm out}\langle\psi|\alpha^\dagger_{\vec{k}\,\rm out}\phi(0)|\chi'\rangle_{\rm in}$$

We thus arrive at

$$\begin{aligned}_{\rm out}\langle\psi|\vec{k}_1\vec{k}_2\chi'\rangle_{\rm in} &= (iZ^{-\frac{1}{2}})^2\int d^4x_1 d^4x_2 e^{-ik_1\cdot x_1 - ik_2\cdot x_2}\\ &\quad\times(\partial^2_{x_1}+m^2)(\partial^2_{x_2}+m^2)\,_{\rm out}\langle\psi|T(\phi(0)\phi(x))|\chi'\rangle_{\rm in} + \text{disconnected}\end{aligned}$$

One can repeat the process until all particles in $|{\rm in}\rangle$ are removed and we are left with $|0\rangle_{\rm in} = |0\rangle$. The argument above can be streamlined by replacing $T(\phi(0)\phi(x))$ for $\phi(0)\phi(x)$ in the line above (5.12)), and this indeed becomes very convenient in completing the argument for arbitrary number of particles.

Similarly we can remove 1-particle states from $_{\rm out}\langle\vec{k}\,\psi|$ using (5.11) repeatedly.

5.7 Perturbation theory, again

We now give a proof of (5.6) that gives us the basis for perturbation theory. Consider

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle\,.$$

Take for definiteness $x^0_1\geq x^0_2\geq\cdots\geq x^0_n$. Recall $\phi(x) = U(t)^{-1}\phi_{\rm in}U(t)$, ans use this in each ϕ in the Green's function:

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0|U^{-1}(t_1)\phi(x_1)U(t_1)U^{-1}(t_2)\phi(x_2)U(t_2)\cdots U^{-1}(t_n)\phi(x_n)U(t_n)|0\rangle\,.$$

Let $U(t,t') = U(t)U^{-1}(t')$. This satisfies

$$U(\infty,-\infty) = S \qquad (5.14)$$

$$U(t,-\infty) = U(t) \qquad (5.15)$$

$$U(t,t')U(t',t'') = U(t,t'') \qquad (5.16)$$

Moreover,

$$i\frac{\partial U(t,t')}{\partial t} = H'_{\rm in}U(t,t'), \qquad \text{with } U(t',t') = 1\,.$$

This is the same equation satisfied by $U(t)$, but with a different boundary condition. So the solution is the same only with different limits of integration,

$$U(t_f,t_i) = T\exp\left(-i\int_{t_i}^{t_f}dt H'_{\rm in}(t)\right)\,.$$

We now have

$$\begin{aligned}G^{(n)} &= \langle 0|U^{-1}(\infty)U(\infty,t_1)\phi_{\rm in}(x_1)U(t_1,t_2)\phi_{\rm in}(x_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)\phi_{\rm in}(x_n)U(t_n,-\infty)U(-\infty)|0\rangle \\ &= \langle 0|U^{-1}(\infty)T\left(U(\infty,t_1)\phi_{\rm in}(x_1)U(t_1,t_2)\phi_{\rm in}(x_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)\phi_{\rm in}(x_n)U(t_n,-\infty)\right)|0\rangle \\ &= \langle 0|U^{-1}(\infty)T\left(U(\infty,t_1)U(t_1,t_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)U(t_n,-\infty)\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\cdots\phi_{\rm in}(x_n)\right)|0\rangle \\ &= \langle 0|U^{-1}(\infty)T\left(U(\infty,-\infty)\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\cdots\phi_{\rm in}(x_n)\right)|0\rangle\end{aligned}$$

Uniqueness of the vacuum means that $|0\rangle$, $|0\rangle_{\rm in}$, $|0\rangle_{\rm out}$, are equal up to a phase. Moreover, $U(\infty)|0\rangle = S|0\rangle$ must be $|0\rangle$ up to a phase. To see this note that S commutes with Poincare transformations, $U(a^\mu,\Lambda)SU^\dagger(a^\mu,\Lambda) = S$ and $|0\rangle$ is the unique state (up to a phase) that is left invariant by a Poincare transformation. Then $U(a^\mu,\Lambda)(S|0\rangle) = U(a^\mu,\Lambda)SU^\dagger(a^\mu,\Lambda)U(a^\mu,\Lambda)|0\rangle = S|0\rangle$ so $S|0\rangle$ is invariant and hence equal to $|0\rangle$ up to a phase. So we have $\langle 0|U^{-1}(\infty) = \langle 0|U^{-1}(\infty)|0\rangle\,\langle 0|$. Using this and

$$U(\infty) = U(\infty,-\infty) = T\exp\left(-i\int_{-\infty}^{\infty} dt\, H'_{\rm in}(t)\right)$$

we finally have

$$G^{(n)}(x_1,\dots,x_n) = \frac{{}_{\rm in}\langle 0|T\left(\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)e^{i\int d^4x\,\mathcal{L}'_{\rm in}}\right)|0\rangle_{\rm in}}{{}_{\rm in}\langle 0|T\left(e^{i\int d^4x\,\mathcal{L}'_{\rm in}}\right)|0\rangle_{\rm in}}$$

Note that we replaced $|0\rangle_{\rm in}$ for $|0\rangle$ since the phases in numerator and denominator cancel.

This is not what we set out to prove. It is better. The denominator corresponds to a set of graphs without external legs. These vacuum graphs can also appear in the numerator, just multiplying any graph with external legs. It is a simple exercise to check that the vacuum graphs in the numerator are cancelled by the graphs in the denominator.

5.7.1 Generating Function for Green's Functions

Let

$$Z[J] = \langle 0|Te^{i\int d^4x\, J(x)\phi(x)}|0\rangle\,.$$

Then

$$G^{(n)}(x_1,\dots,x_n) = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0}$$

and

$$Z[J] = \frac{{}_{\rm in}\langle 0|Te^{i\int d^4x(\mathcal{L}'_{\rm in}+J(x)\phi_{\rm in}(x))}|0\rangle_{\rm in}}{{}_{\rm in}\langle 0|T\left(e^{i\int d^4x\mathcal{L}'_{\rm in}}\right)|0\rangle_{\rm in}}$$

This is a convenient way of summarizing the results above for $G^{(n)}$, all n. Note also that

$$Z[J] = \sum_n \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \, J(x_1) \cdots J(x_n) G^{(n)}(x_1, \ldots, x_n) \, .$$

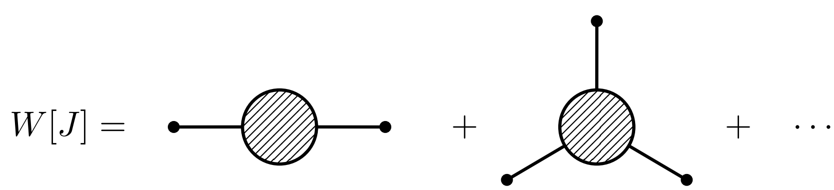
5.7.2 Generating Function for Connected Green's Functions

Similarly we define

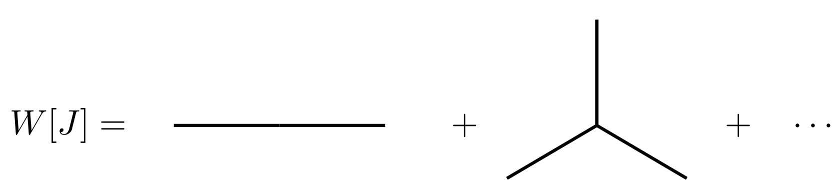
$$W[J] = \sum_n \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \, J(x_1) \cdots J(x_n) G^{(n)}_{\text{conn}}(x_1, \ldots, x_n) \, .$$

We will now show that $Z[J] = e^{iW[J]}$.

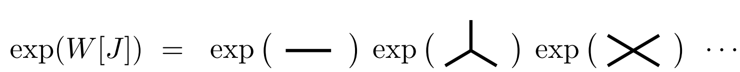
We use a diagrammatic notation to see how this works:



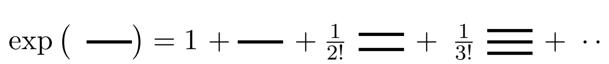
where the heavy dots ﬄ stand for $J(x)$, the hatch circles with n lines stand for $G^{(n)}$ and an integral $\frac{1}{n!}\prod_i \int d^4x_i$ in each term is understood. Let streamline notation for the purposes of this proof: remove the heavy dots (the ends of lines are understood as having them) and reduce the hatch circle to a point, so that the above figure is the same as



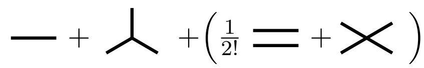
With this notation we consider the exponential of $W[J]$:



Now expand its exponential, as in



and reorganize by powers of J, that is, number fo external legs:

$\exp W[J] = 1 +$  $+ \cdots$

where the ellipses stand for terms with five or more legs. Let's analyze in more detail the term in parenthesis: we want to show that it gives $G^{(4)}$ (times sources, 1/4! and integrated). The "cross" stands for

$$\frac{1}{4!}\int d^4y_1 \cdots d^4y_4\, J(y_1)\cdots J(y_4) G^{(4)}_{\text{conn}}(y_1, \dots, y_4)$$

Now, $\frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}$ of this gives $G^{(4)}_{\text{conn}}(x_1, \dots, x_4)$. Note that the 4! is absent since there are 4! terms from the integral (same as in $\frac{d^4}{dx^4}x^4 = 4!$). Turning to the other term, the disconnected graph, we have

$$\frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}\frac{1}{2!}\left(\frac{1}{2!}\int d^4y d^4z\, J(y)J(z)G^{(2)}(y,z)\right)^2$$
$$= G^{(2)}(x_1,x_2)G^{(2)}(x_3,x_4) + G^{(2)}(x_1,x_3)G^{(2)}(x_2,x_4) + G^{(2)}(x_1,x_4)G^{(2)}(x_2,x_3)$$

If we take the for J-functional derivatives and set $J = 0$ these are the only terms we pick up in the expansion, so we have

$$\frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}e^{W[J]}\Big|_{J=0} = G^{(4)}_{\text{conn}}(x_1, \dots, x_4)$$
$$+ G^{(2)}(x_1,x_2)G^{(2)}(x_3,x_4) + G^{(2)}(x_1,x_3)G^{(2)}(x_2,x_4) + G^{(2)}(x_1,x_4)G^{(2)}(x_2,x_3)$$
$$= G^{(4)}(x_1, \dots, x_4) = \frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}Z[J]\Big|_{J=0}$$

In the general case, the term with J^n in $e^{W[J]}$ is a sum of all possible contributions of the form

$$\frac{1}{n_2!}\left(\frac{1}{2!}\int J_1J_2G_c^{(2)}\right)^{n_2}\frac{1}{n_3!}\left(\frac{1}{3!}\int J_1J_2J_3G_c^{(3)}\right)^{n_3}\cdots$$

such that $2n_2 + 3n_3 + \cdots = n$, in a hopefully obvious condensed notation. For example, $n = 4$ has $(n_2 = 2, n_{\neq 2} = 0) + (n_4 = 1, n_{\neq 4} = 0)$, and $n = 6$ has $(n_6 = 1, n_{\neq 6} = 0) + (n_4 = 1, n_2 = 1, n_{\neq 4,2} = 0) + (n_3 = 2, n_{\neq 3} = 0) + (n_2 = 3, n_{\neq 2} = 0)$. Consider the term with $G_c^{(k)}$: take

$$\frac{\delta^{kn_k}}{\delta J(x_1)\cdots\delta J(x_{kn_k})}\frac{1}{n_k!}\left(\frac{1}{k!}\int J_1\cdots J_kG_c^{(k)}\right)^{n_k}$$

This is completely symmetric under permutations of $x_1, \ldots, x_{kn_k}$. To make this explicit we rewrite it as

$$\frac{1}{n_k!}\left(\frac{1}{k!}\int J_1 \cdots J_k G_c^{(k)}\right)^{n_k} = \frac{1}{n_k!}\frac{1}{(k!)^{n_k}} \times \int \prod_{i=1}^{kn_k} d^4y_i \, J(y_1)\cdots J(y_{kn_k}) G_c^{(k)}(y_1, \ldots, y_k)\cdots G_c^{(k)}(y_{(n_k-1)k+1}, \ldots, y_{n_k k})$$

Taking kn_k J-derivatives we obtain

$$\frac{1}{n_k!}\frac{1}{(k!)^{n_k}} G_c^{(k)}(x_1, \ldots, x_k)\cdots G_c^{(k)}(x_{(n_k-1)k+1}, \ldots, x_{n_k k}) + \text{permutations of } x_1, \ldots, x_{n_k k}$$

This contains many repeated terms. We need to count the number of inequivalent permutations. For each $G_c^{(k)}$ there are $k!$ equivalent permutations of the arguments; this gives $(k!)^{n_k}$. Then we can permute the $G_c^{(k)}$ among themselves; there are $n_k!$ such permutations. So we obtain

$$\frac{\delta^{kn_k}}{\delta J(x_1)\cdots\delta J(x_{kn_k})}\frac{1}{n_k!}\left(\frac{1}{k!}\int J_1\cdots J_k G_c^{(k)}\right)^{n_k} = G_c^{(k)}(x_1, \ldots, x_k)\cdots G_c^{(k)}(x_{(n_k-1)k+1}, \ldots, x_{n_k k}) + \text{inequiv-perms}$$

Finally combine all terms and symmetrize over $x_1, \ldots, x_n$. We obtain all possible combinations of G_c's that can make $G^{(n)}$. But that is precisely what we intended to show.