1. Compute $a_{\vec{k}}|f\rangle = a_{\vec{k}} N e^{\int (dk') f(\vec{k}') a^\dagger_{\vec{k}'}} |0\rangle$

To this end, recall $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 2E_k \delta^3(\vec{k}-\vec{k}')$

Now $e^{-B} A e^{B} = A + [A,B]$ if $[A,B]$ is a c-number. So

$$a_k e^{\int (dk') f(k') a^\dagger_{k'}} = e^{\int (dk') f(k') a^\dagger_{k'}} \left(a_k + [a_k, \int (dk') f(k') a^\dagger_{k'}] \right)$$
$$e^{\int (dk') f(k') a^\dagger_{k'}} (a_k + f(k))$$

$\Rightarrow a_{\vec{k}}|f\rangle = f(k)|f\rangle$

(i) $\phi^{+}(x)|f\rangle = \int (dk)\, e^{ik\cdot x} a_{\vec{k}} |f\rangle = \int (dk)\, e^{ik\cdot x} f(k) |f\rangle = f^{(+)}(x) |f\rangle$

where $f^{(+)}(x) \equiv \int (dk)\, e^{ik x} f(k)$.

(ii) $\langle f|f\rangle = 1 \Leftrightarrow N^2 \langle 0| e^{\int (dk') f(k')^* a_{k'}} e^{\int (dk) f(k) a^\dagger_k} |0\rangle = 1$

Now $e^A e^B = e^{A+B} e^{-\frac{1}{2}[A,B]}$ (Baker-Campbell-Haussdorf formula, for $[A,B]$ a c-number)

$\Rightarrow e^B e^A = e^{A+B} e^{\frac{1}{2}[A,B]} \Rightarrow e^A e^B = e^B e^A e^{-[A,B]}$

$\Rightarrow N^2 \langle 0| e^{\int (dk) f(k) a^\dagger_k} e^{\int (dk) f(k')^* a_{k'}} |0\rangle \; e^{-[\int (dk) f^*(k) a_k, \int (dk') f(k') a^\dagger_{k'}]} = 1$

Use $a_k|0\rangle = 0$, $[a_k, a^\dagger_{k'}] = (2\pi)^3 2E\, \delta^3(k-k') \rightarrow N^2 e^{-\int (dk) |f(k)|^2} = 1$

(iii) $e^{-iHt}|f\rangle = \exp\left[\int (dk) f(k)\, e^{-iHt} a^\dagger_k e^{iHt} \right) e^{-iHt} |0\rangle$

Now, use $H|0\rangle = 0$, and from $H = \int (dk)\, E_k a^\dagger_k a_k \rightarrow [H, a^\dagger_k] = E_k a^\dagger_k$

so that $a^\dagger_k H = (H - E_k) a^\dagger_k \rightarrow a^\dagger_k H^n = (H-E_k)^n a^\dagger_k \Rightarrow e^{-iHt} a^\dagger_k e^{iHt} = e^{-iHt} e^{i(H-E_k)t} a^\dagger_k = e^{-iE_k t} a^\dagger_k$

$\Rightarrow e^{-iHt}|f\rangle = N \exp\left(\int (dk) f(k)\, e^{-iE_k t} a^\dagger_k \right) |0\rangle$

The state $|f_t\rangle$ has $f_t(k) = f(k) e^{-iE_k t}$ and normalization $N_t^2\, e^{-\int (dk) |f(k) e^{-iE_k t}|^2} = 1$. But $|e^{-iE_k t}|^2 = 1$

so this is the same normalization as for the $t=0$ state. So in fact

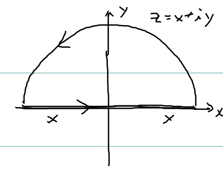
$e^{-iHt}|f\rangle = |f_t\rangle$

2 (i) $G_{ret} = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{(k^0+i\epsilon)^2 - \vec{k}^2 - m^2}$

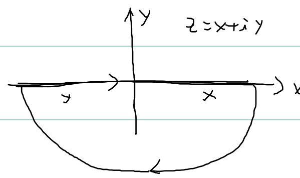
Do first the k^0 integral: $\int dk^0 \frac{e^{-ik^0x^0}}{(k^0+i\epsilon)^2 - E^2}$, where $E = +\sqrt{\vec{k}^2+m^2}$, has poles at $k^0 = \pm E - i\epsilon$.

So consider the contour integral

$$\oint_C dz \frac{e^{-izx^0}}{(z+i\epsilon)^2 - E^2}$$ with C:



for $x^0 < 0$



for $x^0 > 0$

Along the semi-circle of radius $|z| = R$ the integrand vanishes exponentially as $R \to \infty$ (upper semi-circle for $x^0<0$, lower for $x^0>0$), so

- for $x^0 < 0$: $0 = \oint_C dz \frac{e^{-izx^0}}{(z+i\epsilon)^2 - E^2} \to \int_{-\infty}^{\infty} dk^0 \frac{e^{-ik^0x^0}}{(k^0+i\epsilon)^2 - E^2} \Rightarrow G_{ret} = 0$
- for $x^0 > 0$: $\int_{-\infty}^{\infty} dk^0 \frac{e^{-ik^0x^0}}{(k^0+i\epsilon)^2 - E^2} \leftarrow \oint dz \frac{e^{-izx^0}}{(z+i\epsilon)^2 - E^2} = -2\pi i \left[\frac{e^{-iEx^0}}{2E} + \frac{e^{iEx^0}}{(-2E)} \right]$

where we've used Cauchy's theorem and then set $\epsilon = 0$ (as per statement of the problem).

We are left (for $x^0 > 0$) with

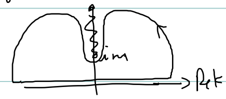
$$G_{ret} = 2\pi i \int \frac{d^3k}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} \frac{1}{2E} \left(e^{-iEx^0} - e^{iEx^0} \right)$$

Using $d^3k = d\phi\, d\cos\theta\, k^2 dk$, the integrand is ϕ-independent ($\int_0^{2\pi} d\phi = 2\pi$) and

$$G_{ret} = \frac{i}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{2E} \left(e^{-iEx^0} - e^{iEx^0} \right) \int_{-1}^{1} d\cos\theta\, e^{ik|\vec{x}|\cos\theta} = \frac{1}{(2\pi)^2} \frac{1}{|\vec{x}|} \int_0^\infty dk \frac{k}{2E} \left(e^{-iEx^0} - e^{iEx^0} \right) \left(e^{ik|\vec{x}|} - e^{-ik|\vec{x}|} \right)$$

$$= \frac{1}{(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{k\,dk}{2E} \left(e^{-iEx^0} - e^{iEx^0} \right) e^{ikr} \quad (\text{with } r = |\vec{x}|)$$

Since $r > 0$, close the contour in upper half plane. But there is a branch point from $E = \sqrt{k^2+m^2}$ at $k = im$; take the branch cut on $\operatorname{Im} k \geq m$:



Then

$$\int_{-\infty}^{\infty} \frac{k\,dk}{E} e^{ikr \pm iEx^0} = \int_{circle} + \int_{cut}$$ but $\int_{circle} \to 0$ as radius $\to \infty$ provided $r > x^0$ (space-like)

and $$\int_{cut} = \int_{i\infty+\epsilon}^{im+\epsilon} + \int_{im-\epsilon}^{i\infty-\epsilon} \frac{k\,dk}{E} e^{ikr \pm iEx^0} = i \int_m^\infty \frac{k\,dk}{\sqrt{k^2-m^2}} e^{-kr} \left[e^{\mp\sqrt{k^2-m^2}\,x^0} + e^{\mp\sqrt{k^2-m^2}\,x^0} \right]$$ (up to an overall sign)

Combining $e^{-iEx^0} - e^{iEx^0}$ terms, $i\int_m^\infty \frac{k\,dk}{\sqrt{k^2-m^2}} e^{-kr} \left[\left(e^{\sqrt{k^2-m^2}\,x^0} + e^{-\sqrt{\cdots}\,x^0} \right) - (\text{same}) \right] = 0$.

(ii) for $m=0$ this is much easier:

$$G_{ret} = \frac{1}{2}\frac{1}{(2\pi)^2}\frac{1}{|x|}\int_0^\infty dk\ \left(e^{-ikx^0} - e^{ikx^0}\right)\left(e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}\right) = \frac{1}{4}\frac{1}{(2\pi)^2}\frac{1}{|x|}\int_{-\infty}^{\infty} dk\ \left(e^{-ikx^0} - e^{ikx^0}\right)\left(e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}\right)$$

which is a sum of delta functions, $\int_{-\infty}^{\infty} dk\ e^{\pm ikx^0} e^{\pm ik|\vec{x}|} = (2\pi)\,\delta(\pm x^0 \pm |\vec{x}|)$

$\Rightarrow G_{ret} = \frac{1}{4}\frac{1}{2\pi}\frac{1}{|\vec{x}|}\left[2\delta(x^0 - |\vec{x}|) - 2\delta(x^0 + |\vec{x}|)\right]$; Now $x^0 > 0$ so $\delta(x^0 + |\vec{x}|) = 0$. Likewise

$\delta\left((x^0)^2 - |\vec{x}|^2\right) = \frac{1}{2|\vec{x}|}\delta(x^0 - |\vec{x}|)$, and we have $G_{ret} = \frac{1}{2\pi}\delta(x^{0\,2} - |\vec{x}|^2) = \frac{1}{2\pi}\delta(x^2)$ (for $x^0 > 0$)

Combining $x^0 < 0$ & $x^0 > 0$ $\Rightarrow$ $\underline{G_{ret} = \frac{1}{2\pi}\delta(x^2)\,\theta(x^0)}$

(iii) $\Delta_+(x) = \int (dk)\, e^{-ik\cdot x} = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\, e^{-iEx^0 + i\vec{k}\cdot\vec{x}}$, up to a factor of $i\theta(x^0)$, the first term in G_{ret}.
This will be non-vanishing for $x^2 < 0$ since in G_{ret} vanishing is from cancellation among two terms.

Copying from $-iG_{ret}$ above, $\Delta_+(x) = \frac{-i}{(2\pi)^2 r}\int_{-\infty}^{\infty}\frac{k\,dk}{2E}\, e^{ikr - iEx^0}$

For $m=0$, $i\Delta_+(x) = \frac{1}{(2\pi)^2 r}\,\frac{1}{2}\int_{-\infty}^{\infty}\frac{k\,dk}{|k|}\, e^{ikr - i|k|x^0} = \frac{1}{(2\pi)^2 2r}\left[-\int_{-\infty}^{0} dk\ e^{ik(r+x^0)} + \int_0^\infty dk\ e^{ik(r-x^0)}\right]$

Now $\int_0^\infty dk\ e^{ika} = \lim_{\epsilon\to 0+}\int_0^\infty dk\ e^{ik(a+i\epsilon)} = \frac{i}{a+i\epsilon}$, $\int_{-\infty}^{0} dk\ e^{ika} = \lim_{\epsilon\to 0+}\int_{-\infty}^{0} dk\ e^{ik(a-i\epsilon)} = -\frac{i}{a-i\epsilon}$

so $i\Delta_+(x) = \lim_{\epsilon\to 0+}\frac{1}{(2\pi)^2 2r}\left[\frac{i}{r-x^0+i\epsilon} + \frac{i}{r+x^0-i\epsilon}\right] = \frac{i}{(2\pi)^2 2r}\,\frac{2r}{r^2 - (x^0 - i\epsilon)^2}$

or $\Delta_+(x) = \frac{1}{(2\pi)^2}\,\frac{1}{x^2}$

(iv) $[\phi(x), \phi(0)] = \int (dk)(dk') \left[\alpha_k e^{-ik\cdot x} + \alpha_k^\dagger e^{ik\cdot x} , \alpha_{k'} + \alpha_{k'}^\dagger \right]$

$= \int (dk) \left(e^{-ik\cdot x} - e^{ik\cdot x} \right)$

$= \Delta_+(x) - \Delta_+(-x)$

We must show $\Delta_+(-x) = \Delta_+(x)$, for $x^2 < 0$, by explicit computation.

Copying from the computation of G_{ret},

$$\Delta_+(x) = \frac{1}{(2\pi)^2 2r} \int_m^\infty \frac{k\,dk}{\sqrt{k^2-m^2}} e^{-kr} \left(e^{\sqrt{k^2-m^2}\,x^0} + e^{-\sqrt{k^2-m^2}\,x^0} \right)$$

is symmetric under $x^0 \to -x^0$ and $\vec{x} \to -\vec{x}$.

3. With $\phi_f(\vec{x},t) = \int d^3y\, f(\vec{x}-\vec{y})\, \phi(\vec{y},t)$

$$\langle 0|\, \phi_f^2(\vec{x},t)\,|0\rangle = \int d^3y\, d^3z\, f(\vec{x}-\vec{y}) f(\vec{x}-\vec{z}) \langle 0|\, \phi(\vec{y},t)\ \phi(\vec{z},t)\,|0\rangle$$

$$= \int d^3y\, d^3z\, f(\vec{x}-\vec{y}) f(\vec{x}-\vec{z})\ \Delta_+(\vec{z}-\vec{y},0)$$

$$= \int (dk) \int d^3y\, d^3z\ e^{i\vec{k}\cdot(\vec{z}-\vec{y})} f(\vec{x}-\vec{y})\, f(\vec{x}-\vec{z})$$

$$= \int (dk)\ \tilde{f}(\vec{k})\ \tilde{f}(-\vec{k})$$

$$= \int (dk)\ |\tilde{f}(\vec{k})|^2$$

where $\tilde{f}(\vec{k})$ is the Fourier transform of $f(\vec{x})$.

For $f(\vec{x}) = \frac{1}{(2\pi b^2)^{3/2}} e^{-\vec{x}^2/2b^2}$, $\tilde{f}(\vec{k}) = \int d^3x\ \frac{e^{i\vec{k}\cdot\vec{x} - \vec{x}^2/2b^2}}{(2\pi b^2)^{3/2}}$

Change variables $\vec{x} \to b\vec{x}$, $\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x\ e^{-\frac{1}{2}\vec{x}^2 + ib\vec{k}\cdot\vec{x}}$

Complete squares and shift $\frac{1}{2}\vec{x}^2 + ib\vec{k}\cdot\vec{x} = \frac{1}{2}(\vec{x}+ib\vec{k})^2 + \frac{1}{2}b^2\vec{k}^2$

$$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\ e^{-\frac{1}{2}b^2\vec{k}^2} \int d^3x\, e^{-\frac{1}{2}\vec{x}^2}$$

Since $\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} = \left[\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} dy\ e^{-\frac{1}{2}y^2}\right]^{1/2} = \left[\int_0^\infty \rho\, d\rho \int_0^{2\pi} d\phi\ e^{-\frac{1}{2}\rho^2}\right]^{1/2} = \left[2\pi \int_0^\infty \frac{1}{2} d\rho^2\ e^{-\frac{1}{2}\rho^2}\right]^{1/2} = \sqrt{2\pi}$

we have $\tilde{f}(\vec{k}) = e^{-\frac{1}{2}b^2\vec{k}^2}$.

For $m=0$ $\langle 0|\, \phi_f^2(\vec{x},t)\,|0\rangle = \int \frac{d^3k}{(2\pi)^3\, 2|k|}\ e^{-b^2k^2} = \frac{4\pi}{(2\pi)^3}\frac{1}{2} \int_0^\infty k\,dk\ e^{-b^2k^2}$

$$= \frac{1}{(2\pi)^2}\ \frac{1}{b^2} \int_0^\infty k\,dk\ e^{-k^2}$$

$$= \frac{1}{8\pi^2 b^2}$$