**Problem 10.2**

(a) $W = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$. At $t_1 = d/c$, $x \geq d = ct_1$, so $\mathbf{E}=0$, $\mathbf{B}=0$, and hence $\boxed{W(t_1)=0.}$

At $T_2 = (d+h)/c$, $ct_2 = d+h$:

$$\mathbf{E} = -\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{z}}, \quad \mathbf{B} = \frac{1}{c}\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{y}},$$

so $B^2 = \frac{1}{c^2}E^2$, and

$$\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \epsilon_0\left(E^2 + \frac{1}{\mu_0\epsilon_0}\frac{1}{c^2}E^2\right) = 2\epsilon_0 E^2.$$

Therefore

$$W(t_2) = \frac{1}{2}(2\epsilon_0)\frac{\mu_0^2\alpha^2}{4}\int_d^{(d+h)}(d+h-x)^2\,dx\,(lw) = \frac{\epsilon_0\mu_0^2\alpha^2 lw}{4}\left[-\frac{(d+h-x)^3}{3}\right]_d^{d+h} = \boxed{\frac{\epsilon_0\mu_0^2\alpha^2 lwh^3}{12}.}$$

(b) $\mathbf{S}(x) = \frac{1}{\mu_0}(\mathbf{B}\times\mathbf{E}) = \frac{1}{\mu_0 c}E^2[-\hat{\mathbf{z}}\times(\pm\hat{\mathbf{y}})] = \pm\frac{1}{\mu_0 c}E^2\,\hat{\mathbf{x}} = \boxed{\pm\frac{\mu_0\alpha^2}{4c}(ct-|x|)^2\,\hat{\mathbf{x}}}$

(plus sign for $x>0$, as here). For $|x|>ct$, $\mathbf{S}=0$.

So the energy per unit time entering the box in this time interval is

$$\frac{dW}{dt} = P = \int \mathbf{S}(d)\cdot d\mathbf{a} = \boxed{\frac{\mu_0\alpha^2 lw}{4c}(ct-d)^2.}$$

Note that no energy flows out the top, since $\mathbf{S}(d+h)=0$.

(c) $$W = \int_{t_1}^{t_2} P\,dt = \frac{\mu_0\alpha^2 lw}{4c}\int_{d/x}^{(d+h)/c}(ct-d)^2\,dt = \frac{\mu_0\alpha^2 lw}{4c}\left[\frac{(ct-d)^3}{3c}\right]_{d/c}^{(d+h)/c} = \boxed{\frac{\mu_0\alpha^2 lwh^3}{12c^2}.}$$

Since $1/c^2 = \mu_0\epsilon_0$, this agrees with the answer to (a).

**Problem 10.4**

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -A_0\cos(kx-\omega t)\,\hat{\mathbf{y}}(-\omega) = \boxed{A_0\omega\cos(kx-\omega t)\,\hat{\mathbf{y}},}$$
$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\sin(kx-\omega t)] = \boxed{A_0 k\cos(kx-\omega t)\,\hat{\mathbf{z}}.}$$

Hence $\nabla\cdot\mathbf{E} = 0$ ✓, $\nabla\cdot\mathbf{B}=0$ ✓.

$$\nabla\times\mathbf{E} = \hat{\mathbf{z}}\frac{\partial}{\partial x}[A_0\omega\cos(kx-\omega t)] = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}}, \quad -\frac{\partial\mathbf{B}}{\partial t} = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}},$$

so $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ ✓.

$$\nabla\times\mathbf{B} = -\hat{\mathbf{y}}\frac{\partial}{\partial x}[A_0 k\cos(kx-\omega t)] = A_0 k^2\sin(kx-\omega t)\,\hat{\mathbf{y}}, \quad \frac{\partial\mathbf{E}}{\partial t} = A_0\omega^2\sin(kx-\omega t)\,\hat{\mathbf{y}}.$$

So $\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$ *provided* $\boxed{k^2 = \mu_0\epsilon_0\omega^2,}$ or, since $c^2 = 1/\mu_0\epsilon_0$, $\boxed{\omega = ck.}$

## 10.5

Ex. 10.1: $\nabla \cdot \mathbf{A} = 0$; $\dfrac{\partial V}{\partial t} = 0$. $\boxed{\text{Both Coulomb and Lorentz.}}$

Prob. 10.3: $\nabla \cdot \mathbf{A} = -\dfrac{qt}{4\pi\epsilon_0} \nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^2}\right) = -\dfrac{qt}{\epsilon_0}\delta^3(\mathbf{r})$; $\dfrac{\partial V}{\partial t} = 0$. $\boxed{\text{Neither.}}$

Prob. 10.4: $\nabla \cdot \mathbf{A} = 0$; $\dfrac{\partial V}{\partial t} = 0$. $\boxed{\text{Both.}}$

**10.7** Consider $\rho(\vec{r}, t) = q(t)\delta^3(\vec{r})$ and $\vec{J}(\vec{r}, t) = -(1/4\pi)(\dot{q}/r^2)\hat{r}$, where $\dot{q} \equiv dq/dt$.

**(a)** The continuity equation is $\nabla \cdot \vec{J} = -d\rho/dt$.

$$\frac{d\rho}{dt} = \dot{q}\,\delta^3(\vec{r})$$
$$\nabla \cdot \vec{J} = -\frac{1}{4\pi}\dot{q}\left(4\pi\delta^3(\vec{r})\right)$$

**(b)** The scalar potential in Coulomb gauge is

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\vec{r'}, t\right)}{\mathfrak{r}} d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r}$$

The differential equation for $\vec{A}$ in Coulomb gauge is

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \mu_0\epsilon_0\nabla \cdot \left(\frac{\partial V}{\partial t}\right)$$

The right-hand side of the above equation vanishes for the charge and current distribution under consideration, which implies $\vec{A} = 0$. (Mathematically this can be accomplished by requiring $\vec{A}$ to vanish at spatial infinity.)

**(c)** The physical fields are

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2}$$
$$\vec{B} = \nabla \times \vec{A} = 0$$

**10.8** The vector potential for a uniform magnetostatic field is $\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}$.

**(a)**

$$\frac{d\vec{A}}{dt} = -\frac{1}{2}\left(\frac{d\vec{r}}{dt} \times \vec{B} + \vec{r} \times \frac{d\vec{B}}{dt}\right)$$

But $d\vec{B}/dt = 0$ for a magnetostatics field, so

$$\frac{d\vec{A}}{dt} = -\frac{1}{2}\frac{d\vec{r}}{dt} \times \vec{B} = -\frac{1}{2}\vec{v} \times \vec{B}$$

**(b)** Confirm

$$\frac{d}{dt}\left(\vec{p} + q\vec{A}\right) = -q\nabla\left(V - \vec{v} \cdot \vec{A}\right)$$

give the correct equation of motion.

$\vec{E} = -\nabla V$ in statics problems,

$$\begin{aligned}\frac{d\vec{p}}{dt} - \frac{q}{2}\,\vec{v} \times \vec{B} &= q\vec{E} + q\nabla\left(\vec{v} \cdot \vec{A}\right) \\ &= q\vec{E} - \frac{q}{2}\nabla\left(\vec{v} \cdot \left(\vec{r} \times \vec{B}\right)\right) \\ &= q\vec{E} + \frac{q}{2}\nabla\left(\vec{r} \cdot \left(\vec{v} \times \vec{B}\right)\right)\end{aligned}$$

Both $\vec{v}$ and $\vec{B}$ are independent of position since it's a uniform magnetic field, so the last term in the above equation can be rewritten using Product Rule (4) from Griffith's including only the terms where derivatives are acting on the position vector.

$$\frac{d\vec{p}}{dt} - \frac{q}{2}\,\vec{v} \times \vec{B} = q\vec{E} + \frac{q}{2}\left[\left(\vec{v} \times \vec{B}\right) \times (\nabla \times \vec{r}) + \left(\left(\vec{v} \times \vec{B}\right) \cdot \nabla\right)\vec{r}\right]$$

Note that

$$\begin{aligned}\nabla \times \vec{r} = 0, &\quad \text{and} \\ \left(\vec{C} \cdot \nabla\right)\vec{r} = \vec{C} &\quad \text{for any } \vec{C}\end{aligned}$$

Putting all of this together, we find

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\,\vec{v} \times \vec{B}$$