1

Show $\vec{E} \cdot \vec{B} = 0$ for TE and TM waves.
For TE waves, $E_z = 0$, and Maxwell's equations yields:

$$\begin{aligned} E_x &= \frac{i}{(\omega/c)^2 - k^2} \omega \frac{\partial B_z}{\partial y} \\ E_y &= -\frac{i}{(\omega/c)^2 - k^2} \omega \frac{\partial B_z}{\partial x} \\ B_x &= \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial B_z}{\partial x} \\ B_y &= \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial B_z}{\partial y} \end{aligned}$$

Thus,

$$\begin{aligned} \vec{E} \cdot \vec{B} &= E_x B_x + E_y B_y + E_z B_z \\ &= \left(\frac{i}{(\omega/c)^2 - k^2} \right)^2 \omega k \left(\frac{\partial B_z}{\partial y} \frac{\partial B_z}{\partial x} - \frac{\partial B_z}{\partial y} \frac{\partial B_z}{\partial x} \right) + 0 B_z \\ &= 0 \end{aligned}$$

An analogous argument works for the TM case.

2

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\text{Re}\left(\vec{\tilde{E}}\times\vec{\tilde{B}}^\star\right)$$

Note that because the x and y components of $\tilde{E}$ and $\tilde{B}$ are purely imaginary, the only non-zero component of the Poynting vector is in the z direction. So

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\frac{\omega k\pi^2 B_0^2}{\left((\omega/c)^2-k^2\right)^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{z}.$$

$$\int\langle\vec{S}\rangle\cdot d\vec{a} = \frac{1}{8\mu_0}\frac{\omega k\pi^2 B_0^2}{\left((\omega/c)^2-k^2\right)^2}ab\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]$$

In the last step I used

$\int_0^a \sin^2(m\pi x/a)\,dx = \int_0^a \cos^2(m\pi x/a)\,dx = a/2$; $\int_0^b \sin^2(n\pi y/b)\,dy = \int_0^b \cos^2(n\pi y/b)\,dy = b/2$.]

Similarly,

$$\begin{aligned}\langle u\rangle &= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)\\ &= \frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\ &+\frac{1}{4\mu_0}\left\{B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right.\\ &\left.+\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right\}.\end{aligned}$$

$$\int\langle u\rangle\,da = \boxed{\frac{ab}{4}\left\{\frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\frac{B_0^2}{4\mu_0}+\frac{1}{4\mu_0}\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]\right\}.}$$

These results can be simplified, using Eq. 9.190 to write $[(\omega/c)^2-k^2] = (\omega_{mn}/c)^2$, $\epsilon_0\mu_0 = 1/c^2$ to eliminate ϵ_0, and Eq. 9.188 to write $[(m/a)^2+(n/b)^2] = (\omega_{mn}/\pi c)^2$:

$$\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\omega kabc^2}{8\mu_0\omega_{mn}^2}B_0^2;\quad \int\langle u\rangle\,da = \frac{\omega^2 ab}{8\mu_0\omega_{mn}^2}B_0^2.$$

Evidently

$$\frac{\text{energy per unit time}}{\text{energy per unit length}} = \frac{\int\langle\mathbf{S}\rangle\cdot d\mathbf{a}}{\int\langle u\rangle\,da} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2-\omega_{mn}^2} = v_g \text{ (Eq. 9.192).}\quad \text{qed}$$

3

Using Product Rule #5, Eq. 10.43 $\Rightarrow$

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-1/2}\\
&= \frac{\mu_0 qc}{4\pi}\mathbf{v}\cdot\left\{-\frac{1}{2}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\right\}\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left\{-2(c^2t-\mathbf{r}\cdot\mathbf{v})\nabla(\mathbf{r}\cdot\mathbf{v})+(c^2-v^2)\nabla(r^2)\right\}.
\end{aligned}$$

Product Rule #4 $\Rightarrow$

$$\begin{aligned}
\nabla(\mathbf{r}\cdot\mathbf{v}) &= \mathbf{v}\times(\nabla\times\mathbf{r})+(\mathbf{v}\cdot\nabla)\mathbf{r},\ \text{but } \nabla\times\mathbf{r}=0,\\
(\mathbf{v}\cdot\nabla)\mathbf{r} &= \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}})=v_x\,\hat{\mathbf{x}}+v_y\,\hat{\mathbf{y}}+v_z\,\hat{\mathbf{z}}=\mathbf{v},\ \text{and}\\
\nabla(r^2) &= \nabla(\mathbf{r}\cdot\mathbf{r})=2\mathbf{r}\times(\nabla\times\mathbf{r})+2(\mathbf{r}\cdot\nabla)\mathbf{r}=2\mathbf{r}.\ \text{So}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left[-2(c^2t-\mathbf{r}\cdot\mathbf{v})\mathbf{v}+(c^2-v^2)2\mathbf{r}\right]\\
&= \frac{\mu_0 qc}{4\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left\{(c^2t-\mathbf{r}\cdot\mathbf{v})v^2-(c^2-v^2)(\mathbf{r}\cdot\mathbf{v})\right\}.\\
&\quad\text{But the term in curly brackets is}: c^2tv^2-v^2(\mathbf{r}\cdot\mathbf{v})-c^2(\mathbf{r}\cdot\mathbf{v})+v^2(\mathbf{r}\cdot\mathbf{v})=c^2(v^2t-\mathbf{r}\cdot\mathbf{v}).\\
&= \frac{\mu_0 qc^3}{4\pi}\frac{(v^2t-\mathbf{r}\cdot\mathbf{v})}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}.
\end{aligned}$$

Meanwhile, from Eq. 10.42,

$$\begin{aligned}
-\mu_0\epsilon_0\frac{\partial V}{\partial t} &= -\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}qc\left(-\frac{1}{2}\right)\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\ \times\\
&\quad\frac{\partial}{\partial t}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left[2(c^2t-\mathbf{r}\cdot\mathbf{v})c^2+(c^2-v^2)(-2c^2t)\right]\\
&= -\frac{\mu_0 qc^3}{4\pi}\frac{(c^2t-\mathbf{r}\cdot\mathbf{v}-c^2t+v^2t)}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}=\nabla\cdot\mathbf{A}.\ \checkmark
\end{aligned}$$