Physics 161: Black Holes

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ABSTRACT

Introduction to Einstein's General Theory of Relativity as applied especially to black holes. Aimed at upper division Physics Majors. Taught as Physics 161 at UCSD. Last taught Spring 2014.

Contents

1. Introduction

This is Physics 161, Black Holes. This course is not a prerequisite for anything, so I am assuming everyone is taking it for interest. This also means that we can modify the content to what you want to learn about. So be sure to communicate to me what you are liking and not liking. I hope this course will be a fun application of your physics, math, and engineering skills, applied to bizarre situations that really occur in nature. It's a big Universe out there and the kind of things going on are pretty amazing. I also find it amazing that we humans – dots on a tiny planet orbiting one of 100 billion stars in one of 100 billion galaxies – can actually work out what the Universe is like. You will hopefully find that the tools needed to do this are within your grasp. Astrophysics cuts across all disciplines, so you will be using much of what you learned previously. The prerequisites for this course are the entire Physics 2 or Physics 4 sequence. Since we will be studying Einstein's Theory of General Relativity (GR) and this is based upon the simpler Special Relativity, it is crucial that you have studied Special Relativity previously. For example, if you only tool Physics 2A,B,C and missed out 2D, then you shouldn't take this course.

GR was Einstein's crowning achievement and in my opinion is one of the greatest achievements humans have yet produced. It is almost never taught at the undergraduate level, so there is really no adequate book. I've developed a way to get to the essential physics without using graduate level math, but we will stretch your math skills a little. I will teach the extra math you need, so don't worry.

My plan for the course is to do a mixture of hand waving explanations, analogies, examples, and mathematical calculations, some of which will be optional for you to learn. So the level of presentation will vary. For this reason interaction, questions and feedback from you will be essential. If you aren't following something, please just raise your hand and say so. GR and Black holes (BH) will definitely blow your mind, and you will discover that many "dumb" questions have very subtle answers. So be sure to ask all the dumb questions you come up with. You can and will understand black holes, curved spacetime, etc. by the end of the quarter.

The grading of this course will be a little different. There will not be any tests. There will be graded homeworks which will account for 60% of the grade. The other 40% will be from a final paper or talk given during the final. Everyone must attend the final and hear the talks. There is a handout that tells about the grading. Note that I am very concerned about academic integrity. You may not copy anything from anybody else, or from any source whatsoever. You must do all the work yourself. You may talk to others on how to do homework problems, but you must work out 100% of everything yourself; otherwise you are cheating and if caught you will suffer very severe consequences. Please read and reread the handout for details of the grading what is expected.

Finally, note that I have not been able to find an appropriate book for this class. I'm on my 3rd book this quarter. The one I used last time was too easy; I've used the current required textbook before, but it may be somewhat too hard. So it is essential that you come to class. That is where you will learn at the level needed to do the homework.

1.1. Tour of the Universe

We will start with a slide show of the Universe. This is so that you get oriented to what is out there in the Universe, and so when I say the word "galaxy" or "supernova" you have a feeling for what I am talking about.

1.2. Black Holes

Black Holes and the expansion of the Universe (covered in Physics 162) are two subjects that rest completely on Einstein's General Relativity. We will not be able to cover GR in depth, but we will understand the essential concepts at a level even most PhD physicists do not. And we will do actual calculations of what happens around and even inside black holes. Most physicists don't study GR because it only differs from Newton's gravity theory and from Special Relativity in a few cases. But GR is Nature's choice – whenever GR differs from Newton, GR has shown to be right. It is how Nature actually works, and requires a radical rethinking of physical reality. GR and Quantum Mechanics are the two subjects I know that are most likely to surprise you.

1.3. Curved Spacetime

GR says that gravity is not really a "force"; but instead is curved spacetime. What does that mean?

Galileo and Newton view motion with respect to a rigid Euclidean reference frame that extends throughout all space and endures forever. Within this ideal frame, there exists the mysterious force of gravity – a foreign influence. Einstein says, "there is no such thing". Climb into a spaceship and see for yourself – no gravity there! Suppose you are floating in a space ship with no windows. Can you tell you whether you are out in the middle of free space or orbiting the Earth? Not really!

This is the starting point of GR, *Physics is locally gravity free.*All free particles move in straight lines and constant speed. In an inertial frame, physics looks simple. But such frames are inertial only in a limited region, i.e. *local*. Complications arise when motion is described in nearby local frames. Any difference between direction in one local frame and a nearby frame is described in terms of "curvature of spacetime". Curvature implies it is impossible to use a single Euclidean frame for all space. In a small region, curvature is small, that is it looks flat. Einstein adds together many local regions and has a theory with no gravity force. Newton has a single flat space and an extra force. These are radically different views. Einstein is right, but usually Newton's view is good enough for calculation.

1.4. Example of two surveyors

Fig 1: Surveyors on Earth going north.

Let me give an example that is extremely helpful in understanding what I just said. Consider two surveyors standing 100 meters apart on the equator. They both decide to start out perfectly parallel towards the north by rolling a big ball directly north. Some time later as they roll their balls, one notices that the distance between the two balls is less than the initial 100m. "Hey" one surveyor calls, you aren't going straight, you are coming towards me. The other says "I'm going perfectly straight, it's you that's moving." After a lot of checking they decide they both are rolling the balls straight, but that there must be some mysterious force that is pulling the balls toward each other. (What is happening of course, is that both balls are approaching the north pole, and would hit each other there.) They try the experiment with bigger balls and discover that the big balls come closer as the go north by the same amount. Since $F = ma$, the bigger balls require a bigger force and thus they decide this force is proportional to the mass of the object. In fact, it seems all objects moving north attract all other objects with a force proportional to their mass. "We have made a great discovery; let's call this force gravity", the surveyors decide.

The surveyors think they have a new force because they think they are moving on a flat surface, but in reality are on the large curved surface of the Earth. They don't realize the reason for the balls coming together is the curvature of the Earth's surface. In fact, you can do the math for the radius of the Earth and even find the value of the effective "Newton's constant G" (not the same of course as our normal G, and this "gravity" does not fall-off as r^{-2}.)

From Einstein's view, there is no force. The movement together of the balls is proof that the Earth's surface is curved. Einstein says the same thing with regard to actual gravity that pulls the falling apple toward the Earth. No force, but curved spacetime. Note in the example of the surveyors only space (Earth surface) was curved; in GR both space and time are curved. This view in fact explains a major mystery of Newton's law. Newton had two types of mass: $m = F/a$ is "inertial mass", telling how hard it is to accelerate things, while the m in $F = GMm/r^2$ is the gravitational mass, telling how much gravity comes off the object. Why are these masses the same? In Coulomb's law, the source of the force is the charge, and it is not the same as the mass. This is a mystery, but it has been tested carefully many times and the two masses are always equal. Einstein's answer is that there is only the inertial mass, which curves spacetime. Gravity as a force, doesn't exist.

1.5. Tidal force as curvature

The principle of relativity you learned in Special Relativity says physics is the same in all inertial frames. Consider traveling in a moving train or plane. Drop a ball; it falls just like when standing on the ground. You can play catch or pour wine on a plane, even though for someone watching from outside the plane the ball or wine would travel in a parabola. The principle of relativity says one cannot tell whether or not one is moving in a frame with constant velocity (except by looking outside at someone else). So consider a mass floating in an orbiting rocket ship; not touching anything, just floating. Where does it get it marching orders from? Newton says both the mass and ship get their orders from the distant Earth. Einstein says the mass gets its orders locally. A free falling frame is a "local inertial frame" so since there is nothing inside the spaceship pushing on the mass, it stays still with respect to the spaceship. In fact, according to Einstein both the space ship and mass are sampling the local curvature of spacetime which is what is causing them to orbit. Things move in "straight lines" in inertial frames; the mass can veer, but only responding to structure of spacetime right there. Newton says the mass would go "straight" in his ideal all pervading reference frame but the Earth deflects it.

How do you tell if a frame is inertial? Easy, just check every particle, light ray, etc. to see if they move in straight lines at constant speed. So inside the space ship it is an inertial frame and everything moves simply. Simple? Too simple! Where is gravity at all? How do we see the curvature?

Fig. 2: Balls in a space ship

Consider two balls in a space ship. We put them side by side 25 m apart. If the space ship is in orbit, the balls just float there. They don't move apart or together, and if there were no windows, there would be no way to tell they were in orbit above the Earth or in the middle of space far from any star or planet. Now, instead of in orbit, drop the entire space ship from a height of 250 meters above the earth. The ship and balls both fall straight down, and will hit the ground 7 seconds later ($t = \sqrt{2d/g}$). While falling, the balls still seem to be floating in deep space away from all forces. However, if you check carefully there is a small effect. Going straight down towards the Earth's center, the balls are about 1×10^{-3} m closer together when they hit. $l = \theta r$, $dl/l = dr/r$, $dl = ldr/r = (25)(250)/6.4 \times 10^6 = 1 \times 10^{-3}$m. Watching this from the ground it is clear what is happening, but inside it seems as if the balls are attracting each other. After 7 seconds they have moved about 1 mm closer. This is not actually the gravity attraction between the two balls, but is the "tidal" force and in fact proves that the space is curved. Note that if your measuring instruments had an accuracy of worse than 1 mm, then this attraction could not be detected. We say that to an accuracy of 1 mm and a time under 7 seconds this 25 m wide space is a local inertial frame, but for longer times, or better accuracy, it is not. Smaller size ships and shorter times give more approximately inertial frames. However, if you add enough small frames together you can detect the curvature. Consider a ring of balls above the Earth's surface each separated by 25 m and drop them all together. After 7 seconds they are all 1 mm closer. In each frame you can't see it, but by adding up all the frames you see that entire circle around the Earth has shrunk. The factor is 1 mm/25000 mm = 1/25000, and the distance from the center of the Earth shrinks by same factor $(1/25000)(6.4 \times 10^6)m = 250m$. Note this is just like the distance around a line of latitude shrinks for surveyors rolling balls toward north pole. The smallness of this effect in a single spaceship actually shows the smallness of the curvature of spacetime, which is part of the reason GR is not easy to experimentally distinguish from Newton.

2. The Metric

In GR the key concept is the metric. GR replaces gravity with curvature of spacetime. The metric tells how to measure distances in space and time. The metric contains all the info about curvature in a simple formula. It is the key to understanding GR and to be able to calculate anything.

Examples of metrics[1]

- 3-D flat space metric: $ds = \sqrt{dx^2 + dy^2 + dz^2}$, (or $ds^2 = dx^2 + dy^2 + dz^2$). This is just the Pythagorian theorem! (We use dx rather than x because we want to talk about "local" curvature. We find x from dx by integrating.)
- 2-D flat space metric: $ds^2 = dx^2 + dy^2$.
- 4-D flat space: $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$. w is "4th" dimension; note how easy this is to write mathematically, but it is hard to visualize! Note the number of dimensions described by a metric is the number of different differentials.
- 2-D curved space metric (surface of sphere):

  Fig. 3: Coordinates for surface of sphere

  $ds^2 = r_0^2 d\theta^2 + r_0^2 \sin^2\theta d\phi^2$, where r_0 is a constant. Note if you define $dx' = r_0 d\theta$ and $dy' = r_0 \sin\theta d\phi$ then *locally* $ds^2 = dx'^2 + dy'^2$, and it looks like flat space in small enough areas. However, if you move far then θ changes, and the distance between points on a sphere is *not* given by a flat space formula: $s \neq \sqrt{(x_0 - x')^2 + (y_0 - y')^2}$. You need to do the integral:

  $$s = \int_{\theta_0,\phi_0}^{\theta_1,\phi_1} ds = \int_{\theta_0,\phi_0}^{\theta_1,\phi_1} \sqrt{r_0^2 d\theta^2 + r_0^2 \sin^2\theta d\phi^2}.$$

  This is a longer distance than the flat space distance between these points.
- 3-D curved metric. Above we had a curvature in 2-D space. How about curved 3-D space? Easy to write mathematically, but hard to visualize. For example, $ds^2 = \sqrt{x^2 + y^2 + z^2} dx^2 + dy^2 + dz^2$ is a curved 3-D space. Just add almost any mathematical function to the flat 3-D metric and

it will be curved. We can try to visualize using "embedding". In the curved 2-D metric example we can visualize the curvature by drawing a 3-D sphere, that is by curving the 2-D surface through 3-D space. Similarly we could visualize curved 3-D space as curved through a 4-D space! But note

[1] Technically the metric is a rank two tensor, e.g. a matrix, and what we are showing here is called the "line element". We will use the terms "metric" and "line elemement" interchangeably in this course, since they contain the same information, but in a more advanced course the distinction would be made.

the curvature of 3-D space does not need the 4th dimension. That sounds hard, but if we can ignore one of the 3-D spatial dimensions, we again have curved 2-D embedded in 3-D. We will show examples later. Note that the space around the Earth is curved, but we can't see that curvature! We can measure its effects however and prove it is there! It becomes an experimental question.

2.1. Using a metric to find distances, areas, etc.

The left hand side of a metric (aka line element) gives the real distance measured in meters. The right hand side gives the coordinates, specified by the differentials, as well as functions and constants. The coordinates may or may not be in units of distance, so to find an actual distance, area, or volume, the left hand side must be used. For example, consider the 3-D metric

$$ds^2 = f(x,y,z)dx^2 + g(x,y,z)dy^2 + h(x,y,z)dz^2,$$

where $f(x,y,z)$, $g(x,y,z)$, and $h(x,y,z)$ are functions that specify the curvature of the space. To find a distance in the x direction, one would hold y and z constant. That is one would set $dy = 0$ and $dz = 0$, to get $ds \equiv dl_x = \sqrt{f}dx$. The actual distance from x_0 to x_1 would therefore be $l_x = \int_{x_0}^{x_1} \sqrt{f}dx$. Likewise to measure a distance in the z direction you would set $dx = dy = 0$ and integrate $dl_z = \sqrt{h}dz$. The distance along an arbitrary curve can be found by factoring out, say dx, and integrating: $l = \int dl = \int \sqrt{f + g(dy/dx)^2 + h(dz/dx)^2}dx$, where dy/dx and dz/dx are calculated along the integration path.

Finally, while in flat space a small area is given by $dA = dxdy$, in curved space one must use $dA = dl_x dl_y = \sqrt{fg}dxdy$ (or $dl_y dl_z$, etc.). The volume element is thus $dV = dl_x dl_y dl_z = \sqrt{fgh}dxdydz$.

2.2. Metrics Continued

It was mathematicians who first wondered whether a consistent system could be created with different (non-Euclidean) metrics. Gauss, Bolyai, Lobachevski, Reimann, etc. worked it all out. When Einstein came along with his great intution that this might be relevant to the real world he applied this math, "differential geometry" in his GR.

GR takes the matter and energy in a system and predicts the metric. This is done by solving Einstein's field equations, something we will not do in this course. Once you have the metric you can calculate distances, times, and motions of particles. We will do this. GR gives some surprising results, and so far every prediction that has been tested has been experimentally verified.

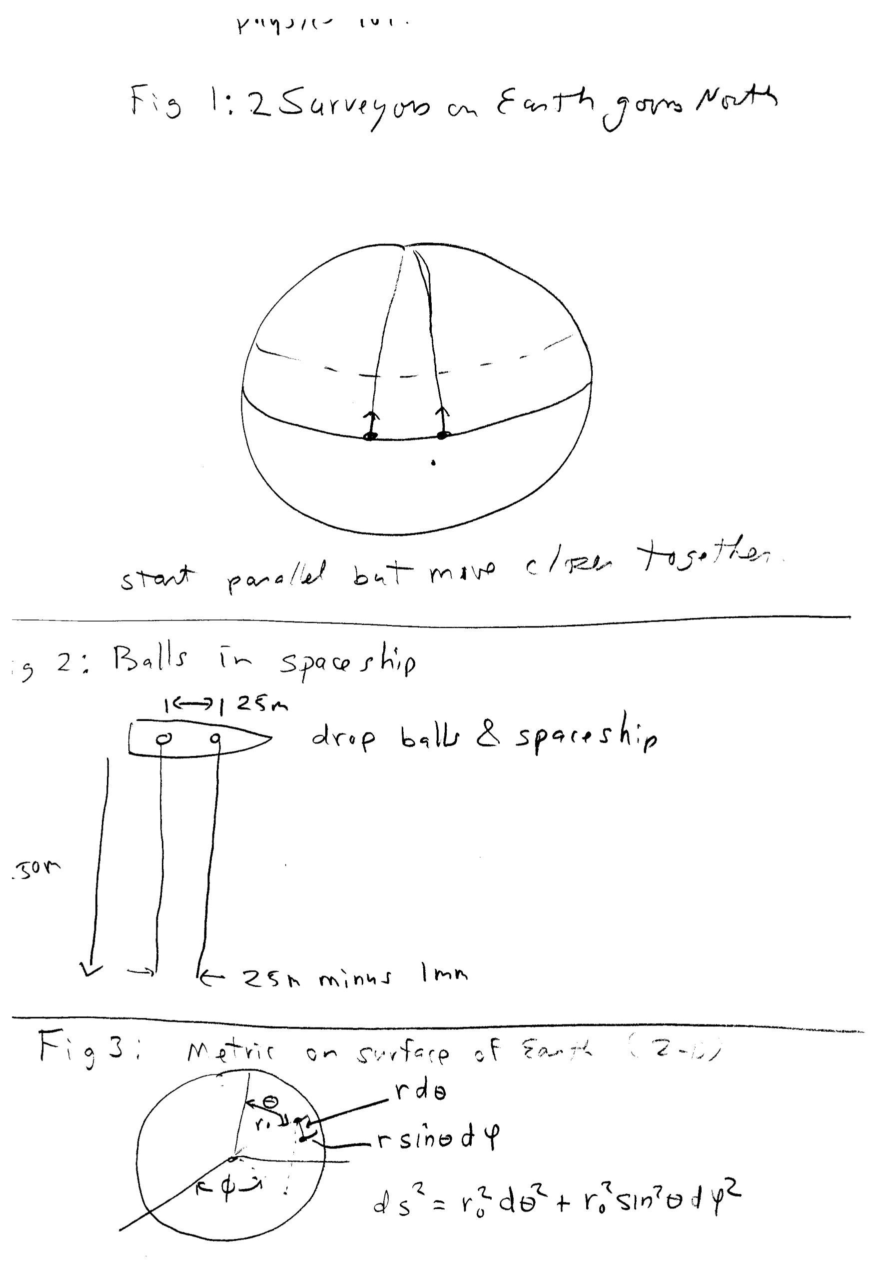

Fig. 1.— Figure for Chapter 1

2.3. Expanding Universe and FRW metric

For example, consider the entire Universe filled uniformly with matter. One can solve the field equations for the metric in this case and find (for the right amount of matter):

$$ds = R(t)\sqrt{dx^2 + dy^2 + dz^2},$$

where the *scale factor*, $R(r) = (t/t_0)^{1/2}$ at early times and $R(t) = (t/t_0)^{2/3}$ or $R(t) = \exp(Ht/t_0)$ at late times depending on the cosmology. Here $t_0 = 13.7$ Gyear is the age of the Universe today. This is called the Friedmann, Robertson, Walker (FRW) metric and was quite shocking to Einstein when he first realized this is what his GR theory predicted for uniform matter. The shocking thing is that $R(t)$ changes with time. Thus the distance between objects changes with time, even when they are "at rest"! Consider two galaxies a distance D apart today (one at (x_0, y_0, z_0) and the other at (x, y, z). Distance between them is $s = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, with $t = t_0$. A few years from now they are farther apart by a factor $(t/t_0)^{2/3}$ even though their positions, x, x_0, y, y_0 etc. haven't changed!

This is the expansion of the Universe. Note it is *not an explosion* that happened long ago and caused everything to blast apart. Note also what happens when $t = 0$ in this metric. $R(0) = 0$, implying $s = 0$ no matter what the values of x, y, etc. are. That is, at $t = 0$, everything in the Universe is touching everything else! This is the big bang. This metric predicts the big bang happened at $t = 0$, and that the Universe expanded since then because the metric is changing with time.

2.4. Spacetime metrics and nomenclature

Spacetime metrics combine the 3-D space metric with the time metric. The flat spacetime metric, also known as the Minkowski metric, is written in Cartesian coordinates as:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.$$

This combines the flat space 3-D metric with the flat time metric: $c^2 dt^2$. The two are combined together so that ds is a Lorentz invariant, that is if the time between two events is dt, and the distance between those same two events is $\sqrt{dx^2 + dy^2 + dz^2}$, then the spacetime interval between these two events has the same value (ds), *no matter which inertial reference frame is used to make the measurement.* It is crucial for this to work that the "time part" of the metric have a different sign that the "space part".

One uses the spacetime metric to meausure the "invariant spacetime interval", ds, between two events. One also uses the space part of the metric to measure real distances (distances measured by a meter stick). The time part of the metric is used to measure real (that is measured by a clock) times between events. The distance between two events is best measured when the time of the events is the same, that is measured at the same time, or by setting $dt = 0$. Thus the **proper distance** between two objects is defined as $dl = \sqrt{ds^2}$, with $dt = 0$. To measure times between events, the events should be at the same position, i.e. one clock at different times. So we set $dx = dy = dz = 0$. The **proper time** between events is defined as $d\tau = \sqrt{-ds^2}/c$, Note, that because the time part of the metric has a minus sign, you have to add a minus sign to cancel it out. While ds^2 can be positive, negative, or zero, the invariant interval, ds is always positive. You should never get an imaginary number out of a metric!

Actually the sign of ds^2 is very important:

- $ds^2 < 0$ **implies a time-like interval**, meaning the two events can be causally related, that is, it is possible for the event with the earlier time to have caused the event with the later time.
- $ds^2 > 0$ **implies a space-like interval.** It is impossible for either of these events to have caused the other. (Like the ends of the same ruler.)
- $ds^2 = 0$ **implies a null or light-like interval.** These two events can be connected by light rays only. Note in this case the definitions of proper time and proper distances above don't apply because you can't set $dt = 0$ or $dx = 0$. The proper time between lightlike separated events is defined as $ds = 0$.

Note that with the Minkowski metric, the variable t measures time and the variable x measures spatial distance. However, in more general metrics, the variables used in the metric *do not represent time or space directly.* To get actual time or space measurements you have to use the left-hand side of the metric, i.e. ds, as described above. The variables in the metric are called the "coordinate time" or "coordinate distances", or even just the coordinates, and may or may not correspond to clock time, or meter-stick distance.

Also, note that many books (and I myself) set the speed of light, c, equal to one. This makes the units easier to work throughout all of Special and General Relativity. It means if you are measuring time in years, you are measuring distance in light-years, or if you are measuring distance in meters, you are measuring time in units of the time for light to travel one meter. For example, one can say this 50 minute class session lasts about 900 billion meters. If we are halfway through it we have about 450 billion meters to go. A useful thing to know is that light travels about 1 foot in one nanosecond.

When using these units, you have to add back in powers of c in order to get to useful units. There is *always* a unique way to do this. In these units velocities are dimensionless, so if you want a velocity in m/s just multiple by $c = 3 \times 10^8$ m/s. Energies and masses are related by $E = mc^2$, so if you know the mass and want an energy just multiply by c^2. [e.g. $v = .001$ means, $v = .001c$, or $v = 300$km/s.]

Finally note that in many books the space part of the metric has the minus signs, and the time part is positive. As long as there is a relative minus sign between space and time, it doesn't matter which has the minus sign, but it is very important to pick one convention and keep it. This is called the "signature of the metric". We will use the "East coast" signature (- + + +), while others use the "West coast" signature (+ - - -). With the opposite signature metric, the definitions of proper distance and proper time change: e.g. $d\tau = \sqrt{ds^2}/c$, $dl = \sqrt{-ds^2}$, and several other formulas change as well.

2.5. Schwarzschild metric

The FRW metric is valid on very large scales where matter is distributed approximately uniformly. On scales the size of the Earth, Solar System, or even galaxies, matter is concentrated in a central source, so FRW is not the proper solution of Einstein's field equation and is not the correct metric. The simplest case is for a spherical object, such as the Earth, Sun, (or black hole!). Solving Einstein's field equations for the region outside of a spherical object of mass, M gives the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

where this is in spherical coordinates with $r^2 = x^2 + y^2 + z^2$. There are several new features here and we will spend a lot of time on this metric.

- First note that flat 3-D spatial metric in spherical coordinates is $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = dr^2 + r^2 d\Omega^2$, **Note $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is shorthand for the angular part.** This looks similar to, but is different that the 2-D curved metric on the surface of the sphere because here r is a variable not a constant. You can tell the variables because they are differential (i.e. dr not only r).
- Second, I included the time part of the metric dt. This makes it a spacetime metric, and not just space. We will talk more about this.
- Third, the time part of the metric has a minus sign! We will come back to what that means and how to deal with it.
- Fourth, the metric does not change with time. Unlike the FRW metric, the variable t does not appear explicitly anywhere in the Schwarzschild metric. The dt doesn't count since that is just tells how to measure time. Thus this metric is not expanding or contracting and just sits there. (Good thing).
- Fifth, the metric is spherically symmetric, which is why we switched to spherical coordinates.
- Sixth, there are factors in front of both dr and dt which mean that both the space and the time are curved in this metric. If those factors were equal to 1, then this would be the 3-D flat spacetime Minkowski metric in spherical coordinates.
- Seventh, note that something weird happens when $2GM/rc^2 = 1$. The dt^2 term goes to zero and the dr^2 term goes to infinity. This happens when a mass M is squeezed into a ball of radius

  $$r_{SC} = \frac{2GM}{c^2},$$

  where r_{SC} is called the Schwarzschild radius. This is the event horizon radius of a black hole, and we expect some weird things to happen at that radius. We will discover that whenever the size of an object is smaller than its Schwarzschild radius it is a black hole. For the mass of the Sun this radius is

$$r_{SC} = \frac{(2)(6.67 \times 10^{-11}\mathrm{m}^3/\mathrm{kg\ s}^2)(2 \times 10^{30}\mathrm{kg}\ M/M_\odot)}{(3 \times 10^8\mathrm{m/s})^2} = 3\mathrm{km}\ \frac{M}{M_\odot},$$

where $M_\odot$ is the mass of the Sun. Thus if you could jam the entire mass of the Sun into a sphere of radius 3 km, it would be a black hole.

- Finally, note that if the key factor $2GM/rc^2$ is small, then the Schwarzschild metric is very close to the 3-D flat space metric. For the Sun, $2GM/rc^2 = 3$ km$/7 \times 10^5$km $= 4 \times 10^{-6}$. So even at the surface of the Sun, spacetime is only 4 parts in a million away from being flat. The smallness of this number is one reason why it is hard to measure the effects of GR that differ from Newton's law. Around the Earth spacetime is even closer to flat. Note that far away from the mass, $r \to \infty$, the Schwarzschild metric turns into exactly the flat space metric. This is what we expect of course; there is no gravity infinitely far away from a mass.

3. General and Special Relativity

GR is based upon **the equivalence principle: All local inerital frames are equivalent for performance of all physical experiments.** Remember that a **local inertial reference frame is one that is free falling and non-rotating**, where particles move at constant velocities in straight lines (or stay still). A very important point is that Special Relativity is how spacetime behaves in any inertial frame. This is equally true for someone out in deep space far from anything, for someone orbiting the Earth in a spaceship, or for someone falling in an elevator with broken cables. All three someones are in inertial reference frames and would find the same result for any experiment they did; that is they could not tell which of the situations they were in, without looking outside (looking outside makes the frame bigger, i.e. non-local). The influence of the Earth's gravity on the guy falling in the elevator is not measurable by local experiments.

The metric of Special Relativity (SR) is the Minkowski metric $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$, and GR is an extension of SR. So we start by understanding the SR metric. In SR the relations between any two local inertial frames are the Lorentz transformations which you have seen before. These, and all of SR, can be derived from two postulates:

1. The principle of relativity: results of experiments don't depend on velocity of lab.

2. All inertial frames measure the same speed of light which is **exactly** $c = 299792458$ m/s! This postulate is the same as requiring the invariant interval ds to be the same in every frame.

These postulates can be used to derive the Lorentz transformations.

3.1. Spacetime diagrams

A very nice tool i sthe spacetime diagram. These will be indespensible when we try and go inside a black hole, so it is worthwhile trying to understand them.

Figure: Intro to Spacetime diagram

Points:

- Axes are x vs. t. Points are events, lines represent observers moving through spacetime (worldlines) or extended objects (if spacelike).

- The slope of an observer's worldline is $dt/dx = 1/v$.

- With $c = 1$, an observer moving at the speed of light is a 45^0 line. Nothing can move at a shallower slope, since that would mean $v > c$. Thus only events within the "lightcone" can be casually connected. [We suppress the y and z directions; if we include y then it is a cone.

- Straight lines represent inertial observers (each related by a Lorentz transformation.) A line straight up is someone standing still in the lab frame.
- Curved lines represent accelerated observers.

3.2. More special relativity: how time and space appears to other observers: Lorentz transformation

Now let's use spacetime diagrams to understand some of the weirdness of special relativity. Suppose observer O uses (t, x) and is in the lab, while observer O' uses (t', x') and is moving in the x-direction with velocity v.

Figure: O' moving at v in O coordinates

Where do the t' and x' coordinate axes go in this diagram? Well, the worldline of O' is their t' axis, since this is $x' = 0$ to them (and also $y' = z' = 0$). Where is the x' axis? To find this consider O' spacetime diagram from O''s point of view.

Figure: O' spacetime diagram in O' coordinates: reflecting light

From O' view t' axis goes straight up and x' axis is at right angles. Consider a photon emitted at $(t', x') = (-a, 0)$ in the +x' direction. It reaches the x' axis at $t' = 0$, and then is reflected back (by a mirror). It returns to the t' axis ($x' = 0$) at $(t', x') = (a, 0)$. Consider other photons emitted at different times, $-b$, etc. and we see that we could *define* the x' axis as the locus of points that will reflect photons emitted at $t' = -a$, to reach back at $t' = a$.

This should be a frame-independent definition, so we can find the x' axis in the O frame using it.

Figure: t' and x' axes in O coordinates: reflecting light

To do this we use postulate two: light travels at $c = 1$ in all frames, that is, the light pulses O' uses to find the x' axis travel at 45^0 also in the O spacetime diagram. We see that the x' axis is *not* the same as the x axis! So events which are simultaneous to O' (i.e. happen at the same values of t', e.g. $t' = 0$ on the x' axis), are not simultaneous to O (don't happen at same values of t, i.e. are not parallel to the x axis.) This is the famous failure of simultaneity in SR, and is made very obvious in spacetime diagrams.

With some work, from the above postulates one can derive the famous **Lorentz tranformations**:

$$t = \gamma(t' + \frac{v}{c^2}x')$$

$$x = \gamma(vt' + x')$$

$$y = y'$$

$$z = z',$$

where the Lorentz factor is the greek letter gamma:

$$\gamma = 1/\sqrt{1 - v^2/c^2}.$$

Note that with $c = 1$ these simplify:

$$t = \gamma(t' + vx')$$

$$x = \gamma(vt' + x')$$

$$y = y'; z = z',$$

with $\gamma = 1/\sqrt{1 - v^2}$.

One can easily find the inverse transforms, just by setting $v \rightarrow -v$ in the above: **Inverse Lorentz transformation:**

$$t' = \gamma(t - vx)$$

$$x' = \gamma(-vt + x)$$

In the homework you will use these to prove that Δs between two events is the same in every frame.

3.3. Calibrating the axes in a spacetime diagram: hyperbolas=circles!

Figure: Hyperbolas on space time diagram

How does one calibrate the axes in a spacetime diagram? Consider the equation: $t^2 - x^2 = a^2$. All the points on this hyperbola have the same invariant "distance" from $(0, 0)$. $\Delta s^2 = -\Delta t^2 + \Delta x^2 = -a^2$; e.g. $x = 0$ and $t = a$, etc. Thus this hyperbola is the equivalent of a circle in normal Euclidian geometry. The proper time is the time that someone traveling along the line would measure on their wristwatch. So no matter what the speed, all travelers from the origin to the hyperbola take the same time to get there (i.e. they age the same amount, a). Note that $-a^2$ is the invariant interval, and since $ds^2 = -a^2 < 0$ this hyperbola is for timelike separations.

Similarly one can plot the equation: $t^2 - x^2 = -b^2$, at get the sideways hyperbola which represents spacelike separations. These points all have the same proper length from the origin.

Figure: Events A and B on space time diagram

To calibrate the axes, we consider two events, A and B. A lies at $(t, x) = (1, 0)$ on the O t axis, while event B lies at $(t', x') = (t', 0)$ on the O' t' axis. We pick B to lie along the hyperbola so the invariant interval (proper time) from the origin is the same to both events A and B. Since the interval Δs is invariant (same in all frames), and event A has $t = 1, x = 0$, we have $\Delta s^2 =$

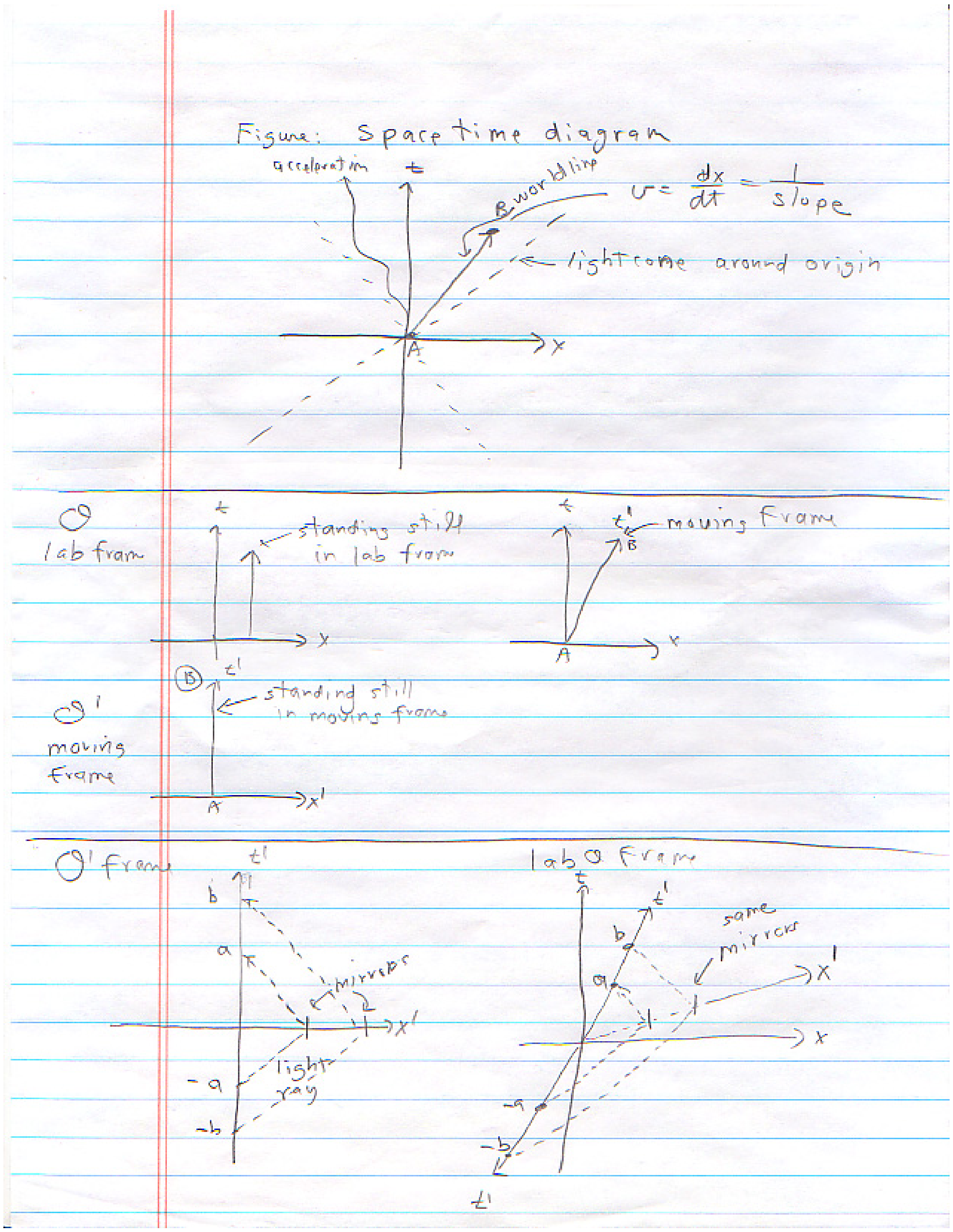

Fig. 2.— Figure for Chapter 3 A

$-t^2 + x^2 = -1$. Therefore in moving frame, -$t'^2 + x'^2 = -1$ also. Event B occurs at $x' = 0$, which implies $t' = 1$. But on the spacetime diagram, the distance from the (0,0) to event B looks looks farther than the distance from (0,0) to event A! The point here is that one can't use Euclidian intuition about distances on spacetime diagrams: It is Minkowski geometry. The proper distance is $\Delta s^2 = -\Delta t^2 + \Delta x^2$, *not* $\Delta t^2 + \Delta x^2$ as we are used to. Now you see the importance of the minus signs in the spacetime metric. They change the geometry from Euclidian to Minkowski.

In the above, the moving frame observer, O', says $t' = 1$, but the lab frame says the clock tick happened at a longer time. (Just read off the t time from the t axis!)

This says that the moving person's time runs slowly, when observed from the lab frame.

Finally, note that on a space time diagram we can easily see which events are separated by time-like intervals and which are separated by space-like intervals. Thus we can see which events can and cannot be causally connected. **The 45^0 lines are called the lightcone.** If we add say the y dimension you see why it is called a cone, but if we add the y and z dimensions it is actually a cone in 4-D! The cone opening towards larger time is called the future lightcone and contains all future events that can be influenced by it. The cone opening towards the past is call the past lightcone and contains all the events that could have influenced the event at the apex of the cones. You can draw a lightcone around any event and see which events can influence which other events.

3.4. Time dilation and length contraction by spacetime diagram

Last time we saw how to calibrate spacetime diagrames. We can use that result to derive the famous time dilation: moving clocks run slow to stationary observers.

Numerically we can calculate for time dilation for event B, considering starting a clock at the origin (event A). $\Delta s^2 = -\Delta t'^2 + \Delta x'^2 = -\Delta t^2 + \Delta x^2$. For a clock tick at event B, $\Delta t = t$, $\Delta x = x$, $\Delta t' = t'$. In the O, lab frame: Event B occurs $x = \Delta x = v\Delta t$, and time occurs at $t = \Delta t$. In O', the moving frame, $\Delta x' = x' = 0$ since events B and A occur at the some place (O' is not moving in its own frame). So $-\Delta t'^2 + 0 = -\Delta t^2 + \Delta t^2 v^2 = -(1 - v^2)\Delta t^2$, or $\Delta t = \Delta t' / \sqrt{1 - v^2} = \gamma \Delta t'$. This is the time dilation you learned before in special relativity.

One can similarly find out the length contraction you learned about in SR. If you remember the derivation, you will remember that it is more tricky than the time dilation calculation. The reason for the trickiness is easy to see in the spacetime diagram. Consider the origin as point A and a ruler of length a laid out on a spacecraft with its far end at point B. How does that look? It lies along the x' axis, which we have just seen how to find (the set of points which all have $t' = 0$. Now draw the sideway hyberbola of all proper distances a. If we just project the point B onto the x axis we find the wrong answer, that the ruler looks *longer* in the lab frame. What have we done wrong? By length of ruler in the lab frame we mean the distance between the two ends measured at the same lab frame time t. Thus the distance between points A and B is not the distance we are after. We need to follow point A up in time until it reaches the same time t as point B has. We know the t' axis has the same angle from the 45^0 line as the A-B segment, so we can just go along the t' axis until we get as high as point B. Now the length of the ruler as measured in the lab frame is the difference in x of the two ends measured at the same time. We see the ruler is shorter as measured in the lab frame as expected.

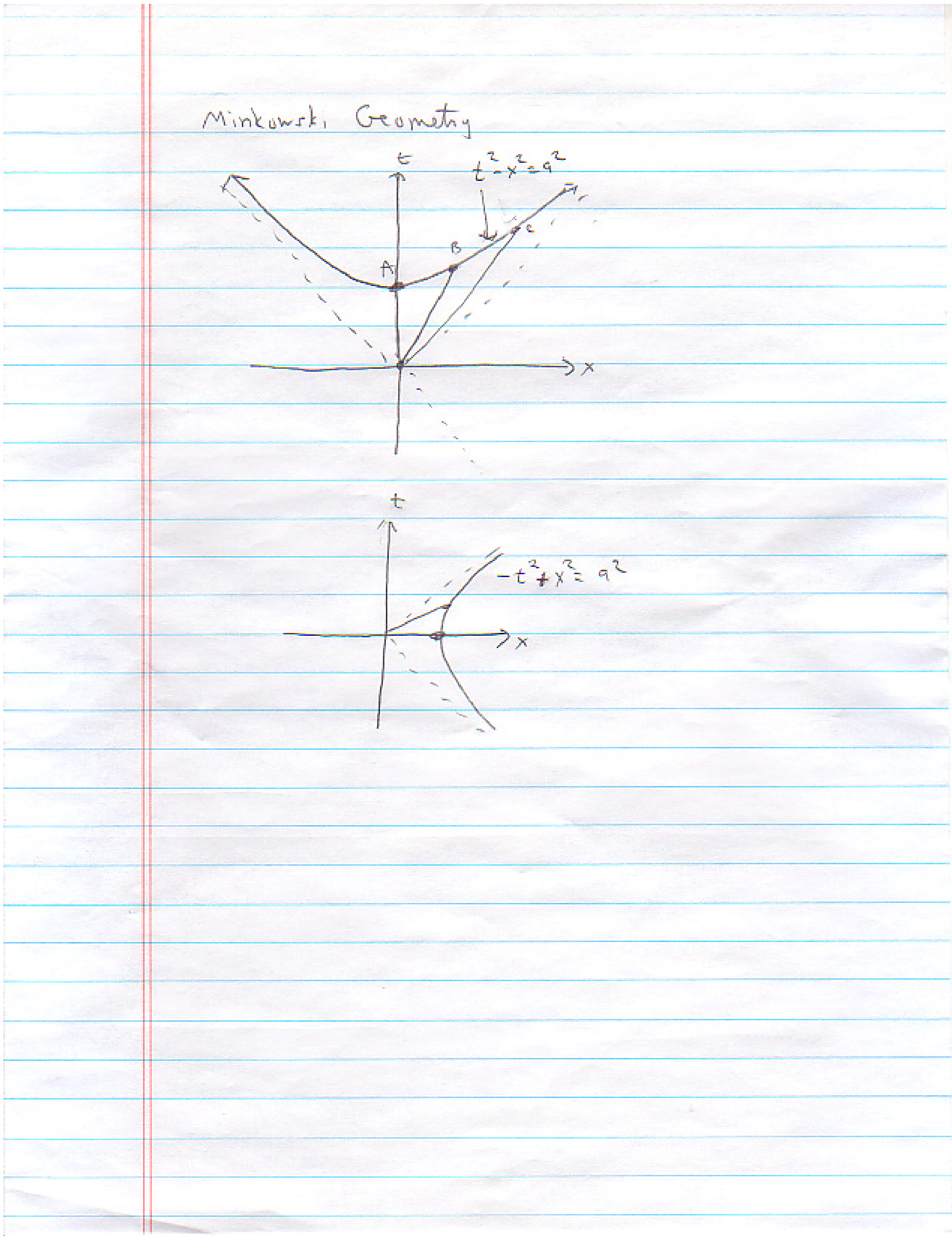

Fig. 3.— Figure for Chapter 3 B

3.5. Other special relativity you need to know

This class has special relativity as a prerequisite, so I'm not going to cover much more. However, you should look over the book to refamiliarize yourself with several topics. For example, you should understand $E = mc^2$. In fact, your need to know that the real equation is $E^2 = (mc^2)^2 + (pc)^2$, where p is the momentum. With $c = 1$ this is more suggestively written: $m^2 = E^2 - p^2$, and we see that the rest mass is an invariant in much the same way that the interval ds is. You also should be familiar with relativistic energy and momentum: $E = m\gamma c^2$, and $p = mv\gamma$. Try Taylor expanding the relativistic expression for kinetic energy $T = E - mc^2 \approx \frac{1}{2}mv^2$, and find the next order correction to the famous Newtonian formula.

3.6. Time Dilation in a gravitational field

Remember the metric of flat spacetime is

$$ds^2 = -c^2 dt^2 + dx^2,$$

where we have supressed the y and z dimensions. In this metric, we find the proper time by setting $dx = 0$ and using $d\tau = \sqrt{-ds^2}/c = dt$, and see that coordinate t is actually the proper time between events. Likewise the proper distance is $dl = dx$, so that the coordinate x is the proper distance. Compare this to the the Schwarzschild metric

$$ds^2 = -c^2 \left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Here the proper time is found by setting $dr = d\theta = d\phi = 0$. Note then $ds^2 \leq 0$, that is a time-like separation. The proper time is again

$$d\tau = \sqrt{-ds^2}/c = \sqrt{1 - \frac{2GM}{rc^2}} dt.$$

So if $dt = 1$ sec, $d\tau = \sqrt{1 - \frac{2GM}{rc^2}}$ sec, which is a smaller (i.e. shorter) time. This proper time is the clock time measured at a distance r from the Earth (or a black hole). Thus we see that clocks run slower in a gravitational field than in deep space (where $r \to \infty$ and $dt = d\tau$).

Let's see how big effect this is for us living on the surface of the Earth. $M_{\rm Earth} = 6 \times 10^{27}$ gm, and $r_{\rm Earth} = 6.38 \times 10^8$ cm, and a useful number to calculate is the Schwarzschild radius of the Earth $r_{SC} = 2GM_{\rm Earth}/c^2 = 0.886$ cm. Then $2GM_{\rm Earth}/r_{\rm Earth}c^2 = .886/6.38 \times 10^8 = 1.4 \times 10^{-9}$. Using a Taylor expansion, this says $d\tau = \sqrt{1 - 1.4 \times 10^{-9}} \approx 1 - \frac{1}{2}1.4 \times 10^{-9}$. Or $d\tau = (1 - 6.9 \times 10^{-10})dt$. So time runs slower here on Earth (as measured from deep space by about 1 sec every 45 years! Not a big effect, but measurable. Gravity curves time.

However in other environments this factor can be bigger. Especially wierd is what it does when $r = r_{SC}$, Then $\sqrt{1 - 2GM/rc^2} \to 0$, and the measured time seems to stop! This occurs at the horizon of a black hole.

3.7. Old Idea of Black Holes

The idea of black holes has been around since the 1700's when Laplace, Mitchell, and others thought about giant stars. One can even derive the Schwarzschild radius using Newton's laws and get the right answer (by coincidence I think?) Consider a big spherical ball of mass M and radius r. The gravitational potential energy at the surface of the ball is $V = -GMm/r$, where m is just some small test mass. Since V goes to zero at infinity, the **escape velocity** from the surface is found by setting this potential energy to the the kinetic energy $E = \frac{1}{2}mv^2$, and solving for v. This gives $v_{esc} = \sqrt{2GM/r}$. This escape velocity increases as the mass increases, or the radius decreases. And one could ask what happens when the escape velocity is equal to or larger than c. Then according to Newton's corpuscular theory of light (which is incorrect!) light would get trapped in such an object, and the giant star would be dark. Setting $v_{esc} = c$ one finds the radius of such an object is $r = 2GM/c^2$, precisely the Schwarzschild radius. This Newtonian calculation gives the right answer, but is wrong. For example, light particles are massless so $E \neq \frac{1}{2}mv^2$. There are several errors in this calculation which just happen to cancel. The correct description requires the Schwarzschild metric.

4. Geodesics: Moving in "Straight Lines" Through Curved Spacetime

We saw that gravity curves time and space. A very important result is how things move in curved space. The basic principle is that things move on paths that are as "straight" as possible. These can be defined as the paths which extremize the distance between two points, and are called **geodesics**. In 3-D Euclidian geometry a straight line is defined as the shortest distance between two points, and Newton's law say in the absense of outside forces particles move along such lines. It is similar in GR, but sometimes it is the "maximum" invariant interval which is relevant, not the minimum. That is why we say "extremizes" the distance instead of minimizes.

To get an idea of this, consider the 2-D analogy of an ant crawling on the surface of an apple. The ant is forced to follow the curved 2-D space of the apple skin, but suppose it walks as "straight" as possible, i.e. not veering to the left or right. As it walks from one side of the apple to the other coming near the stem, the ant will be "deflected" and come away from the stem at a different angle than it approached the stem. If the ant is close to the stem it could even circle the stem continually while still following the "geodesic". This should seem similar to the analogy of the two surveyors.

Now consider two points on opposite sides of the stem and ask which path is the shortest between them (the geodesic). Note that without the curved surface the shortest path would be quite different. However, the metric requires us to stay on the surface.

If we want to calculate the motion of objects in GR, we need to be able to find the geodesics. By finding them we will be able to derive Newton's laws from the Schwarzschild metric, as well as Einsteinian corrections to Newton's laws. We can find out the real answer obeyed by actual objects in the solar system. We will also be able to find out how objects move around black holes.

4.1. Geodesics and Calculus of Variations

GR says that the motion of a particle that experience no external forces is a geodesic of the spacetime metric. One can summarize GR in two statements: 1. Matter and Energy tell spacetime how to curve. 2. Curved spacetime tells matter and energy how to move. In solving for the geodesics we are finding how matter and energy (light) move.

In 3-D Euclidian space the definition of a geodesic, aka "straight line" is the shortest distance bewteen two points. Mathematically this can be found from calculus of variations on the metric distance. The differential distance $ds = \sqrt{dx^2 + dy^2 + dz^2}$ can be integrated between two end points to find

$$s = \int_a^b ds = \int_a^b \sqrt{1 + (dy/dx)^2 + (dz/dx)^2}dx,$$

where we factored out a dx so the integral is done over the x-axis. Note now the integrand contains the derivatives of the the functions $y(x)$ and $z(x)$ which define the path. How do find the path (functions $y(x)$ and $z(x)$) that minimize s? We need a general method. This method is called the "calculus of variations" and you probably learned it in calculus. However, I will remind you of how it is done. If you haven't seen it before, that's ok, since I'll show you all you need. This is beautiful and fun math that is used throughout advanced physics.

The basic idea is to minimize ds the same way you minimize any function is calculus: take its derivative, set it to zero, and solve. This solution will be the geodesic, that is the extremal path. This method works on curved surfaces since ds measures the actual distance following the curved surface. It works the same way in GR; you just add the time part of the metric.

4.2. Geodesics as equations of motion

A geodesic on pure spatial manifold (e.g. curved surface) is a line. For example, straight lines on a plane, or great circles on a sphere surface. In GR the geodesics include time and so are actually the equations of motion! You can understand this by remembering the distinction between time and space. One has choice in spatial motion, but one is forced by nature to move forward in the time direction: 1 sec per sec. You will hit Janurary 1 of next year no matter what you do. Now in flat space a possible geodesic is one in which you don't move in space at all: $dx = dy = dz = 0$. Then $ds = dt$ is an extremal path and you just sit there getting older. However, if spacetime is curved, then motion in dt can *require* motion in dx! Think of the surveyor analogy, where motion north (z) required motion in the x and y direction to stay on the surface of the Earth. Thus since motion in t direction is forced, motion in the x (or other spatial) direction will also be forced. Thus you have the geodesic requiring $dx/dt \neq 0$. dx/dt is a velocity, so you can't stay at rest and be on the geodesics. This is how gravity attracts things and why geodesics near a mass require bodies to fall towards the mass center. Let's do some math.

4.3. Euler-Lagrange Equations

To get used to calculus of variations, let's do a calculation some of you have already done: Lagrange's equations in classical mechanics. Then we will do it on metrics. Doing it several times is necessary and I highly recommend you go over all these calculations at home several times.

We want to extremize the integral $s = \int ds$, which can be written $s = \int (ds/dt)dt$, where we have factored out a t. We can factor out any variable we want so as to make the integrating easier. As an example, consider the general "least action" integral

$$S = \int L dt,$$

where L is called the Lagrangian. In our example $L = ds/dt$, but in regular classical mechanics the Lagrangian is $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$, where $\dot{x} = dx/dt = v_x$, T is the kinetic energy, and $V(x)$ is the potential energy. (Here we used only the x and t dimensions.) Thus for 1-D classical mechanics $S = \int_a^b (\frac{1}{2}m\dot{x}^2 - V(x))dt$ is called the action, and the equations of motion, $F = ma$, with $F = -dV/dx$ are found minimizing this action (principle of "least action"). In our problem we are trying to find the path that gives the shortest (or longest) distance along the path between two fixed points.

Let's do it first in general and get the equations.

$$S = \int L(x, y, z, \dot{x}, \dot{y}, \dot{z}, \lambda)d\lambda,$$

where we explicitly put in the velocities v_x, v_y, and v_z as variables that have to be solved for, and have factored out the general variable λ (usually just time). This lambda is called the "affine parameter" and is the variable you use to trace along the geodesic path. In the above, the dot means differentiating with respect to λ, i.e. $\dot{x} = dx/d\lambda$. Now we take the "variation" of this integral δS using the chain rule and set it to zero: $\delta S = 0$, and solve.

$$\delta S = \int \left(\frac{\partial L}{\partial x}\delta x(\lambda) + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial y}\delta y + \cdots \right) d\lambda = 0.$$

Next note that $\delta(\dot{x}) = \delta\frac{dx}{d\lambda} = \frac{d}{d\lambda}\delta x$, where $\delta x(\lambda)$ is a small deviation from the extremal path. Now integrate by parts the term with the $\delta\dot{x}$, by using:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du.$$

$$\int_a^b \frac{\partial L}{\partial \dot{x}}\delta\dot{x} d\lambda = \int_a^b \frac{\partial L}{\partial \dot{x}}\frac{d}{d\lambda}(\delta x)d\lambda = \frac{\partial L}{d\dot{x}}\delta x|_a^b - \int_a^b \frac{d}{d\lambda}\frac{\partial L}{d\dot{x}}\delta x d\lambda.$$

We want a path defined by $x(\lambda)$, $y(\lambda)$, etc. that goes from point a to point b. Thus we want $\delta x(\lambda)$, the variation of the path from the geodesic to be zero at the end points. That is $\delta x(\lambda = a) = 0$ and $\delta x(\lambda = b) = 0$ (and similarly for δy and δz). Thus the first term on the right hand side of above equation vanishes, and the equation becomes:

$$0 = \int \left(\frac{\partial L}{\partial x} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}} \right) \delta x d\lambda + \text{y and z parts.}$$

The final step in getting our Euler-Lagrange equations is to note that the variation in the path $\delta x(\lambda)$, $\delta y(\lambda)$, and $\delta z(\lambda)$ is completely arbitrary. Thus for the integral as a whole to vanish, the integrand itself must vanish everywhere. That is, to be true for every possible function $\delta x(\lambda)$, the parts of the intregrand multiplying δx, δy, and δz must be zero. Thus we have the **Euler-Lagrange equations**:

$$\frac{\partial L}{\partial x} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}} = 0.$$
$$\frac{\partial L}{\partial y} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{y}} = 0.$$
$$\frac{\partial L}{\partial z} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{z}} = 0.$$

4.4. First Example of Euler-Lagrange equations: classical mechanics

As a first example of the use of The Euler-Lagrange equations, let the Lagrangian be $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$ and let $\lambda = t$. Then $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F$, where we used the normal definition of force as the derivative of the potential energy. Since $v = \dot{x}$, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, and $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = ma$, where the acceleration $a = dv/dt$. Thus the Euler-Lagrange equations found by extremizing the action is just $F = ma$. This might seem like a lot of work to get something you already know, but the beauty of the method is that it works in difficult situations and in difficult coordinate systems. It is usually a lot easier to write down the kinetic and potential energy than it is to use the vector form of $F = ma$ in complicated situations.

4.5. Second example of Euler-Lagrange equations: Flat space geodesics

Now let's extremize the 3-D flat space metric to see if the shortest distance between two points is indeed a straight line! So $s = \int ds$, with

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + \dot{y}^2 + \dot{z}^2}dx,$$

where we have chosen $\lambda = x$, and $L = \sqrt{1 + \dot{y}^2 + \dot{x}^2}$. The y Lagrangian equation thus reads:

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0,$$

or

$$\frac{d}{dx}\left(\frac{1}{2}(1 + \dot{y}^2 + \dot{z}^2)^{-1/2}2\dot{y}\right) - 0 = 0,$$

or

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = \text{constant}.$$

For the z equation we find similarly,

$$\frac{\dot{z}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = \text{constant}.$$

Dividing the y equation by the z equation we get $\dot{y}/\dot{z} = \text{constant}$, so $\dot{z} = c_1\dot{y}$. Substituting this into the y equation, we get $\dot{y}/\sqrt{1 + \dot{y}^2 + c_1^2\dot{y}^2} = \text{constant}$. Since the only variable in this entire equation is $\dot{y}$, solving this equation for $\dot{y}$ will give a constant. Thus we find the Euler-Lagrange equations for the extremal distance between two points are $dy/dx = m_y$, and $dz/dx = m_z$, where m_y and m_z are some constants found by the boundary condition. Thus $y = m_y x + b_y$ and $z = m_z x + b_z$, the equation for a straight line in 3-D. This again might seem like a lot of work to prove that the shortest distance between two points is a straight line, but the method is general.

Before going on to extremize the invariant interval in a spacetime metric, I want to do the last problem again and show you a useful trick. In the above we used x as the affine parameter.

Instead we could have used s itself. Thus we write our integral as $s = \int ds$, with the Lagrangian $L = 1$! However we write this 'one' in a special way:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dx}{ds})^2} ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} ds,$$

where now $L = 1 = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. As before in the Euler-Lagrange equations the term $\partial L/\partial x = 0$, and similarly for the y and z equations, so they reduce to:

$$\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0,$$

and similarly for the y and z equations. But since $L = 1$ is a constant it comes out of the differential, becoming

$$\frac{d}{ds}\left(\frac{\dot{x}}{L}\right) = d\dot{x}/ds = 0.$$

This says that $\dot{x} = dx/ds =$ constant, or $x = m_x s + b_x$, and similarly, $y = m_y s + b_y$, and $z = m_z s + b_z$. Again the equation for a straight line, with s being the distance traveled along the line, but with less algebra.

4.6. Geodesics in Minkowski spacetime

Next we use the Euler-Lagrange equations to extremize the invariant interval, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, between two events in Minkowski spacetime. This will give us the equations of motion of Special Relativity! Let's use proper time τ as the affine parameter: $d\tau = \sqrt{-ds^2}$. We write $s = \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} d\tau$, where $\dot{t} = dt/d\tau$, $\dot{x} = dx/d\tau$, etc. Since basically $\tau = s$, here again the Lagrangian $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$.

Noting that $\partial L/\partial t = 0$, and similarly for $\partial L/\partial x = 0$, etc. the Euler-Lagrange equations become

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} = 0,$$

and

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} = 0,$$

and similarly for y and z. Thus as we move along the path (varying τ to move along the geodesic) we have 4 conserved quantities, $\frac{\partial L}{\partial \dot{t}} =$ constant, $\frac{\partial L}{\partial \dot{x}} =$ constant, etc. Explicitly, these give $\partial L/\partial \dot{t} = \dot{t}/L = \dot{t} =$ constant, and $\partial L/\partial \dot{x} = \dot{x} =$ constant, so our equations are $\dot{t} = c_t$, $\dot{x} = c_x$, $\dot{y} = c_y$, and $\dot{z} = c_z$, or $t = c_t\tau + t_0$, $x = c_x\tau + x_0$, etc., where the c_t,c_x, t_0, x_0, etc. are constants.

To find the values of the constant let's simplify to only the x direction and let time t start with $\tau = 0$, so $t = c_t\tau$. Then $x = c_x\tau + x_0 = (c_x/c_t)t + x_0$, and we recognize the combination $c_x/c_t = v_x$ the velocity in the x direction. Thus $c_x = c_t v_x$. Now evaluate the Lagrangian $L = 1 =$

$\sqrt{\dot{t}^2 - \dot{x}^2} = \sqrt{c_t^2 - c_x^2} = \sqrt{c_t^2 - c_t^2 v_x^2}$, or $1 = c_t\sqrt{1-v_x^2}$. Thus $c_t = 1/\sqrt{1-v_x^2} = \gamma$, the Lorentz factor, and we have as our geodesics the equations of special relativity: motion in a straight line with time dilation included: $t = \gamma\tau$, $x = v_x t + x_0$, etc.

Note that it is possible to have all the constants $v_x = v_y = v_z = 0$, so just $t = \tau$ (standing still aging) is a geodesic.

In summary, what did here is extremize (in fact maximize) the proper time between two events to find the geodesics. Thus the geodesic is that path for which the **maximum** time passes on the wrist watch of the observer traveling that path. [It is maximum, rather than minium due to the minus sign in the metric.] Note that this is basically the answer to twin paradox. The twin that went out and then back did not travel a geodesic, they accelerated three times, while the stay at home twin did not accelerate and therefore followed a geodesic. Thus we understand why the stay at home twin ages more (in fact ages maximally!). Whenever someone accelerates, they leave their geodesic and therefore are aging less! The solution to the twin paradox is also easily understood using a spacetime diagram. Both twins start together at the origin. One twin stays on Earth (worldline is geodesic going straight up). The other accelerates close to the speed of light (worldline close to 45^0). The proper time for the speedy twin is very small; remember the hyperbola of constant proper time. Halfway out, the speedy twin accelerates again (leaves the geodesic again), and speeds home, meeting up with the first twin. From this it is clear that if one minimized the proper time, it would require accelerating to the speed of light for half the time and then coming back with a total proper time approaching 0. We see that actual geodesics maximize the proper time.

4.7. Conserved quanties in the Euler-Lagrange formalism: Energy and Momentum

Note that as we did the derivations of the geodesics above we came across quantities that did not change as we traced along the affine parameter, that is we found conserved quantities. This is a general and important thing to watch for in using the Euler-Lagrange formalism. In general there will be a conserved quantity whenever the Lagrangian L does not depend explicitly on one of the variables. Actually there will be one conserved quantity per variable. This is easy to see. To be general, let's use the 4-vector notation x_μ, where $x_0 = t$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, so μ runs from 0 to 3 and x_μ can represent any of the spacetime variables. In this notation, all the Euler-Lagrange equations can be written in one line:

$$\frac{\partial L}{\partial x_\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial d\dot{x}_\mu} = 0.$$

So if L does not depend on one variable, call it x_μ, we have $\frac{\partial L}{\partial dx_\mu} = 0$, and the Euler-Lagrange equation reads: $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}_\mu} = 0$. Thus along the geodesic (running over values of the affine parameter λ), the quantity $\frac{\partial L}{\partial \dot{x}_\mu}$ is a constant.

In general we define p_μ, the **conjugate momentum**, of a variable x_μ, as

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}.$$

Thus we see that if the Lagrangian does not depend explicitly on a variable, then that variable's conjugate momentum is conserved. This p_μ may be an actual momentum, but it could be some other conserved quantity. If one takes $L = \frac{1}{2}m\dot{x}^2$, then in fact $p_x = mv_x$, which is conserved since L does not depend upon x. If instead we took $L = \frac{1}{2}m\dot{x}^2 - V(x)$, then the conjugate momentum is still $p_x = mv_x$, but it is not conserved along the geodesic as the affine parameter (t) changes. Of course this is because there is a force due to the potential energy.

For the Minkowski metric, let's see what these conjugate momenta are. We saw above that the metric did not depend explicitly on any of the x_μ; t,x, y, or z, thus we expect to have 4 conserved quantities. Consider the t equation with affine parameter proper time τ. We have $p_t = \frac{\partial L}{\partial \dot{t}} = \dot{t}$. Likewise for the x equation we find $p_x = \dot{x}$, $p_y = \dot{y}$, etc. During the above calculation we found that the constant for the Euler-Lagrange equations (which we called c_t) was equal to the Lorentz factor γ. Thus we see that

$$p_0 = p_t = \gamma = E/m,$$

where we noticed that in special relativity the energy of a particle is $E = m\gamma$, where m is the rest mass. Thus the conserved quantity associated with the t variable is nothing other than the energy. (It is energy per unit mass actually, but since p_t is a constant we can multiply by another constant m and still have a constant.)

We also previously found the momentum conjugate to the x variable, $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} = c_x$. Using $c_x/c_t = v$ from before, we have $p_x = v_x c_t = v_x p_t = v_x\gamma$. But in special relativity the momentum is just $mv\gamma$, so see that $p_x = P_x/m$, is the momentum per unit mass. (We call the actual momentum P to distinguish it from the momentum per unit mass.) Thus the 4 conserved quantities we discovered are just the energy and 3 components of momentum.

We can find the relation between these quantities from our definition of the Lagrangian: $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$. Substituting in $p_t = \dot{t} = E/m$, $p_x = \dot{x} = P_x/m$, etc. we find

$$1 = \sqrt{(E/m)^2 - (P_x/m)^2 - (P_y/m)^2 - (P_y/m)^2}.$$

Squaring, multiplying through by m^2, and using $P^2 = P_x^2 + P_y^2 + P_z^2$, we find the well known result $E^2 = P^2 + m^2$. Finally using units to put back the c's, $E^2 = P^2c^2 + mc^4$, which reduces to the famous $E = mc^2$ in the limit of zero velocity (zero momentum).

5. Equivalence Principle, Gravitational Redshift and Geodesics of the Schwarzschild Metric

5.1. Gravitational Redshift from the Schwarzschild metric

We already discussed the curvature of time in the Schwarzschild metric. Clocks run slower in a gravitation field (at smaller values of r). This is one of the more measurable effects of GR. Measurements are possible because atomic transitions act like small clocks which send messengers (photons) to us here at Earth. Recall the proper time in the Schwarzschild metric:

$$d\tau = dt\sqrt{1 - \frac{2GM}{rc^2}},$$

where $d\tau$ is tick of a clock measured at r, and dt is the length of that same tick measured far away ($r \to \infty$).

Now consider some atomic transition, for example, the $n = 3$ to $n = 2$ transition of atomic Hydrogen. This is called the H-alpha transition and results in emission of a photon with energy $E_\gamma = 13.6\text{eV}(\frac{1}{3^2} - \frac{1}{2^2}) = 1.89$ eV. Using $\lambda = c/\nu$, and $E_\gamma = h\nu$, we find the corresponding wavelength $H\alpha : 6563.5$ **Angstroms**. Now in the frame at r, the photon has a fixed frequency ν_0, and each oscillation is like a little clock ticking with period $d\tau_0 = 1/\nu_0$. Thus the time between crests in the outgoing photon measures $d\tau$. That same photon travels out to us at $r = \infty$. Then that same period is observed as dt. Thus the $d\tau_\infty \equiv d\tau_{obs} = dt = d\tau_0/\sqrt{1 - 2GM/rc^2}$ is longer. Using the definitions above of ν and λ we see that the frequency decreases by the same factor and the more easily observed wavelength is longer at when observed at infinity than the emitted wavelength at r by

$$\lambda_{obs} = \frac{\lambda_0}{\sqrt{1 - \frac{2GM}{rc^2}}} = \frac{\lambda_0}{\sqrt{1 - \frac{r_S}{r}}},$$

where we used the definition of the Schwarzschild radius r_S.

The redshift of an emission line is defined as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} = (1 - \frac{r_S}{r})^{-1/2} - 1,$$

where λ_0 is the emitted wavelength in the rest frame of the atom. Note for small values of $2GM/rc^2 = r_S/r$, this expression can be Taylor expanded as $z \approx GM/rc^2 = \frac{1}{2}r_S/r$.

This gravitational redshift is totally different from either the normal Doppler shift redshift caused by an atom moving away from the observer, or the cosmological redshift caused by the expanding Universe. Let's see how big the effect is by calculating it for some typical astronomical objects. First how about emission from the surface of the Sun? We expect the wavelengths recieved by us on Earth to be longer than the emitted wavelengths since the emitted photons have to climb out of the potential well of the Sun where clocks run slower. Using the approximation above

$z_\odot \approx \frac{1}{2}3\text{km}/7 \times 10^5\text{km} \approx 2 \times 10^{-6}$. This is not a very big effect; a clock on the surface of the Sun loses about 30 seconds/year. For H-alpha the line redshift would be only 0.014 Angstroms.

How about for a white dwarf star? The mass of a white dwarf is about $0.6M_\odot$, and the radius is about 5500 km, near to the Earth radius. Thus $z \approx \frac{1}{2}(.6)(3\text{km})/5500\text{km} \approx 1.6 \times 10^{-4}$, and the wavelength of H-alpha emission recieved on Earth would be $6563.5(1 + z) = 6564.6$ Angstroms, quite easily distinguishable from local H-alpha lines. This result agrees well with the measurements from the nearby white dwarf star Sirius B. A clock on the surface of Sirius B would lose about 0.6 second/hour.

Next consider the case of a neutron star, with $M = 1.4M_\odot$ and $R = 10$ km. Now $r_S = (1.4)(3\text{km}) = 4.2$ km, so really should not use the approximation, but calculate $z = [1-(4.2\text{km}/10\text{km})]^{-1/2} - 1 = 0.313$. This is big shift: the H-alpha line would appear at 8618.3 Angstroms and clocks would run 31% slower.

Finally what about light emitted from the Schwarzschild radius of a black hole? With $r = r_S$, $z \to \infty$! Light would be invisible since it would be redshifted to infinitely low energies. Clocks sitting at that location would appear to have stopped!

5.2. Light bending and the Equivalence Principle

To find proper answers in GR we calculate using the metric, in our case the Schwarzschild metric, but there is an interesting, more intuitive way that uses the Equivalence Principle directly. Let's look briefly at two examples. Imagine someone in a falling elevator. The equivalence principle says that experiments done in this falling frame will give the same results as experiments done floating in free space far from any massive object. Suppose the guy in the falling elevator shines a laser beam horizonally from one side of the elevator to the other. That light must follow geodesics and therefore go "straight" across the elevator; if the beam leaves from a height of 3 feet above the elevator floor, it must arrive on the other side also three feet above the floor. From the point of view of the elevator man, the light must travel exactly horizontally. But now think of someone watching through the (glass) walls of the elevator. The man shoots the laser from a height of 3 feet above the floor, but then while the light is traveling across, the elevator continues to fall. Thus when the light hits the opposite wall 3 feet above the floor, the wall has moved down. Therefore the light cannot go straight in the Earth frame. The light must also fall. This gedanken experiment shows the light must fall in a gravitational field in the same way as everything else. Of course, from the point of view of GR, light is just following the geodesic through a curved spacetime; we will calculate those soon, but here we forsee what that calculation will give. In particular, light from distant stars travelling near the limb of the Sun must be "deflected" towards the Sun. Sir Author Eddington did this experiment in 1919, and it garnered world wide fame for Albert Einstein, because it was one of the first proofs of Einstein's GR theory. In the homework, I am asking you to calculate how far the light falls in a gravitational field, that is how much it bends.

5.3. Gravitational Redshift again

The equivalence principle can also be used to show that light must gravitationally redshift. Again consider our man in a falling elevator. This time he shoots his laser from the floor to the ceiling.

First think about it from the point of view of someone outside watching through the glass walls. In the Earth frame the freely falling elevator picks up speed while the light is in transit, and so there is Doppler shift caused by the speeding elevator ceiling. The time the photons spend in transit is $t = h/c$, where h is the height of the elevator, and the speed increase of the elevator is $\Delta v = gt = gh/c$, where $g = GM_{\rm Earth}/r^2 = 9.8{\rm m/s}^2$. We could find the expected Doppler shift from the relativistically correct formula $\nu_{obs} = \nu_0\sqrt{(1+\Delta v)/(1-\Delta v)}$, or since $\lambda \propto \nu^{-1}$, $\lambda_{obs} = \lambda_0\sqrt{(1-\Delta v)/(1+\Delta v)}$, but since a Taylor expansion of this is good enough for the slow speeds involved here we have: $\lambda_{obs} \approx \lambda_0(1-\frac{1}{2}\Delta v)(1-\frac{1}{2}\Delta v) \approx \lambda_0(1-\Delta v)$, or $\lambda_{obs} - \lambda_0 = -\Delta v\lambda_0$, or $z \approx -\Delta v$. Thus in the Earth frame we expect the light beam to be blue shifted by $\Delta\lambda/\lambda_0 = -\Delta v/c = -gh/c^2$, where I put the $c's$ back in. However, the equivalence principle says that this cannot be! The man in the elevator can't tell whether he is falling or floating freely in deep space, so he expects correctly that the frequency of light recieved at the elevator's ceiling is the same as the frequency he sent from the floor. Thus there must be another effect to cancel this one! The light traveling up must be gravitationally redshifted to counterbalance this effect. Thus there must be a factor of $\Delta\lambda/\lambda_0 = v/c = gh/c^2 = GMh/r^2c^2$. This is not quite the result we got before from the metric, since there is a factor of r^{-2} rather than r^{-1}. Before we found the redshift between $r = r_0$ and far away ($r = \infty$) and here we found it only between r_0 and r. Thus to compare to this equivalent principle result we should use our previous calculation twice. We had $z \approx \frac{1}{2}r_s/r$ as the gravitational redshift between r and ∞, so to find the redshift betweent $r = r_{\rm Earth}$ and $r = r_{\rm Earth} + h$ we should find $\Delta z \approx \frac{1}{2}r_s[1/r_{\rm Earth} - 1/(r_{\rm Earth}+h)]$, or saying that h is much less than $r_{\rm Earth}$ this gives $\Delta z \approx \frac{1}{2}r_s h/{r_{\rm Earth}}^2 = GMh/r^2c^2$, precisely what we found above from the equivalence principle.

5.4. Geodesics of Schwarzschild metric from Euler-Langrange

Let's return to our work on geodesics and apply the Euler-Lagrange equations to find the geodesics of the Schwarzschild metric. Remember that the Schwarzschild metric is the unique metric around stationary, spherically symmetric, uncharged objects, so what these geodesics do is tell us how thing move around the Earth, around the Sun, and around uncharged, non-spinning black holes. These are the General Relativistic extension of Newton's and Kepler's laws of motion. Recall the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.$$

As above, we set $c = 1$, and take affine paramter $\lambda = s = \tau$, and extremize $s = \int L d\tau$, with

$$L = 1 = \left[\left(1 - \frac{2GM}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{r}} - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2 \right]^{1/2}$$

The Euler-Lagrange equation for t is then

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0,$$

which since $\partial L/\partial t = 0$, implies there is a conserved quantity we will call energy per unit mass:

$$\frac{\partial L}{\partial \dot{t}} \equiv \frac{E}{m}.$$

Calculating,

$$\frac{\partial L}{\partial \dot{t}} = \frac{1}{2}[\cdots]^{-1/2}(1 - \frac{2GM}{r})2\dot{t},$$

where we abbreviated $L = [\cdots]^{1/2}$. Using $L = 1$, we find our t equation

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m}.$$

We don't yet know that the constant has anything to with energy, but we call it E/m, because of our experience with the Minkowski metric. For $r \to \infty$, the Schwarzschild metric goes to the Minkowski metric and for the Minkowski metric $\dot{t} = E/m$.

Next we find the ϕ equation:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0,$$

so again we have a conserved quantity $p_\phi = \partial L/\partial\dot{\phi} \equiv -l/m$, where we will call this conserved quantity the angular momentum per unit mass. Recall Noether's theorem which says if physics is unchanged by a rotation then angular momentum is conserved. Since the metric does not depend explicitly on the angle ϕ, we get that result here. Doing the differentiation we find

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2\sin^2\theta 2\dot{\phi}) = -r^2\sin^2\theta\dot{\phi}.$$

Thus our ϕ equation reads

$$\frac{l}{m} = r^2\sin^2\theta\dot{\phi}.$$

Note that it makes sense that we called the constant of motion l/m, since this matches the normal definition of angular momentum, $l = \vec{r} \times \vec{P}$, with $v = r\sin\theta\dot{\phi}$.

Next, we consider the θ equation. Here we find that

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \neq 0,$$

thus we do not have a conserved quantity for this equation. We find

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}[\cdots]^{-1/2}(-r^2\dot{\phi}^2 2\sin\theta\cos\theta) = -r^2\dot{\phi}^2\sin\theta\cos\theta,$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2 2\dot{\theta}) = -r^2\dot{\theta}.$$

Thus our θ equation reads:

$$\frac{d}{d\tau}(r^2\dot{\theta}) = r^2\dot{\phi}^2\sin\theta\cos\theta.$$

Finally we come do the r equation, which is kind of messy because of all the explicit r dependence in the metric. However, we don't have to do it, because we can get the fourth equation we need to specify the equations of motion from our definition of the Lagrangian, $L^2 = 1$:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2.$$

Since our object and metric are spherically symmetric we can simplify things by only considering motion in the equatorial plane ($\theta = \pi/2$, and $\dot{\theta} = 0$). Of course we have to remember that we made this assumption later when we use our equations! If we try to consider motion that has a changing θ, or which is not in this plane we would need to come back to the equation above. In this case then, the equation $L = 1$ becomes:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - r^2\dot{\phi}^2.$$

Now eliminate variables other than r, using the constants of motion we have from the above equations: $l/m = r^2\dot{\phi}$, and $E/m = (1 - \frac{2GM}{r})\dot{t}$, to get:

$$1 = \frac{E^2}{m^2(1 - \frac{2GM}{r})} - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - \frac{l^2}{m^2r^2}.$$

We can write this in a nicer form by solving for $m\dot{r}^2$,

$$m\dot{r}^2 = \frac{E^2}{m} - \left(m + \frac{l^2}{mr^2}\right)(1 - \frac{2GM}{r}),$$

Remembering the definition of $\dot{r}$ and using dimensional analysis to put back the c's, we can write this as:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)\left(mc^2 + \frac{l^2}{mr^2}\right).$$

Notice that, as I mentioned in my handwaving description, a step in proper time $d\tau$ forces a step dr in the r direction. Thus this geodesic equation shows that things fall due to the spacetime curvature of the metric. We will look at these geodesic equations in some detail, but for now let's just take one limit of this last equation.

Suppose the angular momentum $l = 0$, which we might expect for radial infall towards a spherical mass. Our equation then is:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right) mc^2 = 0.$$

Next consider a case where you start at rest far from the object so that at proper time $\tau = 0$, $m(dr/d\tau)^2 = 0$, and $r \to \infty$. Our equation becomes:

$$\frac{E^2}{mc^2} - (1 - \frac{2GM}{\infty})mc^2 = 0,$$

or $E^2/mc^2 = mc^2$, or simply $E = mc^2$! So at $\tau = 0$ the total energy is just $E = mc^2$. In Newtonian mechanics the energy at infinity is usually defined as $E = 0$. Isn't it nice how these important results are just built right into the general relativistic view of spacetime. Also since energy E is conserved along geodesics we know that $E = m$ always. This E is not the Newtonian energy; it is *the* conserved quantity, which is better than the sum of $\frac{1}{2}mv^2 + V$. Finally note that if we would have started with some velocity at $r = \infty$ then $E > mc^2$ but it still would have been conserved.

At later times during this radial infall from rest, our equation becomes:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - mc^2 + \frac{2GM}{rc^2}mc^2 = \frac{2GMm}{r}.$$

That is simply

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0,$$

which looks remarkably like the Newtonian case $\frac{1}{2}mv^2 - GMm/r = 0$! But this is not the same equation. It is fully relativistic and the time derivative is τ not t (remember $t = \gamma\tau$ in SR). In general you integrate these 4 equations to get the complete picture of motion near the Earth, Sun, or Black Hole. We will come back to these soon.

6. Distances and Times Around a Black Hole

Suppose you have a very powerful spaceship and fly near to a black hole. Can you notice anything different from deep space? (Actually the calculations and results we get here are also true around the Earth on a very small scale.)

Suppose we fly close to a small black hole of mass $M = 3M_\odot$. (Actually, the smallest black hole we expect to find in nature is around $3M_\odot$, so we pick this number.) We know the Schwarzschild radius for this black hole is about 3×2.95 km $= 8.85$km, so we are careful to stay farther away from the hole than this. Let's suppose we fly all the way around the hole and measure a distance around (circumference) of $C = 2\pi 30$km $= 188.5$ km. How far are we from the hole? Naively, we expect we are 30 km from the hole, but we should check this by using the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

To find the proper distance in the θ direction, we set $dt = dr = d\phi = 0$, and the find the proper length

$$dl_\theta = \sqrt{ds^2} = r d\theta.$$

Note that this is the same as the proper distance in flat space! So there is no curvature in the θ (or ϕ) direction! To find the distance around the black hole we integrate

$$C = \int_0^{2\pi} dl_\theta = 2\int_0^{\pi} r d\theta = 2\pi r,$$

just as in flat space. Thus if we have measured the distance around as 2π 30 km, we know we are at radial coordinate $r = 30$ km.

But does that mean we are 30 km from the center of the hole? We have to use the metric again to find out. This time we want the radial direction, so we set $dt = d\theta = d\phi = 0$, and we get

$$dl_r = \sqrt{ds^2} = \frac{dr}{(1 - \frac{2GM}{r})^{1/2}}.$$

To find the distance from radial coordinate r_1 to radial r_2 we integrate from r_1 to r_2. If we set $r_1 = 8.85$ km and $r_2 = 30$ km, we can find our distance to the black hole horizon itself.

$$\Delta l_r = \int_{r_1}^{r_2} dr \left(1 - \frac{2GM}{r}\right)^{-1/2}.$$

Defining

$$A_i \equiv \sqrt{1 - \frac{2GM}{r_i c^2}} = \sqrt{1 - r_S/r_i},$$

we evaluate the integral as

$$\Delta l_r = \int_{r_1}^{r_2} dl_r = r_2 A_2 - r_1 A_1 + \frac{r_S}{2} \ln\left(\frac{r_2 A_2 + r_2 - r_S/2}{r_1 A_1 + r_1 - r_S/2}\right),$$

where we used the Schwarzschild radius $r_S = 2GM/c^2 = 2.95326 M/M_\odot$ km.

Using this formula we can find the distances. For example starting at r=30 km, and moving in to r = 20 km, we we naively expect to move 10 km, but we actually move 12.51 km. This is very weird, since after moving 12.51 km inward from $r = 30$km, the distance around the black hole would be $2\pi 20$ km *not* $2\pi 17.49$ km. Thus we are seeing directly the curving of space. It is just like the example of the two surveyors, but now the curvature is not in any direction we can experience! In the above, the way we tell what value of r we are at is to travel around the hole and use the circumference. Using the above formula we also find the distance from $r_1 = 10$ km to $r_2 = 30$ is 29.50 km, and the distance from the horizon at $r_S = 8.86$ km to $r = 30$ km is 35.98 km.

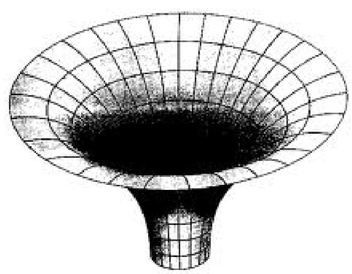

Fig. 4.— Figure for Chapter 6: Embedding diagram of curved space near a black hole

We can visualize the spatial curvature around a black hole by an embedding diagram. The key in this diagram is that the radial coordinate is just the straight 3-D distance, while the proper distance is measured along the curved surface. We continue the embedding diagram even inside the black hole horizon, which turns out to be correct, though we will have to think carefully before understanding why.

Finally, note that this metric and embedding diagram work not only for black holes, but also for the Earth! If the distance to the center of the Earth is $r_{\rm Earth}$, then the distance around the Earth is really not equal to $2\pi r_{\rm Earth}$! Can you figure the difference?

6.1. Can you fall into a black hole?

We found above that the space is flat in the θ and ϕ directions, but curved in the radial direction with proper distance $dl_r = dr/\sqrt{1 - r_S/r}$. Looking at this again we see that moving a small proper distance dl_r implies moving a smaller radial coordinate distance $dr = dl_r\sqrt{1 - r_S/r}$. This seems fine, but look at what happens right near the Schwarzschild radius r_S. When $r \to r_S$, then $dr \to 0$, that is you are not moving at all in the radial coordinate! Does this mean that you can't get into the black hole? No, because it is a square-root singularity, which implies that it is integrable.

Consider the area under the curve $y = 1/\sqrt{x}$. As $x \to 0$, $y \to \infty$, but still the area is $A = \int_0^{x_0} x^{-1/2} dx = 2x^{1/2}|_0^{x_0} = 2\sqrt{x_0}$ which is finite. Likewise $dl/dr = (1 - r_s/r)^{-1/2}$ is integrable and we gave the formula for the integration above. We therefore find that the proper distance from $r = 30$ km to r_S is 35.98 km. So just because a function goes to infinity in a region of interest doesn't always mean there is a problem. Of course, sometimes it does mean there is a problem as we shall see.

6.2. Time to fall into a black hole

Next let's calculate the time to fall into a black hole. There are several ways that we will do this. First, let's return to our geodesics for radial infall. Remember we considered the case with angular momentum $l = 0$, and found that starting at rest from $r = \infty$ meant the conserved energy was $E = mc^2$, and the equation of motion is

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0.$$

Let's find the proper time τ it takes for this freely falling object to go from $r_0 = 30$km to $r = r_S = 8.85$ km, the Schwarzchild radius of a 3 $M_\odot$ black hole. This is the time as measured by the wristwatch of the falling guy. We take the negative square root of this equation since we want to fall in (r decreasing):

$$\frac{dr}{d\tau} = -\sqrt{2GM}r^{-1/2},$$

or

$$\int_{30}^{r} dr r^{1/2} = \int_0^{\tau} -\sqrt{2GM} d\tau,$$

or

$$\frac{2}{3}r^{3/2}|_{r_0}^{r} = -\tau\sqrt{2GM},$$

or

$$\tau = \frac{2}{3c}(1/\sqrt{2GM/c^2})(r_0^{3/2} - r^{3/2}).$$

(Note I put the c back in to get the units right and make calculation easier.) So to go from $r_0 = 30$ km to $r = r_S = 8.85$ km in our $3M_\odot$ black hole takes $\tau = \frac{2}{3}(1/\sqrt{(2.95)(3)\text{km}})((30\text{km})^{3/2} - (8.85\text{km})^{3/2})/3 \times 10^5\text{km/s} = 1.03 \times 10^{-4}$ s. Thus it takes about 0.1 millisecond! It is not clear yet, but this same equation works inside the black hole, so we can also find how long the falling guy has to live before hitting the singularity at the center. Just taking $r = 0$ in the above equation gives

$$\tau = \frac{2}{3}\frac{r_0^{3/2}}{\sqrt{2GM}} = 0.124\text{ms}.$$

Thus our guy gets only an extra 0.021 ms to live inside the black hole!

Now this is the time freely falling starting at infinity. We could also find the time to fall in if we started from rest at $r = 30$ km. For this we go back to the more general geodesic equation before plugging in $E = mc^2$,

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right) mc^2 = 0.$$

Now we want the boundary conditions starting at $\tau = 0$: $dr/d\tau = 0$ at $r = r_0$. Plugging this in, we find the conserved energy is $E = \pm mc^2\sqrt{1 - 2GM/r_0c^2}$. Note this is the quantity that is conserved along the geodesic, and is NOT $E = mc^2$. For the case of falling in we take the negative square root, then integrate the above equation as before. This can be done and we find a more complicated formula. The time to fall into $r = 0$ is a little simplier:

$$\tau = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{r_S}},$$

for a time of 0.29 ms, about three times longer than when you start falling from far away.

Now what does this look like to someone watching from far away. They don't use the proper time τ, but use the coordinate for "far away" time t. We can convert the equations above to coordinate time t by using our time geodesic equation: $\left(1 - \frac{2GM}{r}\right) \dot{t} = \frac{E}{m}$, or $dt/d\tau = (E/m)/(1 - r_S/r)$. Then

$$dr/dt = (dr/d\tau)/(dt/d\tau) = -\frac{m}{E}(r_S/r)^{1/2}(1 - \frac{r_S}{r}).$$

This equation can be solved as before, but we will find some trouble in doing it as we get close to r_S. Consider that limit, $r \to r_S$. Then $1 - r_S/r \to 0$, and a tiny step in dr means an infinite step in dt. This is different than the case before with proper distance because this is not an integrable square root singularity. This is real infinity that cannot be integrated over. In fact, if you try to do the integral you will find you *never* get to $r = r_S$! Time slows down and motion ceases. Everything hangs up at the horizon.

7. Shooting Light Rays into Black Holes, Inside a Black Hole, Orbits in the Schwarzschild Metric, Effective potentials

7.1. Shooting Light into a Black Hole

Let's calculate radial motion into black holes another way, a way that is often very useful, because it bypasses the geodesic equations. Let's calculate how long it takes to shoot a ray of light into black hole.

For light we know the invarient interval $ds^2 = 0$, that is the metric distance, aka proper time, is 0. So for light itself how long does it take to get into a black hole? It takes the same time it take light to go anywhere, zero! From light's point of view no time ever passes since $d\tau = 0$ always. OK, that means we set the metric equal to zero for light. This is the very useful trick. Considering radial infall, we can also set $d\phi = d\theta = 0$, and get

$$dt = \frac{dr}{(1 - r_S/r)}.$$

Integrating both sides from $t = 0$ at $r = r_0$ to $t = t$ at $r = r$, we find

$$t = r_0 - r + r_S \ln\left(\frac{r_0 - r_S}{r - r_S}\right).$$

From our previous work we expect the time for light go from $r = 30$ km to $r = r_S$ to be less than a millisecond. But plugging into the above equation we get a factor $\ln((30 - 8.85)/(8.85 - 8.85)) \rightarrow \ln(\infty) \rightarrow \infty$. There is a logarithmic divergence and again we see we never get into the black hole!

Of course, from the point of view of the light ray, or the falling guy, they get in and are crushed within a millisecond. It is just that time viewed from far away runs differently at the horizon of a black hole. The problem is with the use of this far away time. Also, while it is true that everything that ever falls into a black hole seems to "hang up" at the horizon, it is not that case that someone looking closely at a black hole sees all that junk. Remember the light coming to you from the falling objects redshifts to infinity and therefore those objects become invisible very quickly.

As an interesting aside consider what would happen if you tried to lower yourself slowly into a black hole on a very strong rope. I'm not going to do the calculation but the effective acceleration of gravity you would feel would increase without limit. The effective g, which is $GM/r^2 = 9.8$ m/s^2 here on Earth becomes

$$g = (GM/r^2)(1 - r_S/r)^{-1/2}.$$

Thus the force becomes infinitely strong at $r = r_S$, the rope will break and you will fall to your death.

7.2. Inside the Black Hole

The Schwarzchild metric is the solution to Einstein's GR equations in the vacuum around a spherically symmetric object. Thus we expect these to work even if the object is smaller than r_S. What happens at and inside $r = r_S = 2GM/c^2$? The metric written in the r and t coordinates is bad at $r = r_S$, but actually it is OK inside.

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

First consider lightcones, the causally connected past and future. How did we find these in Special Relativity? Considering just one space and one time dimension: $ds^2 = -dt^2 + dx^2 = 0$, for light. This defines the null geodesics. Solving we find $dt = \pm dr$, or $dt/dr = \pm 1$, which implies the lightcones are lines of 45^0 in a spacetime diagram. Remember also that above the 45^0 lines is the timelike $ds^2 < 0$ future, while outside the lines is the $ds^2 > 0$ spacelike elsewhere. In Special Relativity all lightcones have these 45^0 lines.

How about in General Relativity. Again we set $ds^2 = 0$ to find the null geodesics, giving $-dt^2(1 - r_S/r) + dr^2/(1 - r_S/r) = 0$, or

$$\frac{dt}{dr} = \pm\frac{1}{1 - \frac{r_S}{r}}.$$

Thus the lightcones are not 45^0 lines. As $r \to \infty$, $r_S/r \to 0$, so $dt/dr \to \pm 1$, and the lightcones are at 45^0, but closer to the spherical mass the angles are smaller, $dt/dr > \pm 1$. Thus the lightcones squeeze-up. Thus the path of light as it travels towards a spherical mass in a spacetime diagram is not on a 45^0 line, but on a curved line that gets more steep as it approaches the Schwarzschild radius.

We can calculate the angle of the lightcone at any value of r very simply. Just use

$$\frac{dt}{dr} = \tan\theta.$$

Thus at 30 km from a $3M_\odot$ mass hole the angle of the lightcone is $dt/dr = 1/(1 - 8.85/30) = 1.42$, or $\theta = 55^0$. This angle goes to 90^0 at $r = r_S$. So we see that right at $r = r_S$, $dt/dr \to \infty$, and no progress in r is possible! Not even light can make it into the black when far-away time is used as the coordinate. How does this jive with the fact that we calculated things can fall into black holes in just milliseconds? The point is that t is time measured by someone far away, not the traveler.

Now look what happens to the metric when $r < r_S$. Notice that since $r_s/r > 1$, the terms $(1 - r_S/r)$ become less than zero.

$$ds^2 = -dt^2(1 - r_S/r) + dr^2/(1 - r_S/r) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Thus we see that the t coordinate and the r coordinate terms switch signs! Remember that in GR r is not a distance, it is a coordinate. You have to use the metric to find distances. Same with

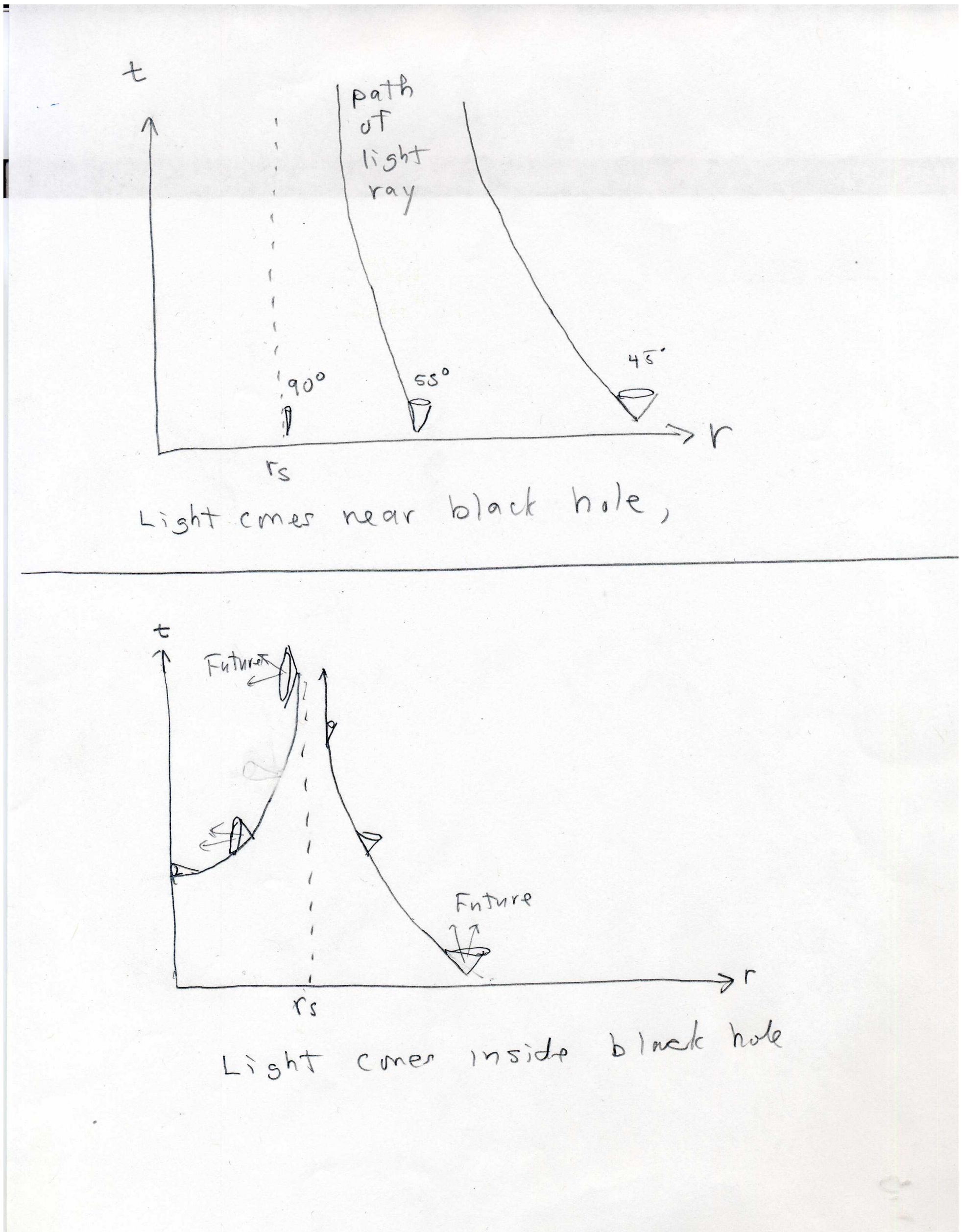

Fig. 5.— Figures for Chapter 7: (a) Path of light as it approaches a black hole and (b) Lightcones around and inside a black hole

t. The minus sign is how we recognize the time coordinate in GR, so this means that r becomes timelike and t becomes spacelike! If you try to find the lightcones inside the black hole you see they flip over sideways!

The same force that moves eveyone towards the future now moves things towards $r = 0$, which is the future lightcone! This one way of understanding the reason that once inside the black hole you can't get out. There is no force in nature that can move you backward in time. Thus inside a hole there is no force in nature that can move something towards larger r! Note that while proper time increases, t actually decreases. This is not a problem since t is no longer a time dimension. It is a space dimension; r is the time. We can see this explicitly by considering two points inside, say near $r = 2$ km and $r = 2.1$ km, so $dr = 0.1$ km. Take $dt = 0$ and calculate the invariant interval: $ds^2 = +dr^2/(1 - 8.85/2) = -0.0029$ km < 0, that is the interval is timelike, and thus these points can be causally connected! Similarly, two events with $dr = 0$ and $dt \neq 0$ have $ds^2 > 0$ and therefore are spacelike separated and cannot be causally connected. Moving straight up in the spacetimem diagram is not allowed.

The angle of the lightcone just inside r_S is still 90^0, but rather than being squeezed, it is now wide open. As you move in towards $r = 0$, the lightcone closes up, but continues to point toward $r = 0$. Inside $dr/dt = \pm(1 - r_S/r)$. As $r \to 0$, $dr/dt \to \pm\infty$, or $dt/dr \to 0$, and the cone squeezes in pointing towards the center of the hole. So while at first when you are inside the hole there some freedom of movement, in the end you are directed directly at the center.

7.3. Orbits in the Schwarzschild metric

Let's go back to our geodesic equations and find the orbits around a black hole or other spherical object. So far we only considered radial orbits. To extend this to circular and eliptical orbits we will use the method of effective potentials, which allows one discover the main types of orbits and the main points of interest without doing the hard work of actually solving the differential equations. This is very general technique and worth learning.

7.4. Effective Potential for Newtonian Orbits

First let's do it in Newtonian mechanics, by writing the total energy $E = T + V = \frac{1}{2}mv^2 - GM/r$ in spherical coordinates. Recall that in 3-D non-relativistic mechanics, $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, where $\dot{x} = dx/dt$, etc., and $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$. Let's simplify by considering motion in the x-y plane so $z = 0$, $\theta = \pi/2$, and $\dot{\theta} = 0$. Then $\dot{z} = 0$, $\dot{x} = \dot{r}\cos\phi - r\sin\phi\dot{\phi}$, and $\dot{y} = \dot{r}\sin\phi + r\cos\phi\dot{\phi}$. Substituting this into the formula for kinetic energy and using $\sin^2\phi + \cos^2\phi = 1$, we find:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

Next defining the angular momentum $l = mv_\phi r = m\dot{\phi}r^2$, so $r\dot{\phi} = l/(mr)$, we can write the total energy as

$$E = \frac{1}{2}m\left(\dot{r}^2 + \frac{l^2}{m^2r^2}\right) - \frac{GMm}{r},$$

or

$$E = \frac{1}{2}m(\frac{dr}{dt})^2 + V_{eff},$$

with

$$V_{eff} = -\frac{GMm}{r} + \frac{l^2}{2r^2m}.$$

Thus we have written the equation of motion as a one dimensional equation in the radial coordinate, r, with energy $E = T + V_{eff}$. The value of doing this is that we are very familar with the the solutions of a one dimensional equation for a particle moving in a potential well with the shape V_{eff}. Depending on the total energy, the particle may escape the well and travel off to infinity, or it might be trapped in the well and oscillate back and forth in the bottom of the well. For small enough energy it might be at rest at the bottom. So we now think of the radial coordinate r as the one dimension and can easily understand the possible Newtonian orbits around a spherical object.

To discover the possible orbits we draw a plot of r vs. the effective potential. We see that V_{eff} drops from infinity at $r = 0$, reaching a minimum at a value of $r = r_{min}$, and then rises slowly to $V_{eff} = 0$ at $r = \infty$. The exact shape depends upon the constants (conserved quantities) l, and E, as well as M.

The easiest solution to this one dimensional problem is the particle as rest at the minimum. The particle just sits there, which means $r =$ constant. This is the circular orbit of Kepler's laws. The radius of the circular orbit is found by finding the minimum of the effective potential by solving $dV_{eff}/dr = 0$. This gives $0 = GMm/r^2 - l^2/(mr^3)$, or $r = GM/v^2$, where we substituted back in $l = mvr$ before solving for r. Note that we did not substitute in for l before taking the derivative with respect to r, since l is a conserved quantity and therefore constant along the orbit. This is the result you get from $F = ma$, aka $GMm/r^2 = mv^2/r$. In this case, the total energy is at its minimum which is less than zero. $E_{min} = -mv^2 + \frac{1}{2}mv^2 = -\frac{1}{2}mv^2$. Note if you had a different value of v, that would imply a different value of l, and a different r_{circ}.

Another obvious solution is the particle having $E < 0$, so the particle oscillates around the bottom between turning points r_1 and r_2 where $E = \frac{1}{2}m\dot{r}^2 + V_{eff}$. These orbits are the elliptical Kepler orbits, and the position and speed of the orbiting object at turning points can be found from the equation above. There are also orbits that have $E > 0$ which can come in from infinity, reach a point of closest approach and then return to infinity. These are the hyperbolic Kepler orbits. Finally, a particle with $E = 0$ will do a similar similar thing, but the orbit will be parabolic.

7.5. Effective Potential for Schwarzschild Orbits

Let's apply the same effective potential technique to the full r geodesic equation we derived earlier.

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right),$$

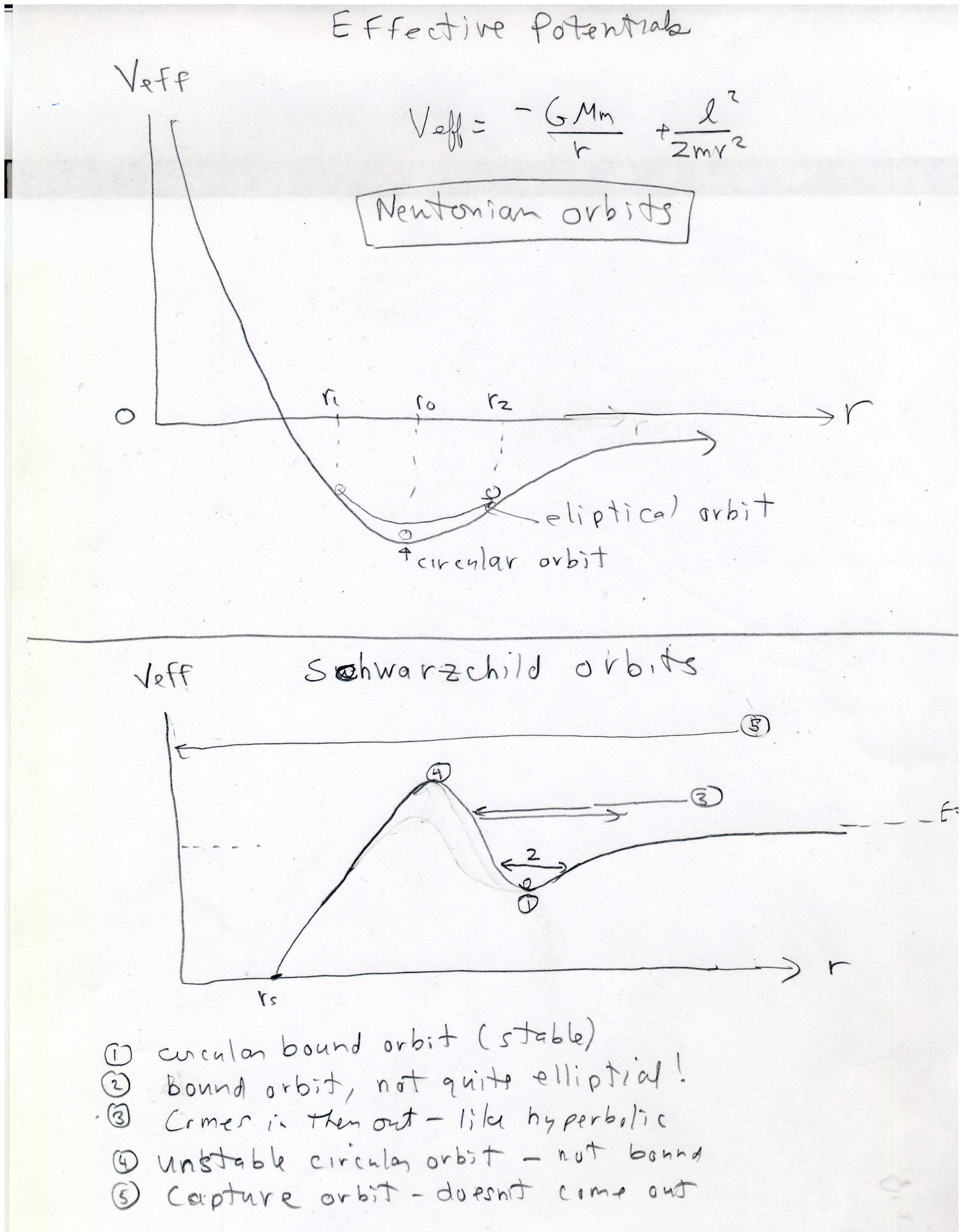

Fig. 6.— Figures for Chapter 7: (a) Effective potential for Newtonian Potential and (b) Effective potential for Schwarzschild Geodesics

where $r_S = 2GM/c^2$. Here the effective potential is

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right) = m - \frac{r_S}{r}m + \frac{l^2}{mr^2} - \frac{r_S l^2}{mr^3},$$

and keep in mind that we are using the proper time τ, not t as the time variable, and l is the angular momentum and E is the total energy. We can spot the different types of orbits as we did for the Newtonian effective potential. Now we see there are five different types of orbits.

1. At the minimum we have a bound circular orbit. The radius is found from $dV_{eff}/dr = 0$ as before.

2. We also have the bound orbits with turning points r_1 and r_2 like in the Newtonian case, but here these turn out to not be elliptical! The orbits don't close. If we do the perhelion advance of Mercury we will show this. The time to between r_1 and r_2 and back is more than the time to go around $\phi = 360^0$, so the perhelion advances.

3. Next, as for Newtonian hyperbolic orbits, we have orbits that start at $r = \infty$, come in, reach a distance of closest approach, and go out again. Close examination shows these are close to, but not exactly the same as, the Newtonian case.

4. Now look at maximum point on top of the hill. We find this point also when we set $dV_{eff}/dr = 0$, since the slope is zero here. If you set a particle there and carefully balanced it, it would stay there. Thus this is another circular orbit, but it is unstable. A tiny perturbation outward and the particle will escape to infinity. A tiny perturbation inward and it will fall in the hole.

5. Finally, there is another new type of orbit not found in the Newtonian case, the capture orbit. If the energy is high enough, the particle goes over the high point and then plunges down into the hole, never to return. In the Newtonian case a particle can never be captured in this way due to gravity alone, even though this is how we think when, for example, a comet hits the Sun. Hitting the Sun's surface invokes non-gravitational forces; a point mass object could never capture anything in Newtonian mechanics.

Let's find the radii of the circular orbits. We can write

$$V_{eff} = m + \frac{l^2}{mr^2} - \frac{r_S m}{r} - \frac{r_S l^2}{mr^3},$$

and differentiate and set to zero.

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{mr^3} + \frac{r_S m}{r^2} + \frac{3r_S l^2}{mr^4} = 0.$$

We can simplify this to a quadratic equation in r by multiplying through by mr^4:

$$r_S m^2 c^2 r^2 - 2l^2 r + 3r_s l^2 = 0,$$

which has two solutions:

$$r_\pm = \frac{l^2}{r_S m^2 c^2}\left(1 \pm \sqrt{1 - \frac{3r_S^2 m^2 c^2}{l^2}}\right),$$

where I've put the c's back in for fun and reference.

Note that if the quantity in the square root is negative we don't have a real solution. Thus there is no circular orbit in that case. This happens when the angular momentum is too small, i.e. there are only capture orbits. If the quantity in the square root in positive, then we have two solutions, that is, two circular orbits at different radii, as we saw in the plot. Thus we have two circular orbit solutions if $l^2 > 3r_S^2 m^2 c^2$, and none if not.

When there are two orbits we can tell if the orbits are stable or unstable by taking the 2nd derivative. If $d^2V_{eff}/dr^2 > 0$, the the curvature of the effective potential is positive and it is a minimum. This means the orbit is stable. On the other hand if if $d^2V_{eff}/dr^2 < 0$, it means there is a maximum in the effective potential at that point, and the orbit is unstable. We can do the math, but we saw before from the picture that the smaller solution (the one with the minus square root) was unstable, and the larger solution with the plus square is a stable orbit.

From the above discussion we see that the smallest possible *stable* circular orbit will happen when the square root term vanishes. This happens when $1 = 3r_S m^2 c^2/l^2 = 0$, or $l^2 = 3r_S^2 m^2 c^2$. The value of the radius for this value of the angular momentum can be found just by plugging this value of l into the formula for $r_\pm$, and we find that the minimum stable circular radius is

$$r_{min} = \frac{l^2}{r_S m^2 c^2} = 3r_S,$$

that is just exactly three times the Schwarzschild radius. Actually we should check that this orbit is stable, since for this value of l^2, the stable and unstable orbits are the same. Plugging in r_{min}, and the value of l^2 above into

$$\frac{d^2V_{eff}}{dr^2} = \frac{6l^2}{mr^4} - \frac{2r_S m}{r^3} - \frac{12r_S l^2}{mr^5},$$

we find that $d^2V_{eff}/dr = 0$ at this combined minimum. Thus this orbit is just really neither stable nor unstable, but neutral. It is called, however, the minimum or last stable orbit, since an orbit just a tiny bit larger is in fact stable.

This last stable orbit value is a very important result. When things fall into black holes they have trouble getting in because of angular momentum. The tend to get ripped apart and form what are called accretion disks. The material in the disk gradually spirals inward. However, when the radius reaches $3r_S$, there is no longer a stable circular orbit, so the material all just flows into the hole. Thus we expect that when we look at real black holes in space we will see material down to $3r_S$ but not any closer. When we detect X-rays from black holes, we hope we are looking at material radiating from the distance $3r_S$. And when we find periodic signals from black hole candidates, we assume this is the radius at which the material is orbiting.

8. Extracting Energy from a Black Hole; Light Orbits

8.1. Extracting Energy from a Black Hole

Now let me tell you why black holes are the most efficient energy generation devices we have yet concieved. Much better than nuclear fission or even nuclear fusion. Of course, there is the small problem of not having a black hole near by and available for use.

First consider throwing some junk into a black hole from far away. We learned before that the total energy of the stuff far from the black hole is just $E = mc^2$, where m is the mass of the stuff. Suppose this stuff doesn't go straight into the hole, but is slowed down in a series of orbits, finally settling into the last stable orbit we just discussed. What is the energy of the stuff now? We can use the radial geodesic equation to find out. On a circular orbit $dr/d\tau = 0$, so we have

$$\frac{E^2}{m} = \left(1 - \frac{r_S}{r}\right)\left(m + \frac{l^2}{mr^2}\right).$$

Next we plug in $l^2 = 3r_S^2 m^2$, and $r = 3r_S$,

$$\frac{E^2}{m} = (1 - \frac{1}{3})(m + \frac{3r_S^2 m^2}{m 9 r_S^2}) = (1 - \frac{1}{3})(1 + \frac{1}{3})m = \frac{8}{9}m.$$

Thus

$$E = \sqrt{\frac{8}{9}}mc^2.$$

Somehow during the transition from far away to the last stable orbit the energy has decreased. Note that in doing this we did *not* follow a geodesic. E is conserved along geodesics, so starting at $r = \infty$ means $E = mc^2$ still at $r = 3r_S$. Thus the stuff must accelerate to get to the smaller stable circular orbit. Typically falling stuff hits other falling stuff and the collisions are what cause the acceleration. Thus, the original energy must have radiated away, and therefore be available for other use. The fractional amount of energy radiated away is $(mc^2 - E)/mc^2 = 1 - \sqrt{\frac{8}{9}} = 5.7\%$.

Let's compare this with other forms of energy generation. When burned, a gallon of gasoline produces 1.32×10^8 Joules of energy and has a mass of 2.8 kg. Thus the fractional mass energy lost when a gallon of gasoline is burned is $\Delta E/E = (1.32 \times 10^8)/((2.8)(3 \times 10^8)^2) = 5.2 \times 10^{-10}$. This is thus also the fraction of the rest mass of the gasoline that is turned into useful energy, a far cry from the 5.7% we get from the black hole.

Next, consider nuclear energy, say the complete fission of 1 kg of pure Uranium 235. This produces 8.28×10^{13} Joules, so $\Delta E/E = (8.28 \times 10^{13})/((1)(3 \times 10^8)^2) = 0.0092\%$, still below the black hole. Finally one usually thinks of the main source of energy in the Universe as nuclear fusion of Hydrogen into Helium (as in the Sun). Actually in large stars, the Helium is then burned to Carbon, Oxygen, etc. all the way up to Iron, which is the most stable element. This whole chain of nuclear fusion releases about 0.9% of the rest mass into usable energy, still more than a factor of six below what you get from throwing your garbage into a black hole. Black holes are the best devices we know of for producing energy. In fact, using a rotating black hole we can turn even more of the rest mass of stuff into useful energy. The energy generation method discussed here is thought to be the power source of quasars, the most luminous and energetic steady objects in the Universe.

8.2. Geodesics and motion of light around a black hole

In one sense light is just the same as a particle with it's rest mass $m \to 0$. To find the geodesics for light, then, we should be able to just take the limit $m \to 0$ in the geodesic equations we already found. As a reminder I reproduce those equations below, in the special case with $\theta = \pi/2$ relevant for motion in the x-y plane.

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m} - \left(1 - \frac{r_S}{r}\right)\left(m + \frac{l^2}{mr^2}\right)$$

$$\frac{d\phi}{d\tau} = \frac{l}{mr^2}$$

$$\frac{d\theta}{d\tau} = 0$$

$$\frac{dt}{d\tau} = \frac{E}{m}\left(1 - \frac{r_S}{r}\right)^{-1}.$$

But we immediately see that if we try to take the limit $m \to 0$ in these equations we get into trouble. We get nonsense like $0 = \infty$, etc. Actually we talked about this trouble before when discussing light. For light the proper time $d\tau$ is always zero. Thus when we chose to use τ for our affine parameter, we excluded the use of these equations for light! What we should do is go back to the Schwarzschild metric, choose a different affine paramter, and use the Euler-Lagrange equations to rederive the geodesics for light. However, we can instead use a trick. This is another of those useful tricks worth learning.

We note that in the limit $m \to 0$, the speed of the particle goes to c and relativistic factor $\gamma \to \infty$. That is why the proper time $\tau \to 0$. However, the combination $m\gamma$ is the energy of the particle and stays fine as $m \to 0$. Thus if we use $\lambda = \tau/m$ as the affine parameter everything may work out fine. We do that by substituting $\tau = m\lambda$ everywhere in the geodesic equations above, and then multiplying through by m:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_S}{r}\right)\left(m^2 + \frac{l^2}{r^2}\right)$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}$$

$$\frac{d\theta}{d\lambda} = 0$$

$$\frac{dt}{d\lambda} = E\left(1 - \frac{r_S}{r}\right)^{-1}.$$

Now we can take the limit $m \to 0$ and get equations that make sense. Only the r equation changes to

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_S}{r}\right)\frac{l^2}{r^2}.$$

Note that the trick we did was to try to rescale our original equation that had a bad affine parameter τ to a different affine parameter that behaved ok. We scaled by a constant, such that we could take the limit we wanted. If the equations we got didn't make sense we might have tried another rescaling. The basic idea of rescaling differential equations is a trick that is often useful when you want to take some limit of the equations.

Note one problem with the above is that we can't use the old definitions of E and l which involve m. We know from the Euler-Lagrange treatment that these quantities are conserved along geodesics because there is no explicit t or ϕ dependence in the metric, so we can still call them energy and angular momentum, but we don't know yet what the formulas for them are when considering light geodesics.

OK, we are now ready to do the effective potential treatment for the photon geodesic equation. We count $(dr/d\lambda)^2$ as the kinetic energy term, and therefore find that

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\frac{l^2}{r^2} = \frac{l^2}{r^2} - \frac{r_S l^2}{r^3}.$$

Plotting this we can discover the light orbits around a black hole.

Using what we learned from the massive case we see that there are three types of orbits: an unstable circular orbit, a coming in, then out orbit, and a capture orbit. The really new thing here is the circular light orbit. In Newtonian mechanics light cannot orbit anything, but here in GR it can! Let's find the radius of the light orbit:

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{r^3} + \frac{3r_S l^2}{r^4} = 0,$$

or

$$r_{lightorbit} = \frac{3}{2}r_S.$$

Thus at $1.5r_S$, just half the last stable orbit for massive particles, light will orbit the hole. Since it is an unstable orbit, a black hole can trap light for some amount of time, and then release it back out to infinity.

Light orbits can produce some pretty wierd effects. Suppose an advanced race built a tube around a black hole just at the light orbit radius. This is outside the trapped surface, so people could go in that tube and still get away from the black hole. If you were standing in such a tunnel, what would you see? The light would go round and round, and you would see the back of your head. Actually further down the tube you would see yourself again. The tube would in fact look

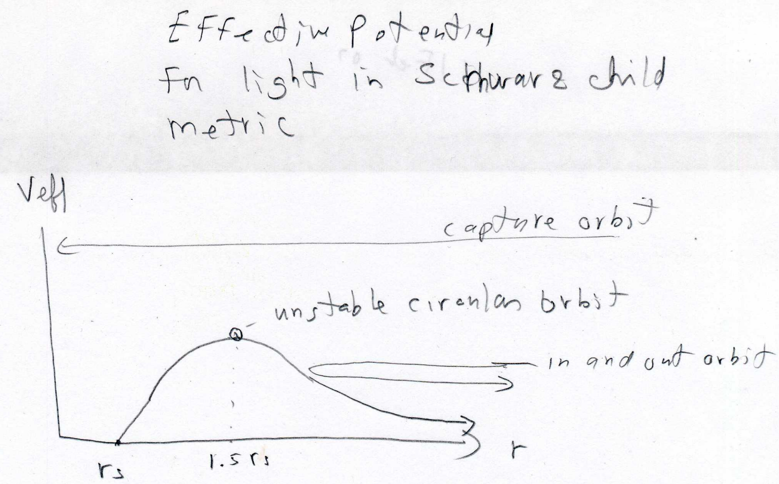

Fig. 7.— Figure for Chapter 8: Effective potential for light orbits in Schwarzschild metric

infinitely long, and perfectly straight. If you tried to measure its length with a laser you would never find the end. So by looking you would not know that you were in a circular tube! It would look perfectly straight to you!

Even weirder would be if you built a such a tube a tiny bit inside the light orbit radius

$r = 1.5r_S$. In this case if you shone your laser beam, it would curve inward towards the hole and hit the inner wall of the tube. Thus to you it would seem that the inner wall was further out than the center. The appearance would be that the entire tube was curving outward, not inward! However, if you got on your skate board and gave yourself a push, you it would not take any energy to coast around the hole, it is all at the same r. It would look like you were going uphill, but feel like you were going level! Finally, what would it be like if you built the tube just outside the light orbit radius? Check out the effective potential; light trying to circle just outside the peak in the potential (light orbit radius) will "fall" out to infinity. Thus the light would go for a way down the tube but eventually hit the ceiling. Thus it would look like the tube case curing down towards the hole (which it is). However, again a skateboard ride would feel like you are going straight, even though it looks like you are going downhill.

9. Light Bending and Gravitational Lenses

9.1. Formulas

Let's calculate the result of the most famous GR experiment. The simple formula we are going to derive for light bending is also the basis of all gravitational lensing, which has become a major industry in modern astrophysics. We start with the geodesics we found previously for light.

$$\frac{dr}{d\lambda} = \pm\sqrt{E^2 - (1 - \frac{r_S}{r})\frac{l^2}{r^2}},$$

and

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}.$$

Here we haven't really discussed what λ is, so let's get rid of it by dividing the second equation by the first: $d\phi/dr = (d\phi/d\lambda)/(dr/d\lambda)$:

$$\frac{d\phi}{dr} = \pm\frac{l}{r^2}\left(E^2 - (1 - \frac{r_S}{r})\frac{l^2}{r^2}\right)^{-1/2} = \frac{\pm 1}{r^2\sqrt{\frac{E^2}{l^2} - (1 - \frac{r_S}{r})\frac{1}{r^2}}}.$$

This is an equation for the angular deviation of the light ray ϕ, as a function of the radial coordinate, r. The good thing is that only the combination l/E appears. We can define this constant as

$$b = \frac{l}{E},$$

and call it the **impact parameter**, or distance of closest approach. This makes sense since for light we expect the angular momentum to be $l = pr\sin\phi$, and $E = p$, so $b = l/E = r\sin\phi$, which is just the normal definition of the impact parameter for something being fired at an object from far away.

This equation tells us how the angle of the light ϕ changes as we approach and then recede from a spherical mass. There are several possible trajectories, including circular, and inspiralling, but for now let's consider trajectories that start far away, come close, and then leave to far away on the other side. We would like to calculate by angle the light is bent because of its passage near to the massive object. The first step is a change of variables from r to $u = r^{-1}$. Then $r = 1/u$ and $dr = -du/u^2$. The differential equation is changed using $d\phi/du = (d\phi/dr)(dr/du)$, with $dr/du = -1/u^2$. Thus we get

$$\frac{d\phi}{du} = \pm\left(\frac{1}{b^2} - u^2 + r_S u^3\right)^{-1/2}.$$

To find the angle we integrate this starting at $r = \infty$ ($u = 0$) with $\phi = \pi$, going to $r = b$ ($u = 1/b$), and ending at $r = \infty$ ($u = 0$) with $\phi \approx 0$. We see a problem, in that the integration variable goes from 0 to 0. We solve this problem by noting that by symmetry the angle of light bending from

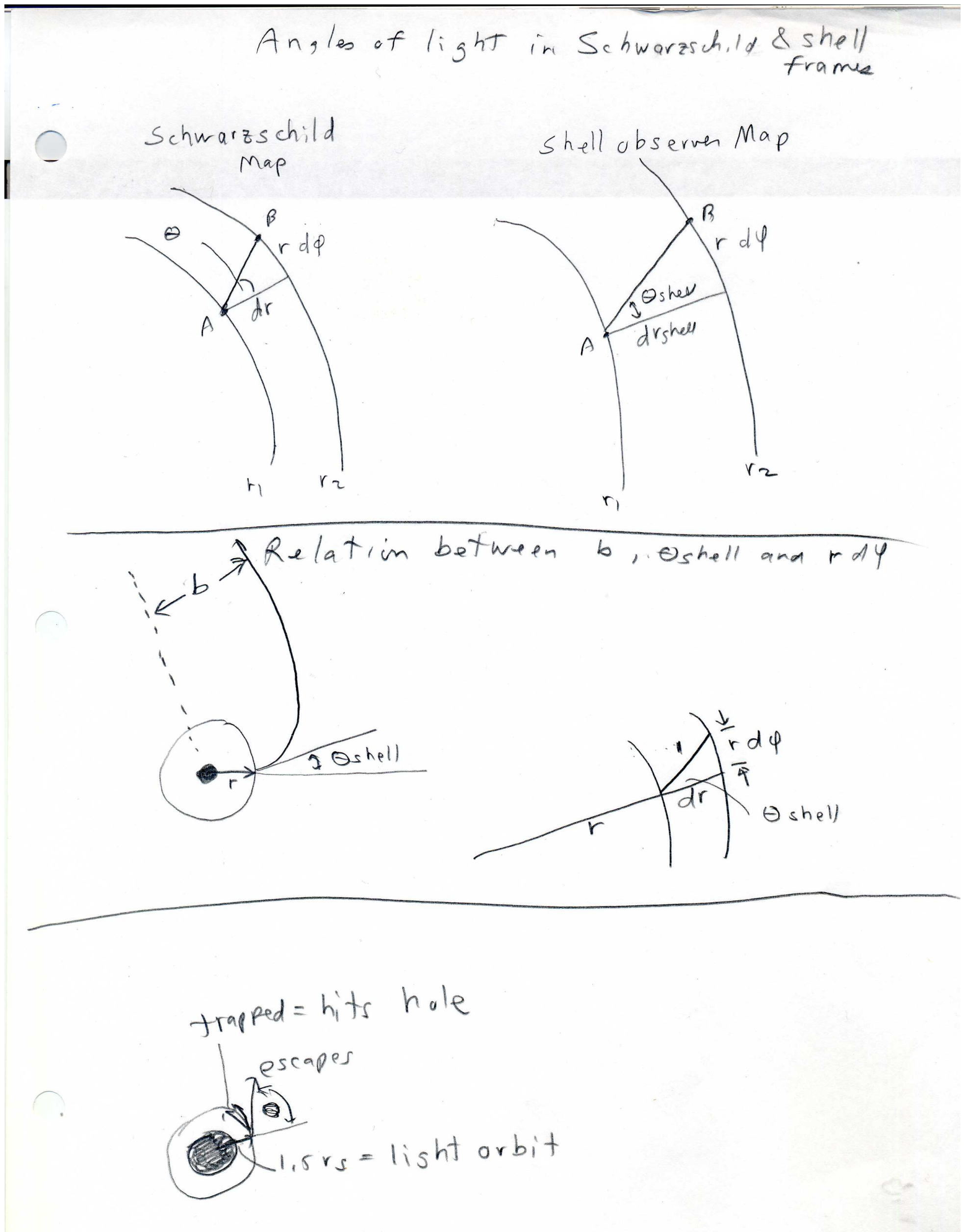

Fig. 8.— Figures for Chapter 9 and 10: (a) Definition of Impact parameter,light bending trajectory, and (b) Angles of light in Schwarzschild and shell frames

infinity to b, must be the same as the angle of light bending from b to infinity. So we can just go from $u = 0$ to $u = 1/b$, and then multiply our final answer by two. So half our angle is:

$$\phi = \int_0^{1/b} \left(\frac{1}{b^2} - u^2 + r_S u^3 \right)^{-1/2} du.$$

This is the complete answer. We can do this integral numerically for any values of b and get all the possible light orbits. However, for almost all cases, the angle of light bending is small and we can find a very important analytic formula. This formula is the one almost always used in graviational lensing. To find it we note that usually the quantity $r_S/r \ll 1$, or equivalently $r_S u^3 \ll u^2$. This will be true if the impact parameter b is much larger tha r_S which is usually true. In this case we can turn this integral into one we can do analytically. We change variables again to y, where

$$y^2 = u^2 - r_S u^3,$$

so $y = (u^2 - r_S u^3)^{1/2} = u(1 - r_S u)^{1/2} \approx u(1 - r_S u/2)$. Then $dy \approx du - r_S u du = (1 - r_S u) du$, and $du/dy \approx (1 - r_S u)^{-1} \approx 1 + r_S u \approx 1 + r_S y$. We do all this so that we can make the integration variable y so we can do the integral, but also keep the first order term that has an r_S in it. The integral becomes

$$\phi = \int_0^{1/b} \left(\frac{1}{b^2} - y^2 \right)^{-1/2} \frac{du}{dy} dy \approx \int_0^{1/b} \frac{1 + r_S y}{(\frac{1}{b^2} - y^2)^{1/2}} dy.$$

This integral can be easily evaluated or looked up in a table to give

$$\phi \approx \sin^{-1}(by)|_0^{\frac{1}{b}} - r_S \sqrt{\frac{1}{b^2} - y^2}|_0^{\frac{1}{b}},$$

or $\phi \approx \frac{\pi}{2} + \frac{r_S}{b}$. The total change in ϕ is twice this or $\pi + 2r_S/b$, for a total light bending angle of

$$\Delta\phi = \frac{2r_S}{b} = \frac{4GM}{bc^2}.$$

We can compare this answer with the experiment Eddington first did. During a total solar eclipse one can see stars near the Sun. For a light ray going near the limb of the Sun the impact parameter would be nearly the Sun's radius, $b = 7 \times 10^5$ km, and we know $r_S = 3$ km for the Sun, so $\Delta\phi = 8.5 \times 10^{-6}$ radians, or 1.75 arcsec. This is close to the value found by Eddingon, and was the first real test of GR. This formula is the basis of all gravitational lensing!

How is this test done? Thing of a bunch of stars about 1/2 degree apart on the sky. At night you carefully measure the angular distance between them. Now suppose the Sun, during the day, comes through that area. Well you can't see the stars because the Sun lights up the sky. But during a total eclipse of the Sun, the Moon blocks out the Sun and you can see the stars. Thus you can measure the effect on the starlight of passing by the Sun. How do the stars appear? They appear farther apart than when the Sun is not present, because the light bends around the Sun. We always think in our minds that light goes straight, so we extrapolate back to the wrong position when the light bends. We see from this that the 1.75 arcsec is not really right. The stars farther from the limb of the Sun bend less than 1.75 arcsec. In fact, what would a single circularly blob of light (e.g. a galaxy) look like when it is gravitationally lensed like this? The light from the part of the blob closest to the Sun will bend more than the light from the far edge of the blob. Thus the circular blob of light will be squashed radially. It will also be stretched tangentially. This is the hallmark of gravitationally lensed objects and can be used to tell whether or not something has been lensed. The limit of this is when the blob of light is directly behind a very small lens which is powerful enough. Then the circular blob of light is stretched into a ring of light around the lens. This is called the Einstein ring, and has been observed. What determines the radius of the ring? It is easily derived from the light bending formula above. It depends on r_S which depends only on the mass of the lens. Thus if we see an Einstein ring, we can know the mass of the lens, even if the lens mass itself is not shining and is invisible. This is one was to search for the dark matter. There is one complication, in that the Einstein radius also depends on the distance of the lens. We may come back to this.

9.2. Powerpoint presentation on Gravitational Lensing

Gravitational Lensing is divided into divided into two basic catagories: strong lensing when two or more images of the source appears and weak lensing, where the image is squashed and magnified, but no additional image is made. These are used for various purposes in astrophysics.

- Strong lensing of galaxies: Einstein rings
- Strong lensing of quasars, Einstein crosses, time delays, dark matter substructure
- weak lensing and dark matter
- weak lensing and the bullet cluster merger proof of dark matter
- Microlensing search for MACHOs
- Microlensing search for extra-solar planets

10. Some Questions and Puzzles Around Black Holes

10.1. How fast is an object going when it enters a black hole?

Suppose we take the case where someone starts from rest at ∞ and falls into the hole. We have already calculated this case. Of course, as viewed by the person falling, there is no speed at all. But as viewed from far away the person's time slows down and then stops as it enters the Schwarzschild radius. The calculation is done starting from our radial and time equations: $dr/d\tau = \pm\sqrt{\frac{2GM}{r}} = \pm\sqrt{\frac{r_S}{r}}$, and $dt/d\tau = (E/m)(1-\frac{r_S}{r})^{-1} = (1-\frac{r_S}{r})^{-1}$, where we used conserved energy $E = m$ which is valid starting at rest from infinity. This $dt/d\tau$ equation might seem funny since we have often seen instead $dt/d\tau = (1-\frac{r_S}{r})^{-1/2}$ coming from the metric proper time at a fixed value r, but the τ in these cases differ. One is for a stationary observer at fixed r, and the other is the proper time for someone falling into the hole from infinity. Dividing these equation we find the relation between r and t, that is the speed as seen from from away:

$$v_{\text{far away}} = \frac{dr}{dt} = -\left(1-\frac{r_S}{r}\right)\sqrt{\frac{r_S}{r}}.$$

We see again that as $r \to r_S$, $v \to 0$. The far away observer never sees the person fall in. Following Wheeler, we define the **shell frame** as the frame that is static at a given value of r, that is someone who is just hovering over the hole at certain radius. We on Earth are in a shell frame, since the Earth's surface holds us here.

How does it look from someone on a shell near the horizon at $r = r_S$?

Distances are measured in this frame by poking a meter stick in the radial direction, and thus are just the proper distances dl_r from the Schwarzschild metric:

$$dl_r = dr_{\text{shell}} = dr\left(1-\frac{r_S}{r}\right)^{-1/2},$$

where $r_S = 2GM/c^2$ as usual. How does time run in the shell frame? Well, r is not changing between clock ticks, so compared to far-away time t, the metric value of $d\tau$ is the shell frame time:

$$dt_{\text{shell}} = dt\left(1-\frac{r_S}{r}\right)^{1/2}.$$

Using the two equations above, we can relate times and distances in the shell frame to those in the far away frame (e.g. Schwarzschild coordinates r and t). We can also find the speed measured in the shell frame:

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = (1-\frac{r_S}{r})^{-1}\frac{dr}{dt}.$$

For the person falling in from far away, we put in the result for dr/dt above to find:

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = -\sqrt{\frac{r_s}{r}}.$$

This gives the remarkable result that to a shell observer, sitting at $r = r_S$ (well, just above that so they can tell us what happened!), the falling objects goes by at $v_{\rm shell} = 1$, that is the speed of light! Isn't it strange that the same object doing the same thing can be moving at 0 speed or c from different vantage points.

We have to ask now what is the energy of that falling object. From far away it is $E = m$ and this stays the same; energy is conserved along geodesics according to our Euler-Lagrange calculation. According to the shell observer, they see something going be at nearly the speed of light, that is $\gamma \to \infty$, so $E = m\gamma \to \infty$. This is actually consistent with what we said before that the redshift of something escaping from the Schwarzschild radius was infinite. That is, it would take an infinite amount of energy to get out, and the escape velocity is c. We once again that in special and general relativity, the velocity and energy of an object are frame dependent, and there is no such thing as the "real velocity" or "real energy" of an object. Also note that in General Relativity it is usually impossible to divide up the energy into kinetic and potential energy in a meaningful way. These are frame dependent also (because velocity is).

There is yet one other velocity to think about. The mixed coordinate speed $dr/d\tau$ which we used above. This is how fast the falling person sees the outside world go by: $dr/d\tau = \pm\sqrt{\frac{2GM}{r}} = \pm\sqrt{\frac{r_S}{r}}$. Wheeler calls this the **rain frame**. Here we see again the speed goes to c at the horizon. Also we see that inside the horizon the speed becomes faster than light! Is this really true? Well the formula is correct, but the interpretation of r changes inside. So there is not really motion faster than c.

10.2. What does it look like standing near or falling into a black hole?

Using our light bending formulas we can describe how things look to someone falling into a black hole, or to someone sitting on a shell at a distance r. We have to be careful what variable we use. So far we mostly discussed things from the point of view of the "far-away" or Schwarzschild coordinate frame. We also discussed the "free-fall" frame. What is the metric in the free fall frame? Minkowski of course; that's called the equivalence principle. There is one other frame we've kind of discussed which is useful. The frame we have here on Earth, sitting stationary at a fixed distance from the center.

So let's figure out what a black hole "looks" like when we are standing still above it at a fixed value of r, i.e. in the shell frame. For someone a coordinate distance r from the hole, looking toward the hole, what will they see? Let's define the angle of looking from directly overhead, that is $\theta = 0$ is looking straight away from the hole and $\theta = \pi$ is looking directly at its center. Looking straight at the center of the hole one sees only blackness, since no light can come out of it (assuming that there is no one else between you and the hole). At a certain angle one would see the "edge" of the black hole, that is a separation between the blackness of the hole and the sky around it. Now because light travels on curved geodesics, one cannot use normal Euclidian geometry to find that angle. One must use the light geodesic equation we derived last time. To find the angular size of the hole we want to find those geodesics that just barely make it into the hole, and those that just barely miss the hole. When we look out at a certain angle, we are specifying a light geodesic. We want to map these angles into the various geodesics and see which go into the hole and which don't. We found earlier that the light geodesics are specified only by the "impact parameter" $b = l/E$, and r_S, so there should be some mapping between angle of looking and b.

We can tell geodesics which go into the hole and which don't using the effective potential for light, which we found from the radial geodesic equation. Remember that for given values of E and l (actually only the combination $b = l/E$ occurs) there are light paths that come in towards the black hole, reach a point of closest approach (which we call the turnaround radius $r_{\rm ta}$), and then return to $r = \infty$. There are also geodesics that go over the hump and plunge into the black hole, and one geodesic at the light orbit radius, $r_{\rm lo} = \frac{3}{2}r_S$ that circles the black hole.

It seems clear that if the turnaround radius is outside the light orbit radius, then the light will escape the hole, and thus the critical geodesic, the light path that just goes in and therefore defines the visible "edge" of the black hole, is the one that has $r_{\rm ta} = r_{\rm lo}$, or $r_{\rm ta} = \frac{3}{2}r_S$.

We find this critical geodesic by first finding the turnaround radius from our radial geodesic equation,

$$\frac{dr}{d\lambda} = \left[E^2 - \left(1 - \frac{r_S}{r}\right)\frac{l^2}{r^2}\right]^{1/2},$$

by setting $dr/d\lambda = 0$. This gives $E^2 = (1 - r_S/r_{\rm ta})l^2/r_{\rm ta}^2$, or

$$b = \frac{r_{\rm ta}}{\sqrt{1 - r_S/r_{\rm ta}}},$$

where we used the definition $b = l/E$. We find the critical geodesic, which is specified by $b_{\rm crit}$, the critical value of b, by setting the turnaround radius to its smallest possible value, the light orbit radius. Thus we find $b_{\rm crit} = \frac{1}{2}\sqrt{27}r_S = 2.598r_S$.

Now we need a way to map this value $b_{\rm crit}$ into a critical viewing angle $\theta_{\rm crit}$. First we want to map angles in coordinate space, r and ϕ, to angles in the shell frame. Consider someone sitting still on a shell at r. Consider a light ray sent out from event A at r to event B at $r + dr$ at some angle θ.

What does this do in the shell frame? Well since the Schwarzschild metric is flat in the angular directions, know that that $dl_\phi = rd\phi$ will be the same to both the far away and shell observers. We also know from the metric and definition of proper length that $dr_{\rm shell} \equiv dl_p = dr/\sqrt{1 - r_S/r}$ is longer than dr. Thus the angles will be different. Thus locally we can write $\tan\theta_{\rm shell} = rd\phi/dr_{\rm shell}$.

Next, we can use our light bending formula, the combination of radial and ϕ geodesics we found last time, to relate these quantites to b,

$$\frac{d\phi}{dr} = \frac{\pm 1}{r^2\sqrt{\frac{1}{b^2} - (1 - \frac{r_S}{r})\frac{1}{r^2}}}.$$

Combining the above two equations we find:

$$\tan\theta_{\rm shell} = \frac{rd\phi\sqrt{1-r_s/r}}{dr} = \pm\frac{\sqrt{1-r_s/r}}{\sqrt{(r^2/b^2)+(r_s/r)-1}}.$$

This is good for any value of r and b, but we can specialize to the critical value of b to find the critical angle

$$\tan\theta_{\rm crit} = \pm\frac{\sqrt{1-r_s/r}}{\sqrt{(4/27)(r^2/r_s^2)+r_s/r-1}}.$$

We can now use these formulas to find the angular size of a black hole as we hover over it. First plug in the light orbit radius $r = \frac{3}{2}r_S$, to find $\tan\theta_{\rm crit} = \infty$, or $\theta_{\rm crit} = 90^0$. This is just as expected, since looking out at 90^0 means looking straight out over the horizon which is the light path. We know light with $\theta_{\rm shell} = 90^0$ or more will never get out, but that light sent at slightly less angle will get out. Thus below you, like an infinite black plane, you see blackness (the hole). Looking out in any direction you see yourself at a distance of $3\pi r_S$, and then another ring of you's at twice this distance, and then again another ring, and so on forever! Above you the entire sky is displayed. Even stars behind the black hole can be seen, since curved light paths bring them to you. The entire celestial sphere is projected onto 1/2 the sky.

Going further out than $r = 1.5r_S$ we will see stars in more than one-half the sky, and the black hole in less. Light launched at angle from the vertical less than the $\theta_{\rm shell}$(critical) will escape the black hole, while other light rays will be captured. You just reverse these light rays to know what the shell observer sees.

For example, using our equation above, (or integrating the light bending equation shows) that at $r = 2.5r_S$, $\tan\theta_{\rm crit} = \pm 1.356$. Now there are actually several angles that satisfy this value of $\tan\theta_{\rm crit}$, since arctan is a multivalued function and there is the $\pm$ from when we took the square root. The easiest thing is to use common sense. Naively taking the inverse tangent gives $\theta = \pm 54^0$, neither of which makes sense, since at this radius we expect the black hole to be smaller than half the sky. But adding 180^0 to -54^0 is also a solution and gives $\theta_{\rm crit} = 126^0$ which makes sense and is the correct answer. Plug in other values for yourself: for example at $r = 1.08r_S$ the angle is 40^0, while for $r = 1.001r_S$ the angle is 4.7^0. Thus as you move closer to the black hole than the light orbit radius, it seems to cover more and more of the sky, almost completely enveloping you as you approach the Schwarzschild radius. The entire sky is still visible, but mapped into a small cone directly above your head. Finally as you cross the horizon to inside the black hole that small opening closes up; at that point there is no shell observer possible and so our equations no longer apply.

11. Death by Black Hole

11.1. The final plunge

We discussed what it looks like for someone standing still in the shell frame as they approach a black hole. Next we ask, what does the observer who free falls into the black hole see. This is more complicated because you have to take into account special relativistic forward beaming and also abberation, that is the effect of the motion of the observer relative to the stars has on viewing angles. The basic result is that as one moves faster and closer to the black hole, everything the shell observer sees gets shifted forward in the direction of motion; that is, what the shell observer sees at 90^0, the free falling observer sees at a larger angle. Thus while the shell observer who approaches close to the horizon sees black everywhere except directly away from the hole, the free falling observer still sees a little more than half the sky. However, almost all the stars are shifted forward in the direction of motion and form a ring very near the edge of the black hole.

For the free falling observer, the stars are visible from inside the black hole and the final view just before death is all the stars moved into a ring at 90^0, with the black hole filling exactly half the forward sky.

This all is described in more detail in a book called "Exploring Black Holes", by Taylor and Wheeler, pages 5-24 to 5-30. Andrew Hamilton of the University of Colorado has a great web page with explanations and simulated movies of orbiting and plunging into a black hole.

11.2. How do you die when you go into a black hole?

We saw before that near a black hole the gravitational acceleration can be very large. Is that true near all black holes? If someone is in free fall, then locally there is no force of gravity and so it should be very comfortable, just floating in space. That is the equivalence principle. However, this depends on the size of the spacecraft, and of the person being small enough so that locally the metric is Minkowski. We measure the amount of deviation from perfect Minkowski space using the concept of **tidal acceleration**. For someone falling feet first into a black hole this can be defined as the difference in acceleration between their feet and their head. The fact that the Moon's attraction of the oceans differ from the side of the Earth near the Moon to the middle of the Earth gives rise to the tides; which is why this is called the tidal force (or tidal acceleration in GR).

This tidal force is what will eventually kill you. If you are falling in feet first, you can see that since your feet are closer to the hole, they will be pulled in faster than your head. You will experience this as a stretching feeling. In addition since everything falls towards the center of the hole, the angle of falling on your left side will differ slightly from the angle of fall on your right side. Thus you will be squeezed side-to-side. The net result of lengthwise stretching and sideways squashing is called **spaghettification**. Things falling into small black holes get stretched and squeezed into long skinny strands, like a spaghetti noodle!

How do we calculate the tidal acceleration? First consider the Newtonian case here on Earth. The acceleration of gravity is given by

$$g_{\rm Newton} = -\frac{GM}{r^2} = -\frac{1}{2}\frac{r_S}{r^2},$$

where we use here the Schwarzschild radius of the Earth $r_S = 3{\rm km}\ (M_{\rm Earth}/M_\odot) = 0.90$ cm! This seems to have units of meter^{-1}, so to get it into meters/second^2, we multiply by c^2. Thus $g_{\rm Earth} = -0.5(0.009/(6.37\times10^6)^2)(3\times10^8)^2 = 9.8\ {\rm m/s}^2$, as expected. Note that if you want you can just remember $r_S({\rm Earth}) = 0.90$ cm, instead of $g_{\rm Earth}$. The tidal force on Earth is the difference of this acceleration between your head and feet, which we will say are seperated by a distance $\Delta r \approx 2$ meter. For small differences we can use calculus to find $\Delta g = g(r+dr)-g(r) = (dg/dr)dr \approx a_{\rm tide}\Delta r$. Thus

$$a_{\rm tide} = \frac{dg}{dr} = \frac{r_S}{r^3},$$

a very simple formula. The difference in acceleration is then $\Delta g = a_{\rm tide}\Delta r$. For our 2 meter tall person and for the surface of the Earth, this is just $\Delta g = (0.009)(6.37\times10^6)^{-3}(2)c^2 = 3\times10^{-6}{\rm m/s}^2$. This is only about 3×10^{-7} gee's, and you can't even feel the stretching force standing here on the Earth's surface.

Now, let's do the same thing for the Schwarzschild metric. We want to use τ rather than t since we want the experience of the free-faller. We will use r for simplicity, but perhaps we should use some other measure of distance, such as the proper length. We could convert using the formulas above, but for now we only want a rough estimate. Start from the geodesic equation for free falling from infinity, $dr/d\tau = -\sqrt{r_S/r}$, and take the derivative with respect to τ, to find the bulk acceleration, $g = d^2r/d\tau^2 = -\sqrt{r_S}(-1/2)r^{-3/2}dr/d\tau = -\frac{1}{2}r_S/r^2$. Then the tidal acceleration is the derivative of this with respect to r.

$$\Delta g = \frac{r_S}{r^3}c^2\Delta r.$$

This is the same as the Newtonian formula, except we must be careful to use coordinate variable r and proper time τ.

We can now see when someone dies when they fall into a black hole. I'm not sure what acceleration between the head and feet would kill someone, but probably 100 gee's would do it. I've heard that jet pilots black out around 7 gee's. So we can ask how close to a hole before reaching that tidal acceleration. $r_{\rm kill} = (r_S\Delta r c^2/\Delta g)^{1/3} = r_S^{1/3}[(2\ {\rm meter})c^2/((100)(9.8{\rm m/s}^2))]^{1/3} = 560{\rm km}(r_S/{\rm km})^{1/3}$. The kill radius in units of the Schwarzschild radius can be found by dividing both sides of the above equation by r_S, giving:

$$\frac{r_{\rm kill}}{r_S} = 560\ {\rm km}\ (r_S/{\rm km})^{-2/3}.$$

Thus for a $3M_\odot$ black hole, the kill radius is about 1200 km, or about 130 times the horizon radius. We see that all the discussion we had before about exploration of a small black hole and coming within 30 km of the center would have to be done by robots, if at all.

Notice that in the above equation, the kill radius in units of r_S goes like an inverse power of r_S. Thus for big enough black holes the tidal field should be managable even at the horizon. For example, consider a 3 billion solar mass black hole (such things are thought to power quasars). This would have $r_S = 3(3 \times 10^9)$km $= 9 \times 10^9$km, about twice the size of the solar system (Neptune orbits 4.5×10^9km from the Sun). Then $r_{\rm kill}/r_S = 1.3 \times 10^{-4}$, or $r_{\rm kill} = 1.1 \times 10^6$km, well inside Mercury's orbit of 58×10^6km, so you are deep within the black hole before being killed. In fact, the tidal acceleration as someone enters such a black holel is only 2.2×10^{-10} gee's, a thousand time less than we experience here on Earth. Thus someone could be falling into a very large black hole and never even feel it! They would certainly be killed at the kill radius above which would happen a few hours later.

12. Where Do Metrics Come From?

We pulled our Schwarzschild metric out of a hat. But to really find a metric you have to solve Einstein's general relativistic field equations. For a simple spherical situation in vacuum like the Schwarzschild case, this is not too hard, but in general it is nearly impossible. Einstein's equations are a set of 10 coupled partial differential equations that are made even more difficult because there are some symmetries, called general coordinate invariance that have to be obeyed. There are really only a very few analytic solutions known for realistic situations.

I want to write down these equations for you so you can see what they are and why it is difficult to solve them. We are not going to try to solve them. In order to understand these equations we will have to learn a little tensor notation. Tensor notation is a way to write complicated equations in a short hand. This is similar to writing Maxwell's equations using the div and curl. Maxwell himself did not have that convienent notation and the equations he wrote were much longer and more complicated to look at.

In tensor notation Einstein's field equations are very simple:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda\eta_{\mu\nu}.$$

We need to go over the terms in this equation slowly and step by step. First remember the idea of 4-vectors, like

$$x^\mu = (t, x, y, z),$$

or

$$p^\mu = (E, p_x, p_y, p_z).$$

Here the index μ is equal to 0,1,2, or 3, with $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $p^0 = E$, $p^1 = p_x$, etc. The zeroth component is the timelike component, and the first through third are spacelike.

Note that we can have indices both above and below the letter. The indices above the letter must not be mistaken for taking a power. When index is above the letter it indicates a 4-vector, when it is below the letter it indicates a 1-form.

In differential geometry one would learn exactly what differental forms are, and how they differ from 4-vectors, but here we will skip all that. One can create a 1-form corresponding to a 4-vector using a metric. I will show that in a minute, but in flat space, the 1-forms corresponding to the 4-vectors above are:

$$x_\mu = (-t, x, y, z),$$

and

$$p_\mu = (-E, p_x, p_y, p_z).$$

The only difference in this case is that the index is lower rather than upper, and the timelike component got a minus sign. If one used the other signature metric, then the time-like component got a minus sign.

The way you create these 1-forms is using the flat space Minkowski metric of special relativity:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We wrote this earlier as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, which technically is called the **line element**. The metric is actually a rank 2 tensor, that is the 4 by 4 matrix given above. We form the line element (what we have been calling the metric) by muliplying the 4 by 4 matrix by the two copies of four vector dx^μ:

$$ds^2 = \sum_{\mu\nu} dx^\mu \eta_{\mu\nu} dx^\nu.$$

Summing first over ν we get

$$\sum_\nu \eta_{\mu\nu} dx^\nu = (-dx^0, dx^1, dx^2, dx^3) = (-dt, dx, dy, dz) = dx_\mu.$$

Then the sum over μ is

$$ds^2 = \sum_\mu dx_\mu dx^\mu = -(dx^0)^2 + (dx^1)^2 + (dx^3)^2 + (dx^3)^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

that is, the line element above that we have been calling the Minkowski metric. Note that in order to get the 1-form (lower the index), we just multiply by $\eta_{\mu\nu}$ and sum over one of the indices. Thus in general in flat space we can use $\eta_{\mu\nu}$ to lower indices:

$$v_\mu = \sum_\nu \eta_{\mu\nu} v^\nu,$$

where v^μ is any 4-vector. We can also raise indices using $\eta^{\mu\nu}$ which is the same as $\eta_{\mu\nu}$. Raising and lowering works for all tensors. As another example try working out $\sum_\mu p_\mu p^\mu = -E^2 + p_x^2 + p_y^2 + p_z^2 = -m^2$. The beauty of the notation is that one can tell what an object is just from the unsummed over indices. If there are none, then the object is a "scalar", and is an invariant like the mass or proper distance; it won't change by a boost, translation, or rotation. If there is one upper index it is a 4-vector, and will tranform like x^μ (a simple Lorentz transformation). If there is one free lower index it means means a 1-form. Two lower indices means a 2-form, etc. and the transformation properties of these are well determined as well.

Now, most people use the **Einstein summation convention** which means to just leave out all the summation symbols! Any index that is repeated is summed over. For example then one writes the equation above as

$$v_\mu = \eta_{\mu\nu} v^\nu,$$

and then $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. We have been calling ds^2 the metric because it contains the same information as $\eta_{\mu\nu}$, the Minkowski, or flat space metric. We go from flat space to full curved space in GR by changing the elements of the 4 by 4 matrix. We use the symbol $g_{\mu\nu}$ for the full curved space metric and write the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

In spherical coorindates we don't use (t, x, y, z), but instead use (t, r, θ, ϕ) as coodinates for (x_0, x_1, x_2, x_3). Thus remembering the Schwarzschild metric we have $g_{00} = -(1-r_S/r)$, $g_{11} = (1-r_S/r)^{-1}$, $g_{22} = r^2$, and $g_3 = r^2 \sin^2\theta$, with all $g_{\mu\nu} = 0$ when $\mu \neq \nu$. In curved space $g_{\mu\nu} \neq g^{\mu\nu}$, but we won't get into that.

Now let's look again at Einstein's equation:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda\eta_{\mu\nu}.$$

We see that since μ and ν can each take any value from 0 to 3, this is actually 16 equations. There are some symmetries that mean that not all 16 equations are independent: basically $G_{\mu\nu}$, $T_{\mu\nu}$, and $\eta_{\mu\nu}$ are all symmetric, meaning $T_{02} = T_{20}$, etc. which makes 6 of the equations redundant, thereby reducing the number to 10. We also see the $\eta_{\mu\nu}$ flat space metric. The constant term Λ multiplying $\eta_{\mu\nu}$ is called the **cosmological constant** and was introduced by Einstein under the false belief it would stop the Universe from expanding. We will come back to this term after looking at the term with the rank 2 $T_{\mu\nu}$ tensor. The other factors in this term are just Newton's constant, G, and the speed of light. The two-index $T_{\mu\nu}$ is called the **Stress-Energy tensor**. This is where you specify what the gravitational sources are: the masses, energies, stresses, etc. that curve spacetime. We have said that in GR, mass and energy curves spacetime, and we specify exactly what those source terms are in this tensor. Note for a black hole the source term would be a delta-function at the origin with a mass, m, while for the expanding Universe case it would be a uniform density of energy throughout the Universe. A fairly general case often considered is the perfect isotropic fluid. In this case $T_{00} = -\rho(t, r, \theta, \phi)$, $T_{ii} = p(t, r, \theta, \phi)$, and for $\mu \neq \nu$, $T_{\mu\nu} = 0$, where ρ is the energy density and p is the pressure. Note that we are using the convention that **greek indices such as μ and ν run from 0 to 3, while latin indices such as i, and j run only from 1 to 3**, that is, latin indicate the space-like dimensions.

It would take quite a bit of work to understand why this is the form of the stress energy tensor, but some idea can be had by thinking in terms of 4-vectors. Time and space are combined into a 4-vector because as one speeds up, they can be transformed in a well known way (by the Lorentz transformation). What is constant during such a "boost" is neither time nor the space but the contraction of the 4-vector, which is called the invariant interval s (or τ). Likewise it is the contraction $p_\mu p^\mu = m^2$ which is constant in the momentum 4-vector. So somehow the source of the gravitational field (curvature) must be 4-vector like. We know that mass, that is mass per unit volume, or density, causes gravity, but mass is really made of energy and momentum. Thus both energy and momentum are sources of gravity. But we need the densities, that is the mass or momentum per unit space (or time). This is because GR is a local theory described by differential equations, so it is only the amount of something at a local point (the density) that comes into the equations. The energy per unit volume is the density and is written ρ above. It can be a function of space and time, but it is an initial condition that is specified before solving the field equations. Since Energy is in the zero position in a 4-vector, the density goes into the zero-zero position of $T_{\mu\nu}$. In the momentum position we put some kind of density of momentum what ever that is. In fact, in GR it is the fluxes of momentum that go in the T_{ii} positions. The flux of momentum is another way of saying the pressure, $\tilde{p}$ (For pressure I put a tilde over the $\tilde{p}$ to distinguish it from momentum, which I just call p). This can be understood by thinking of what pressure is. Pressure is force per unit area: $\tilde{p} = F/A$, and $F = dp/dt$, is the change in momentum, so $\tilde{p} = dp/(dtdA)$, or pressure equals the flux of momentum through a given small surface element. This is a more general definition of pressure than you may be used to, but is great because it works even when the particles causing the pressure are not bouncing off walls. In the case where particles are bouncing off walls, it gives the same number as the normal defintion of pressure.

In general a component of the stress-energy tensor $T_{\mu\nu}$ is the flux of the μ component of momentum (p^μ) across a surface of constant x^ν. So again T^{00} is the flux of $p^0 = E$ across a surface of constant $x^0 = t$, that is the energy density $\rho(x, y, z)$, and T^{11} is the flux of momentum $p^1 = p_x$ across a surface of constant $x^1 = x$, or the x-momentum per time in the y-z direction, that is, the pressure.

For the isotropic fluid case (which covers many cases of interest) the stress-energy Tensor then reads:

$$T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

For non-isotropic cases, we have to specify the off-diagonal terms also. Keeping with the 4-vector idea, these involve energy and momentum fluxes in space and time. Thus these are stresses, e.g. momentum fluxes in the y-direction transfered across the x-direction, i.e. y-forces across x-surfaces. It is surprising, but inevitable that just squeezing a ball gives rise to gravitational fields!

As a little aside, we note that it is energy, not just rest mass, that goes in the stress-energy tensor, so electric and magnetic fields also count. Thus electromagetic fields and radiation give rise to curved spacetime and "gravitational fields". It is a little complicated, but for a pure electromagnetic field one can write out the stress energy tensor in terms of electric field E and magnetic field B as follows:

$$T^{\alpha\beta} = F^{\alpha}_{\gamma} F^{\beta\gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta},$$

where we raise and lower using the metric $g_{\mu\nu}$, and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Throughout we use the Einstein summation convention, so there are many sums in the above equations. In electromagnetic theory, the tensor $F^{\mu\nu}$ is called the field strength tensor, and is related to the 4-vector form of the electric and magnetic fields, A^{μ}. The above tensor notation is quite fun in that it allows Maxwells' equations to be written in a very simple form, which we don't reproduce here.

Now look again at the 3rd term in the Einstein equations. We now see that since Λ is a constant it represents a uniform stress-energy. In fact, it represents an energy density of empty space; that is, remove all matter, radiation, etc. of every sort from the space and Λ represents the energy density then. This seems like a weird concept, and when the expansion of the Universe was discovered, Einstein decided that the cosmological constant term was not needed. In fact, this term is mathematically allowed, and actually is completely acceptable in modern quantum field theory, and so should be included. It is an experimental question what the value of Λ is. If there is no energy density of the vacuum, then one can just set $\Lambda = 0$. When we write $T_{\mu\nu}$ for a black hole we typically don't include Λ and so set $\Lambda = 0$. If we didn't set it to zero then we would not get the Schwarzschild metric, but a metric that actually has two horizons in it, one near the normal Schwarzschild radius, and another at a very great distance corresponding to the eventual "edge" of the visible Universe. Maybe more on that next quarter in Phyiscs 162!

Next look at the first term in Einstein's field equations: $G_{\mu\nu}$ which is called the Einstein tensor. We to define need one more notational convention: We will write derivatives with respect to x^{μ}, as ∂_{μ}. That is $\partial_{\alpha} f(x) = \partial f/\partial x^{\alpha}$, and $\partial^{\alpha} f(x) = \partial f/\partial x_{\alpha}$, where as usual if $\alpha = 0$, this means a time derivative, and if $\alpha = 2$, is means a derivative with respect to y, etc.

Now the Einstein tensor is defined in terms of the metric and contractions of the **Reimann curvature tensor** $R^{\lambda}_{\mu\nu\kappa}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

where R is called the **Ricci scalar curvature**

$$R = g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa}.$$

Don't forget that we are using the Einstein summation convention so there are actually 256 terms in the equation for R! We raise and lower indices of the Reimann and other tensors by contracting with the metric, for example: $R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^{\sigma}_{\mu\nu\kappa}$. The **Ricci tensor** is formed by

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa}.$$

Note that the rank four Reimann tensor is a shorthand way of writing 256 numbers! It is formed from the metric and the rank three affine connection $\Gamma^{\lambda}_{\mu\nu}$ which contains 64 numbers:

$$R^{\lambda}_{\mu\nu\kappa} = \partial_{\kappa} \Gamma^{\lambda}_{\mu\nu} - \partial_{\nu} \Gamma^{\lambda}_{\mu\kappa} - \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta}.$$

The final definition is the **affine connection** also known as the **Christoffel symbol**, which involves derivatives of the metric:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\partial_{\lambda} g_{\mu\nu} - \partial_{\mu} g_{\lambda\nu} - \partial_{\nu} g_{\mu\lambda} \right).$$

Whew! So starting with the the stress energy tensor and the cosmological constant, one solves this set of coupled differential equations for the metric $g_{\mu\nu}$. That is, we need to find what sixteen quantities contained in $g_{\mu\nu}$ can be differentiated and summed over as specified above so as to satify the Field equations. Given how complicated the above formulas are you might guess that this is not an easy task! In fact it is made even more difficult because the solution of the above equations is not unique. There are certain contraints call the **Bianchi Indenties** that are simultaneously satisified

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} = 0,$$

where for any tensor V, the co-variant derivative $V^{\mu}_{;\nu}$ is defined

$$V^{\mu}_{;\nu} = \partial_\nu V^\mu + \Gamma^{\mu}_{\nu\kappa} V^\kappa.$$

The Bianchi identities are where convervation of momentum and energy are encoded into General Relativity. These identities plus the symmetric nature of the tensors, imply extra symmetries in the field equations that reduce the 16 differential equations to effectively only 6, but one must be very careful in consistently picking values of some seemingly arbitrary functions caused by these coordinate invariances. This is called choosing a gauge. The results you get may depend on the value of these arbitrary functions (the gauge you choose), and thus workers who picked different values might get different answers. Luckily, any quantity that can be actually measured in an experiment must be independent of the gauge and therefore the same in every gauge. It is important in GR to calculate things that can be measured.

One can thus see why solving these equations is not easy. For realistic conditions (realistic values of the stress-energy tensor) there are only a few known analytic solutions. Even on a computer these equations are extremely difficult and only recently has progress been made solving them for realistic situations. And when you solve them what you get is the metric. Thus, while there is some value in going through the steps of finding the analytic solution for the black hole stress-energy, in the end, to find the physics you still just start from the metric as we did in this class. Now you see why in this class we decided to skip the field equations and just pull metrics out of a hat.

Hopefully this lecture helped you see some of the full complexity of GR.

13. Inside the Black Hole: Kruskal-Szerkeres Coordinates; General Black Holes

13.1. Coordinate problem at $r = r_S$

Let's look at the point $r = r_S = 2GM/c^2$ in the Schwarzschild metric. The metric blows up here, but we found that for things falling in, there is no problem. Forces don't become large, in fact nothing too bad happens. The problem here is not the physics, it is the coordinates. There is a coordinate singularity at $r = r_S$, but no physical singularity. Besides, coordinates where t runs backward, and r and t switch roles are just confusing.

As a first example of how coordinates can be bad consider polar coordinates on a plane:

$$ds^2 = dr^2 + r^2 d\theta^2.$$

In the angular direction this says the distance is $ds_\theta = r d\theta$. But what look what happens as $r \to 0$. Then $ds/r = d\theta \to \infty$, and the metric doesn't make any sense.

Fig: Grid of coordinates on surface of sphere

As a second example of coordinate problems, consider the surface of the sphere.

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2,$$

where R is a constant (the radius of the sphere). We can plot out these coordinates on a grid ϕ vs. θ and everything looks fine. However, at $\theta = 0$, all the values of ϕ are the same point, that is ϕ is not defined. However, there is nothing special about the actual point $\theta = 0$ on a sphere. It is only the coordinates that are bad at the point $\theta = 0$. In fact it is very easy to fix this problem by just moving the origin to another point on the sphere!

The problem with the Schwarzschild metric at $r = r_S$ is similar; it is not a physical singularity, but only a coordinate singularity. Remember how light cones squeeze up and then flip sideways as one crosses r_S?

Fig: Coordinate singularity of Schwarzschild coordinates

13.2. Kruskal-Szerkeres coorindates

Knowing the problem above for the Schwarzschild coordinates, there was a long hard search for better coordinates to describe the inside and horizon of black holes. These were found in 1960 by Kruskal and Szekeres. These coordinates show the complete geometry inside the black hole and hold some surprizes. Changing coordinates is just changing variables. Here are the definitions: For $r > r_S$ (outside black hole):

$$u = \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/2r_S} \cosh\frac{ct}{2r_S},$$

$$v = \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/2r_S} \sinh \frac{ct}{2r_S},$$

while for $r < r_S$ (inside black hole):

$$u = \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/2r_S} \sinh \frac{ct}{2r_S},$$

$$v = \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/2r_S} \cosh \frac{ct}{2r_S}.$$

With this change of variable the Schwarzschild metric becomes:

$$ds^2 = \frac{4r_S^3}{r} e^{-r/r_S}(-dv^2 + du^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

It it easiest to see this by calculating du and dv in terms of dt and dr and plugging these into the equation above to recover the Schwarzschild metric. Recall the definitions of the hyperbolic sine and cosine: $\cosh x = (e^x + e^{-x})/2$, and $\sinh x = (e^x - e^{-x})/2$, with identity: $\cosh^2 x - \sinh^2 x = 1$. One sees immediately that this metric does not contain any singularity or problem at $r = r_S$. It does however have a problem at $r = 0$. This is true physical singularity. The density and curvature become infinite there, so no change of coordinates can remove that problem. This is a problem for quantum gravity or string theory.

For use later we note the following identity (valid both inside and outside the black hole):

$$u^2 - v^2 = \left(\frac{r}{r_S} - 1\right) e^{r/r_S}.$$

Let's plot out and explore this form of the Schwarzschild metric. We plot u vs. v, and try to understand how particles move in these new coordinates. Everything we did before is still valid, it is just the coordinates that are different; like switching from Cartesian to Polar coordinates on a plane.

Recall that we find the time and space coordinates by the signs of the terms in the metric. We see that θ, ϕ, and u are space-like coordinates, and that v is timelike. So we plot u on the horizontal axis and v on the vertical axis.

- First note that light travels at 45^0 everywhere in these coordinates. Remember we find the null (lightlike) geodesics by setting $ds = 0$. We can also set $d\theta = d\phi = 0$ to consider only radial moving light. The KS metric then reads: $0 = -dv^2 + du^2$, or $dv/du = \pm 1$. Thus lightcones look exactly like they do in flat Minkowski space. This is the best property of these coordinates. There is no squeezing up as one approaches the horizon, or flipping over inside the black hole.
- Regions of constant r are hyperbolas (i.e. $u^2 - v^2$ =constant). Thus particles that don't move at all in the Schwarzschild metric actually move along hyperbolas in these coordinates. This

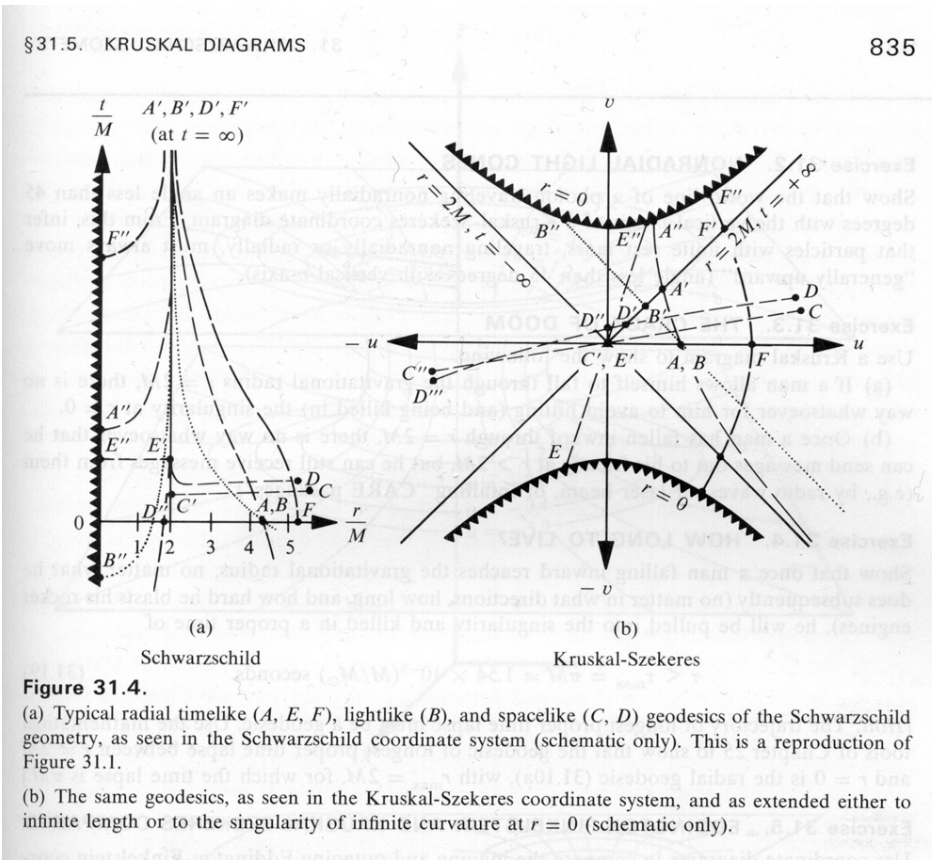

Fig. 9.— Figure for Chapter 13: Kruskal-Szekeres coordinates from Misner,Thorne, & Wheeler

can be seen from the above equation for $u^2 - v^2$ which depends only on r and not on t. So different values of constant r give different hyperbolas. The old variable t advances as you move along one of these hyperbolas. A particle at rest does not move straight up in u and v coordinates, but moves along a hyperbola.

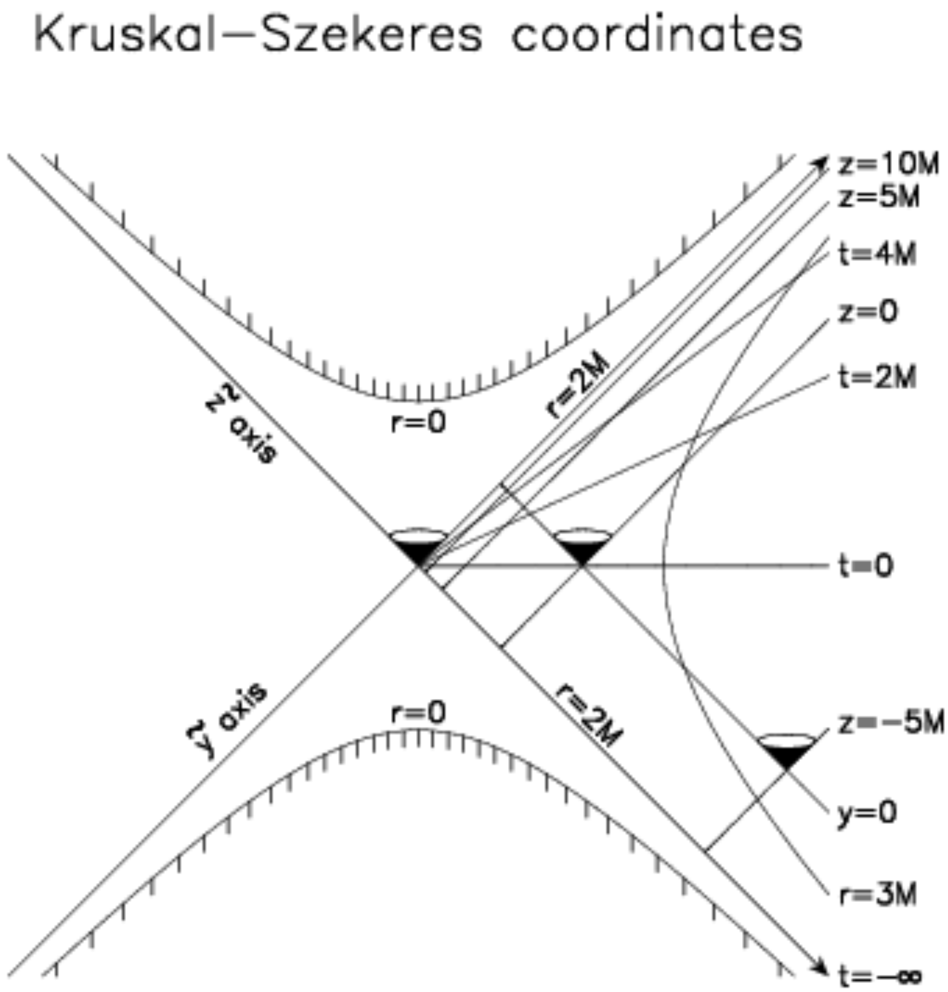

Fig. 10.— Figure for Chapter 13: Kruskal-Szekeres coordinates

- Note that the sigularity point $r = 0$ is also a hyperbola. Plug in $r = 0$ and find $u = \sinh ct/2r_S$ and $v = \cosh ct/2r_S$. Noting that $\cosh^2 x - \sinh^2 x = 1$, we see that $v^2 - u^2 = 1$, is the hyperbola of the singularity at the center of a black hole. Thus in these coordinates the singularity is spread out over u and v. In fact, there are two hyperbolas here. These coordinates show a better picture of the singularity than the Schwarzschild coordinates. We draw singularities as jagged lines; this indicates nothing can pass by and anything that touches will be destroyed. In our plot we have suppressed the theta and phi directions. Really then the singularity is a set of two surfaces! (Rotate our coordinate around the axis to represent say the phi direction.

- To find where lines of constant t are, divide the definitions of u and v. We see that $v/u = \tanh ct/2r_S$ outside the black hole and $v/u = \coth ct/2r_S$ inside. Thus v/u depends only t and not on r, so surfaces that have constant t have constant values of v/u, and actually are straight lines. We can draw these on our coordinate plot. Where these lines cross the hyperbola shows us how time evolves along the hyperbolas. We see that lines with higher slopes represent later times. Actually we knew this already because light travels up at a 45^0 angle toward the future.

- The line $t = \infty$ is the line $u = v$, that is, $\tanh(ct/2r_S) = \pm 1$ has as its solution, $t = \pm\infty$. Remembering that lightcones are still 45^0 angles one can go from a finite time with $v < u$ past $t = \infty$ to where $v > u$. However, after $t = \infty$, we see that all future lightcones will lead to the hyperbola that represents the singularity. Thus the inside of the black hole in these coordinates is the region above the diagonal lines, and one still has to hit the singularity and get killed once you are inside the black hole.

- We see that the horizon of the black hole is thus the null line that goes through the origin. These 45^0 lines divide up the entire spacetime into four regions. The outside to the right is called region I. It is where we start when we fall into the black hole.

  The region on the top is called region II. It contains the inside of the black hole. The singularity towards the future is called the black hole.

  The region to the left is called region III. It is another asymptotically flat region. This region is connected to region I by the wormhole.

  The region on the bottom is called region IV. The singularity on the bottom (towards the past) is called the white hole. Note that the future lightcones show that it is impossible to ever reach the white hole.

- Using the identity above for $u^2 - v^2$, we see that inside the hole lines of constant r are side-to-side hyperbolas like the singularity, but outside the hole lines of constant r are up-and-down hyperbolas. A shell observer outside the hole therefore moves along one of these up-and-down hyperbolas. The world lines of objects at rest are not straight up and the slope of the world lines does not give the velocity. However, moving at c is a 45^0 angle. So if you drop an object into the hole from a shell it would start to accelerate towards the hole, that is, the slope of its world line would shift towards the hole and eventually approach 45^0 as it crossed the horizon. Once inside it would eventually hit the singularity on the top.

So we see that a black hole geometry is more complicated than Schwarzschild imagined. We recognize regions I and II as the normal outside and inside of a black hole, but what are these extra regions? Region III is pretty interesting. If you go far along the -u axis you get to a place where spacetime is flat. This is called asymptotically flat which is like the far-away coordinates we are used to, except this region III is not just the same outside as region I. It seems to be a completely new area. In theory it could be a completely different Universe, or a part of this Universe far from region I. Thus black holes seem to connect different regions. This can be shown using an embedding diagram.

The area of the diagram between regions I and III is called the **Einstein-Rosen bridge, or Schwarzschild throat or Schwarzschild wormhole**. If you could go through it you could enter the black hole from region I, and come out of the black hole into region III, which could be millions of lightyears away, or even part of another Universe!

However, can you get through the wormhole from region I to region III? Try and do it. Start from anywhere in region I and stay inside the future lightcone. You see that because of the 45^0 angle of the lightcone, you can't go from region I to region III. The closest you can come is to start at the $t = -\infty$ null line, but even in that case you can't get out of region II and will hit the singularity. So Schwarzschild wormholes can't be used to get anywhere. We will talk about other wormholes later.

How about region IV, the white hole. Well, things can come out of white holes, but nothing can go in. They only exist in the past lightcones. In fact, the same theorems that say that black holes can't be destroyed also say that white holes can't be created. So unless they were formed at the beginning of the Universe, we don't expect to find any.

Finally, we note that in real black holes we don't expect regions III or IV to exist. We see this by imagining the creation of a black hole due to the collapse of a star. Remember the Schwarzschild metric is valid only outside of spherical objects. Inside objects, one would need a different solution to Einstein's equations. Thus follow the collapse of a star into a black hole. The surface of the star follows a path just like when we dropped a particle into the hole. But only outside this surface does the Schwarzschild metric hold. As the surface collapses it moves up, finally entering region II, which is when the star turns into a black hole. At that point the trapped surface forms, and all the material in the star gets crushed into the singularity. But all this time regions III and IV were not really there. So in the final black hole we find no reason to suppose that these regions exist. They are mathematical solutions to Einstein's equations, but don't exist because they were never formed.

13.3. General Black Holes

So far we've talked only about the simplest black holes: spherical ones, described by the Schwarzschild metric. There are other more general solutions to Einstein equations that are known, but still much is not known. For example, it is *conjectured* by Kip Thorne that black holes are very general, that

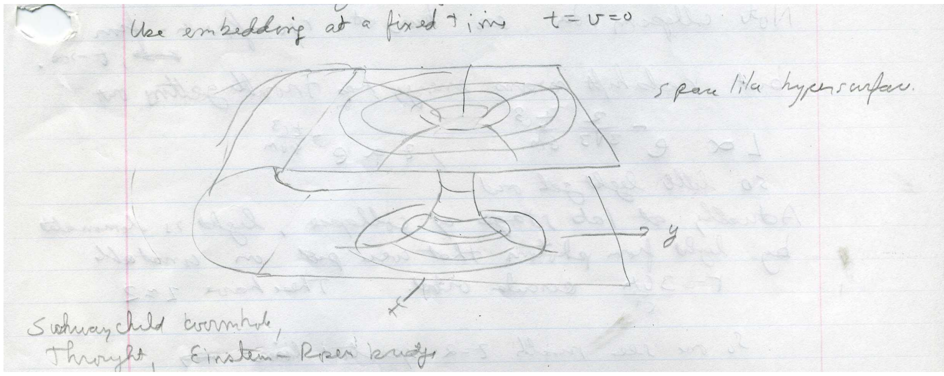

Fig. 11.— Figures for Chapter 13b: Schwarzschild wormhole

whenever a mass M is concentrated inside a region with circumference in any direction smaller than $2\pi r_S$, $(2\pi(2GM/c^2))$, then a horizon forms, i.e. that a region from which light cannot escape comes into existance. This conjecture has not been proved so is more like a physicists rule of thumb. The big problem is configurations that change with time, static situations are more well understood. I list below some general known things about black holes.

- When a black hole forms, the horizon grows from $r = 0$ outward to $r = r_S$. It is not an instaneous thing. This can be seen by following photons out of a star as it collapses. The ones near the center are the first to get trapped.

- Stationary black holes have no hair! This is the common way of stating a famous theorem about black holes: The metric of a black hole is completely specified by three things and only three things: it's mass, it's angular momentum, and its charge. Thus any information about stuff that fell into a black hole is completely lost. The Schwarzschild solution was found in 1916, and soon after Reissner (1916) and Nordstrom (1918) found the metric for a charged black hole. Kerr round the rotating black hole solution in 1963, and the charged rotating solution was found by Newman et al. in 1965. The proof of this theorem is due to Hawking (1972), Carter (1973), and Robinson (1975). This also implies that any intial quadrapole, octopole moments will be radiated away by gravitational waves and you are left with a Kerr hole or Schwarzschild hole.

- Hawking area theorem. In any process involving classical horizons, the area of the horizon cannot decrease. This implies that black holes can never bifurcate. The proof assumes that local energy density is greater than zero; quantum mechanical processes can can violate this assumption.

- Cosmic censorship: naked singularies can't exist. This is a conjecture. It says that whenever there is a singularity (place where curvature goes to infinity) there will be a horizon surrounding it. All known static solutions of GR obey this, but there have been recent claims by people doing numerical GR that there can be exceptions.

14. Rotating Black Holes: the Kerr Metric

14.1. Kerr Metric

Almost all stars rotate. Thus when massive stars collapse into black holes at the end of their lives the resulting black holes should also be rotating. Thus the Schwarzschild metric we have been using is probably not the metric of most real black holes! This was known for a long time, but not until 1963 was the metric for a rotating black hole found by Kerr. For a black hole rotating in the ϕ direction the metric in Boyer-Lindquist coordinates is:

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma}dt^2 - \frac{4aGMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2+a^2)^2 + a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2,$$

where

$$a = J/M$$

is the angular momentum per unit mass of the spinning hole,

$$\Sigma = r^2 + a^2\cos^2\theta,$$

and

$$\Delta = r^2 - 2GMr + a^2.$$

Actually the charged rotating black hole metric is nearly the same. Just replace Δ by

$$\Delta = r^2 - 2GMr + a^2 + Q^2/G,$$

where Q is the charge of the hole (in the proper units (esu)), and one has to add the c's back in to use this equation.

Let's analyze this metric some. First note what happens if we take $a = 0$. Then $\Sigma = r^2$, and $\Delta = r^2 - rr_S$, so the metric becomes

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2,$$

which is just the Schwarzschild metric.

Next, notice the cross term, $dtd\phi$. In the tensor formulation this would correspond to a non-diagonal term in $g_{\mu\nu}$. It means that there is a blending of the the coordinates ϕ and t; the forced motion we all must take in time will somehow be blended with motion in the ϕ direction. We will come back to this in a minute. Finally, note one needs caution using these coordinates since for $a \neq 0$ surfaces of constant t and constant r do not have the metric of a 2-sphere!

14.2. Horizon of Kerr metric: maximum rotation and charge of black holes

Next, how do we find the horizon, that is, the trapped surface or distance from which light can't escape? The proper way is to find the regions of the metric where all future lightcones converge.

This requires more differential geometry than we have, but earlier for the Schwazschild metric we found the horizon by when the sign of the dr term changed. Looking at the metric above, this will happen when the sign of Δ changes. Thus the horizon is at $\Delta = 0$. Solving $\Delta = r^2 - 2GMr + a^2 = 0$, we find two solutions for the horizon of a Kerr hole:

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2} = \frac{r_S}{2} \pm \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}.$$

These are called the inner and outer horizons, and the larger one r_+, turns out to be the trapped surface. The smaller one is called the **inner or Cauchy horizon**. Note that if we take $a = 0$, $r_+ = r_S$ as expected, and that for $a \neq 0$, r_+ is always smaller than the Schwarzschild radius. Thus rotating black holes are smaller! Since there is no θ dependence in this equation we see that the horizons are both spherical.

Also note that if $a^2 > (GM)^2$, then there is no solution. A more careful analysis shows that when $a^2 > (GM)^2$ a black hole (trapped surface) cannot form. Thus there is a maximum angular momentum that a black hole can have:

$$a_{max} = GM = \frac{r_S}{2}, \text{ or } J_{max} = \frac{GM^2}{c}$$

This is called the **maximal black hole**. Also note that from the brief description of the charged rotating black hole above, there is also a maximal charge, which can be found from the condition that a horizon exists.

$$\left(\frac{GM}{c^2}\right)^2 \geq G\left(\frac{Q}{c^2}\right)^2 + \left(\frac{J}{Mc}\right)^2,$$

where I put the c's back in to help with the units, each term is in units of meter^2, and cgs units must be used (with charge in esu). If this inequality is violated no horizon can form. The maximally charged black hole is therefore non-rotating, and the maximally rotating black hole is non-charged. In practice most of the Universe is not heavily charged, and for real black holes we expect the Kerr metric to obtain, and the maximally rotating black hole to be given by J_{max} above. Note for maximal black holes the horizon is a sphere of radius $r_+ = GM = r_S/2$, half the size of a non-rotating black hole of the same mass.

14.3. Singularity of Kerr metric

We would like to find the singularity in this metric, that is, the point that corresponds to $r = 0$ in the Schwarzschild metric. The proper way to do this is to calculate the curvature, as I discussed in a previous lecture, and then look for the places where it goes to infinity. The answer, which makes sense from inspection of the metric is that the singularity occurs at $\Sigma = 0$, or $r_{\rm sing}^2 + a^2 \cos^2\theta = 0$. If $a = 0$, we have $r_{\rm sing} = 0$ as we should, but if $a \neq 0$, we have the singularity only at $\theta = \pi/2$ and $r = 0$. This is peculiar, since why should it matter what θ is when $r = 0$? Actually there is a problem with the coordinates inside the black hole, which means that this is not really the right answer. Consider the region $\Sigma = 0$ in limit $M \to 0$, and $a \neq 0$. Then since there is no mass we know the metric must be equivalent to the Minkowski flat space metric; which it is in fact in spheroidal coordinates. Then $\Sigma = 0$ is just a coordinate singularity of the type we have seen before. So it takes some coordinate transformations and fairly subtle analysis to show that the actual singularity at $\Sigma = 0$ is actually a ring of radius a in the x-y plane. For maximal black holes both the horizon and $a = r_S/2$. Thus the ring singularity is just inside the horizon. If one entered the black hole at the equator one would hit the singularity immediately, but one entered from the north pole, one could all the way to the center without hitting the singularity.

14.4. The Ergosphere

Next, let's find a very interesting new property of spinning black holes. Consider a photon traveling in ϕ direction, so $ds^2 = 0$, and $d\theta = dr = 0$. The Kerr metric gives us the equation:

$$0 = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{4aGMr\sin^2\theta}{\Sigma} dt d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2.$$

Divide this equation through by dt^2, and we get a quadratic equation for $\dot{\phi} = d\phi/dt$:

$$A\dot{\phi}^2 + B\dot{\phi} + C = 0,$$

where

$$A = \frac{\left[(r^2 - a^2)^2 - a^2\Delta\sin^2\theta\right]}{\Sigma}\sin^2\theta,$$

$$B = -\frac{4GMar\sin^2\theta}{\Sigma},$$

and

$$C = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}.$$

This has solution:

$$\frac{d\phi}{dt} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Notice now that if $C = 0$, we have $\dot{\phi} = (-B \pm B)/(2A)$, which has $\dot{\phi} = 0$ as a solution. In this case we have that light itself, traveling in the ϕ direction stands still! (The other solution, $\dot{\phi} = -B/A$ is for light traveling in the same direction as the hole is spinning, while the solution $\dot{\phi} = 0$ is for light traveling opposite to the spin. If even light cannot move against the direction of the hole spin, then massive particles that must move slower than light obviously can't stay still no matter what they do. This "stationary light" solution occurs when $C = 0$, or $\Delta = a^2\sin^2\theta$, or $r_0^2 - 2GMR + a^2 - a^2\sin^2\theta = 0$, or $r_0^2 - 2GMR + a^2\cos^2\theta = 0$, or

$$r_0 = GM + \sqrt{G^2M^2 - a^2\cos^2\theta}.$$

This is an ellipoidally shaped region region that has a minimum radius of $r_0 = GM + \sqrt{G^2M^2 - a^2}$ at $\theta = 0$ (north and south poles), and a maximum radius of $r_0 = 2GM = r_S$ at $\theta = \pi/2$ (equator). The region inside this ellipsoid is called the **ergosphere**. Inside this region nothing can stand still. Everything rotates with the hole. The spinning hole actually drags the spacetime around it with it! Light itself going against the rotation direction will be carried backward around the hole. This is called "frame dragging". If you drop something straight down into a spinning black hole, it will start orbiting the hole even though there is nothing but empty space outside the hole. Actually, this frame dragging occurs even outside the ergosphere, but it is possible to overcome it by using rockets. It actually occurs not just around black holes, but around any spinning object, since the Kerr metric is the description of spacetime in the space around any spinning object. It falls off as r^{-3}. Thus if you put a very precise gyroscope in orbit around the Earth, the frame dragging will cause it to precess. This is called the Lense-Thirring effect. The size of the precession is very small, only about 0.042 arcsec per year, but there was a NASA satellite called Gravity Probe-B, that was in orbit trying to measure this effect. It seemed to have failed due to experimental errors.

We can now picture a spinning hole from the top and from the side.

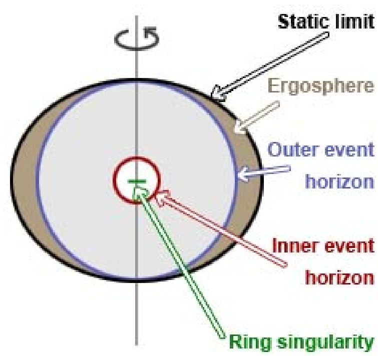

Fig. 12.— Figure for Chapter 14a: Spinning Black hole: Kerr metric: Ergosphere, and horizon

Considering a maximally spinning hole with $a = GM = r_S/2$, from the top there is the horizon at $r = GM = r_S/2$, just half the normal Schwarzschild radius. Then there is the ergosphere which from the top is a circle of radius $2GM = r_S$, but from the side goes from just touching the horizon at the north and south poles, to being out at r_S at the equator. We will discuss in a minute, why the ergosphere is called by that name.

14.5. Light orbits, inmost stable orbits for Kerr metric

In the same way we used the Euler-Langrange formalism to find the geodesics (orbit equations) for the Schwarzschild metric, we can find the geodesics for the Kerr metric. We can then go through the effective potential treatment and find the types of orbits. We won't spend the time to do this, but the result is that there are two closest stable orbits for massive particles and also two for light rays. Recall for the Schwarzschild metric the light orbit was at $r = 1.5r_S = 3GM$, and the closest stable orbit was at $r = 3r_S = 6GM$.

For the Kerr hole these orbits depend upon the value of angular momentum $a = J/M$, but for maximally rotating holes ($a = GM$) the light orbit for light rays going in the same direction as the spin is at $r = GM = r_S/2$ (**prograde light orbit**), This is the same r as the horizon and the singularity! For light rays going opposite to the spin the light orbit is at $r = 4GM = 2r_S$ (**retrograde light orbit**).

For massive particles around a maximally spinning hole, the innermost stable orbit is at $R = GM = r_S/2$ (prograde), and at $r = 9GM = 4.5r_S$ for retrograde orbits. Again the prograde (innermost stable circular orbit (ISCO) is at the same r as the horizon.

For reference, the effective potential for the Kerr metric is

$$V_{eff} = -\frac{\kappa GM}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\kappa - E^2)(1 + \frac{a^2}{r^2}) - \frac{GM}{r^3}(L - aE)^2,$$

where L is the conserved angular momentum per unit mass of the test particle (not J the angular momentum of the hole!), and E is the conserved energy per unit mass of the test particle, and where $\kappa = 0$ for light ray geodesicis and $\kappa = 1$ for massive particles.

Recall that we calculated earlier the amount of energy that could be extracted by throwing garbage into a Schwarzschild black hole. We used the geodesic equations to find that a fraction of the rest mass could be converted to useful energy by gradually bringing the garbage down to the innermost stable orbit and collecting the energy that had to be radiated away to get there. Finally the garbage goes down the hole, but we get out the difference between the energy $E = mc^2$ far away and the energy at that last stable circular orbit. For the Schwarzschild case, we found a rest energy fraction $f_{SC} = 1 - \sqrt{8/9} = 5.7\%$ could be obtained. Doing a similar calculation for a maximally spinning rotating black hole we find the maxmimal energy that can be extracted is a rest energy fraction of $f = 1 - \sqrt{1/3} = 42.2\%$ for the prograde inmost stable orbit and $f = 1 - \sqrt{25/27} = 3.7\%$ for the inmost retrograde stable orbit. Thus we have discovered an even more efficient way of producing energy. Throw garbage into a spinning black hole making sure to do it in the same direction as the hole is spinning. This is the most efficient energy producing device I have heard of.

14.6. Energy extraction from rotating black holes; ergosphere and Penrose process

A very interesting property of rotating black holes is that there exist negative energy particle trajectories. If one finds the geodesics and effective potential as described above one finds the energy to be: (Shapiro and Teukolsky equation 12.7.26)

$$E = \frac{2aGML + (L^2r^2\Delta + m^2r\Delta + r^3\dot{r}^2)^{1/2}}{r^3 + a^2r + 2GMa^2},$$

where E and L are the conserved quantities found from the Euler-Lagrange treatment of the Kerr metric. If one solves for the regions where $E < 0$, one finds that it corresponds to what we called the ergosphere. This property is why it is called the ergosphere, since *ergo* is the Greek word for energy. The ergosphere is also called the static limit, since particles cannot stay at rest inside this surface. The particles which are retrograde inside the ergosphere can have negative energy, but only particles inside the ergosphere can be scattered into geodesics with negative energies. Such negative energy particles always have trajectories that carry them into the black hole, so we can never see them. However, this gives rise to an interesting possible process called the Penrose process that allows one, in principle, to extract energy from a rotating black hole. Suppose, one shoots a massive particle inside the ergosphere, and then at a preprogrammed time it splits into to two particles, one of which has negative energy and one of which has positive energy. Penrose showed that the negative energy particle would go down the hole, but the positive energy particle could escape, carrying with it more energy that it came in with!

$$E_{in} = E_{out} + E_{down},$$

or $E_{out} = E_{in} - E_{down}$, which implies $E_{out} > E_{in}$ since $E_{down} < 0$. This does not violate energy conservation, since it turns out the negative energy particle will slow down the spinning of the hole and reduce its energy. Thus this is a way to extract energy from a spinning hole. In fact, this particular process is probably not too useful astrophysically, because in order to get it to work the relative velocity of the pieces of the splitting particle must be greater than $c/2$.

14.7. Inside the Kerr black hole

Recall that the singularity at $\Sigma = 0$ was a ring of radius a in the equatorial plane. In order to properly investigate the properties inside the Kerr black hole we need some coordinates analogous to the Kruskal-Szekeres coordinates for the Schwarzschild metric. These were found by Kerr and Schild in 1965. They proved that singularity is a ring as described above and also showed that there were other regions inside just like in the Schwarzschild case.

They found that the ring singularity has some very weird properties. As one passes through it one goes through $r = 0$ to negative values of r! And since r becomes timelike near the ring, for small values of r and θ near $\pi/2$, they found orbits in the the ϕ direction that were timelike. Since ϕ a periodic function, this means looping around in ϕ near the ring means looping around in time; time becomes periodic inside the ring! That means that closed time-like loops must exist near the singularity. This is thus a time machine!

Even weirder would be the case the case discussed above where $Q^2/G + a^2 > G^2M^2$, and there would be no solution for a horizon. This would of course violate the cosmic censorship "no naked singularity" conjecture, but this is only a conjecture at this point. So in the pure Kerr solution, there would be no horizon, but there might still be a singularity! In this case, this would seem to be a "naked singularity"; that is a singularity not protected by a horizon. Because of the

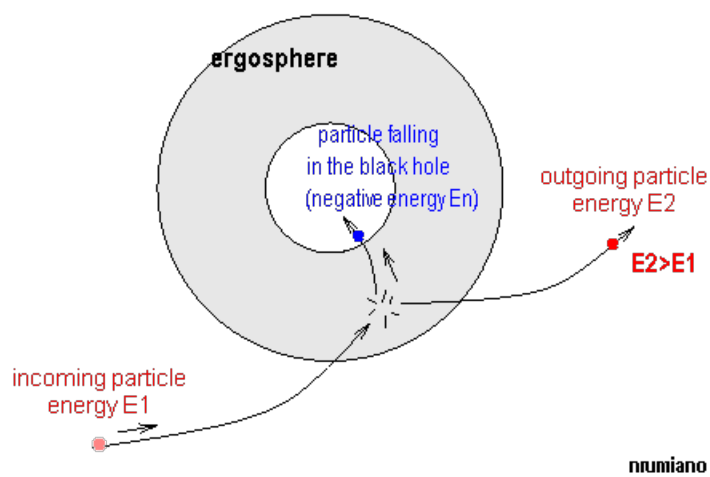

Fig. 13.— Figure for Chapter 14b: Penrose process of extracting energy from rotating black hole

closed timelike loops, one can in principle make use of the causality violation occuring near the ring singularity to go "backwards in time" by an arbitrary amount as measured by t coordinate. However, people think that in real stars this can't happen for reasons we will discuss below.

The full geometry of the Kerr black hole is more complicated and rich than the Schwarzschild geometry, so we want to simply the drawing of the Kruskal-Szekeres coordinates, by using what are called **Penrose diagrams**. Here we stretch time and space some and draw infinity as an edge. Lightcones stay the same, but the edges represent going all the out to infinity in time or space.

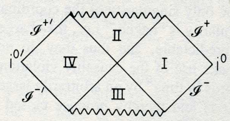

Fig. 12.3. A conformal diagram of the extended Schwarzschild spacetime (see Fig. 6.9), represented in the same manner as used in Figure 12.2. Note that since the extended Schwarzschild spacetime has two distinct asymptotically flat regions, two distinct conformal boundaries are shown.

Fig. 14.— Figure for Chapter 14c: Penrose diagram of Schwarzschild Black hole

Recall region I is where we start and region II is inside the black hole; once here you have to hit the squiggly line which represents the singularity. Region III is the white hole, and region IV is another asymptotically flat region of space time, which you can't get to without traveling faster than c.

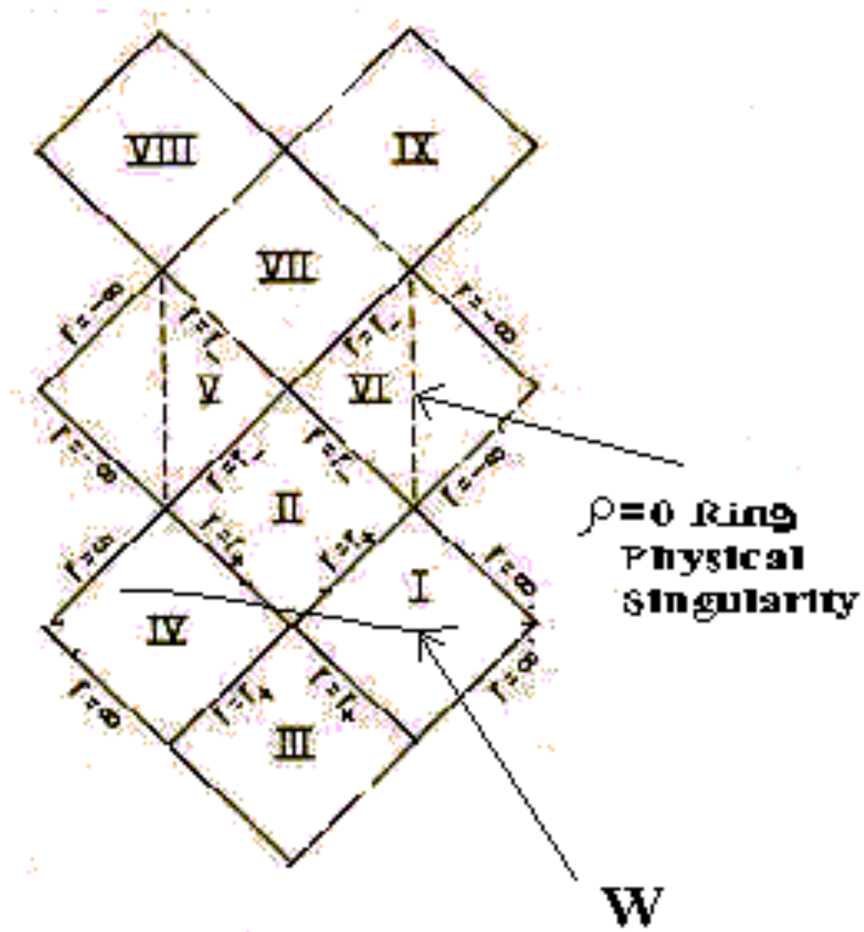

Fig. 15.— Figure for Chapter 14d: Penrose diagram of Kerr Black hole

The Penrose diagram of the Kerr geometry starts out the same as the Schwarzschild diagram, but the inner horizon and the ring singularity makes a difference. The inside of the inner horizon $(r < r_-)$ is split into two regions V and VI. Once you pass through the inner horizon you have

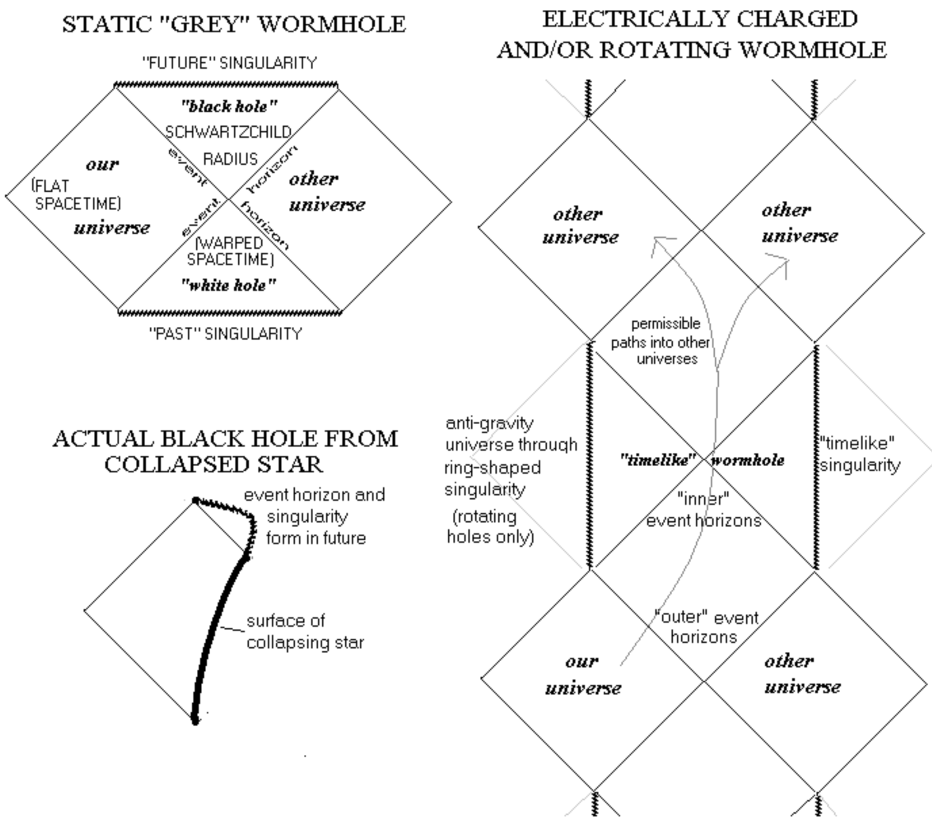

Fig. 16.— Figure for Chapter 14e: Penrose diagrams of Black holes

have a choice. You can either hit the singularity and die, or go through the ring and come out into other regions labeled V' and VI'. Regions V and VI are still inside the black hole, but regions V' or VI' are other asymptotically flat spacetimes, similar to region IV. In these new asymptotically flat regions, the ring singularity is completely "naked" and has negative mass. With respect to the original region II, the singularity of course is inside the black hole, but for someone in region V' or

VI' there is no black hole and it is possible to escape to any distance. At this point one can actually go through the ring again and get a region VII identical in structure to region III, and continue the process ad infinitum! In theory one can extend the structure downward as well through what used to be the white hole.

Note that in the Schwarzschild case one could not travel from region I to region III because the future lightcones all contained the singularity. In the Kerr metric however, this is not true! By going through the ring singularity one can reach another asymptotically flat spacetime without hitting any singularity. So is this a real wormhole? Can a rotating black hole be used as both a wormhole and a time machine? The answer is not yet proved one way or the other! The conjecture is that all the inner structure we have been discussing is unstable. An actual collapse of the spinning star is not perfectly symmetric or smooth. The "no hair" theorems say that one gets to the pure Kerr geometry only after the spacetime settles down. In particular, it is conjectured that the ring singularity is unstable and it is the inner horizon at $r = r_-$ that becomes the actual singularity. In this case, anything going into a Kerr black hole would have to be destroyed. There are several reasons for this and Wald's book has a long description on page 318. One reason is that any small perturbations (light rays or particles) entering the black hole would get an infinite redshift by the time they get to the inner horizon. These would then have to be put into the stress-energy tensor and would destroy the perfect symmetry needed to maintain the ring singularity. The same thing would be true of anyone trying to go through the singularity into another spacetime. Their mass would change the metric so much as to destroy it.

15. Can Anything Escape a Black Hole? Hawking Radiation

In 1974 Stephen Hawking startled the Physics community by proving that black holes are not black; they radiate and lose mass. We have always heard and seen proofs that nothing can get out of a black hole, but here was a proof that radiation must escape! This was not any mistake in previous work – General Relativity is a classical theory, and Hawking used quantum mechanics in his proof. Hawking was considered to be a very smart guy before this result, but this result is what made him famous.

We have considered photons classically following geodesics, but in quantum mechanics the uncertainty principle means that rays cannot be localized to arbitrary precesion. Near the horizon of a black hole this changes their behavior.

To do the Hawking calculation properly one needs to use Quantum Field Theory (QFT), the relativistically correct version of quantum mechanics, and this theory is beyond the scope of this course. So we will just describe things heuristically to give you a conceptual idea of what is going on here.

The uncertainty principle has several forms, one of which is

$$\Delta E \Delta t \geq \hbar,$$

where ΔE is the uncertainty in a particle's energy in a quantum state for a measurment lasting a time Δt. Thus on a very short time scale one can't know precisely the energy of any quantum state. This principle is preserved in Quantum Field Theory, and it means that the vacuum is filled with energy fluctuations that violate energy conservation. As long as the violation lasts for a time less than $\Delta t = \hbar/\Delta E$, everything is OK. Short times implies small distances since particles travel at speeds less than c, so on small scales energy does not have to be conserved. Only over larger times and larger distances is energy conservation valid. This is how particles tunnel under barriers that seem to not allow particles through. Thus in QFT we visualize the vacuum, that is empty space, as being a boiling caldron of particles and anti-particles being continually created and destroyed. These particles are living on borrowed energy and so have to disappear again in time to satisfy the uncertainty relation, unless they can get enough energy somehow to "go on mass shell" and survive. One can actually use this idea to calculate the probability of creating particles in particle accelerators. The particles in the accelerator transfer their energy to the **virtual particles** living on borrowed energy and allow them to become real.

Now consider such fluctuations near the horizon of a black hole. Suppose two photons are spontaneously created, one with energy E and the other with energy $-E$. The particle with $-E$ cannot propagate freely through space and would normally have to pay back its borrowed energy in a short time $\Delta t = \hbar/E$. However the particle can gain energy by falling into the black hole. If the negative energy particle falls in, then that negative energy will be forced down to the singularity and be added to the black hole. Then the positive energy photon can escape to infinity. Recall that inside the black hole forward in time means decreasing r, and that the energies of particles can be either negative or positive depending on whether t is going forward or backward. Over long times the energy has to be conserved, so if $-E$ goes down the hole, $+E$ has to escape to infinity. Another way to say this is that the virtual particle can gain enough energy to live by falling into the hole. If it was originally a negative energy photon, then the hole mass will decrease and the excess energy radiated away.

I realize this is not a very satisfactory explanation. It is not easy to explain without QFT, and even then it is tricky, but careful consideration by many physicists have convinced most everyone that this effect in fact happens. However, it has not been observed experimentally, as we shall see. Also, since all types of particles are created by quantum fluctuations near the horizon, a black hole must emit all types of particles: photons, electrons, quarks, etc.

Their are other ways to get at Hawking radiation that are more general. I like a method that considers what space time looks like in a permanently accelerating frame (like the shell frame we discussed earlier, which is the frame of us sitting still on Earth.) If one uses Rindler coordinates one can show that someone moving with a constant acceleration through empty space, actually sees a thermal bath of particles and also an event horizon. This is called the Unruh effect and the radiation is called Unruh radiation. One can calculate the temperature that is seen using this method, say in the shell frame just above the horizon of a black hole and get Hawking's answer.

In any case, Hawking showed that the radiation coming out of a black hole is a nearly perfect blackbody, characterized by the **Hawking Temperature**

$$T = \frac{\hbar c^3}{8\pi k G M} = 10^{-7}\left(\frac{M_\odot}{M}\right)\ \mathrm{K},$$

where k is Boltzmann's constant and the temperature is given in degrees Kelvin. Recall that the typical energy of a photon radiated from a blackbody is given by

$$E = kT = \frac{\hbar c^3}{8\pi G M}.$$

Also the total energy coming off of a spherical object (like a star or black hole) is proportional to its surface area A and the temperature to the fourth power,

$$L = \sigma A T^4,$$

where $A = 4\pi R^2$, and the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

We can use these formulas to find out the temperature of a black hole, and also how much energy is coming off it. But first we note above that the temperature of the smallest black hole expected to exist in nature, $(3M_\odot)$ is a few times 10^{-7} K. This is a very low temperature!

Of course the Hawking radiation causes the black hole to decrease in mass. One can ask how long will it take to reduce the mass of the black hole to zero, that is to completely evaporate a black hole? Well, currently with a cosmic microwave background radiation filling the Universe with a 2.7K radiation field, black holes are not losing any mass, but are gaining! So black holes would never evaporate in the current environment. However, the Universe is cooling down and eventually may get to below 10^{-7} K, and then black holes could start losing mass. Also if there were black holes with masses less than about the Moon's mass, then their Hawking temperature would be greater than 2.7K and they would be evaporating (as long as other stuff wasn't falling it).

Now the area of a black hole is $A = 4\pi r_S^2 = 16\pi G^2 M^2$, and the energy lost through Hawking radiation is equal to the change in the black hole mass:

$$\frac{dM}{dt} = -L_{\rm hawking} = -\sigma A T^4 = -\sigma 16\pi G^2 M^2 T^4 = -16\pi\sigma G^2 (10^{-7} M_\odot K)^4 M^{-2}.$$

We integrate this from the hole's current mass to zero mass over the black hole's lifetime $t_{\rm life}$,

$$\int_0^{t_{\rm life}} dt = -\int_M^0 \frac{M^2 dM}{B},$$

where $B = 16\pi\sigma G^2 (10^{-7} M_\odot K)^4$. Thus we find

$$t_{\rm life} = \frac{M^3}{3B} \approx 10^{10}\text{years} \left(\frac{M}{10^{12}\text{kg}}\right)^3 \approx 10^{66}\text{years} \left(\frac{M}{M_\odot}\right)^3.$$

Since the Universe is only 1.37×10^{10} years old we see that for normal black holes, their lifetimes are way longer than the age of the Universe. These holes are going to be around for a very long time.

But since the temperature goes inversely with the mass, lighter holes evaporate faster and also have less mass to evaporate. Thus their lifetimes are much shorter. We can ask how light would a black hole have to be in order to just be evaporating today?

I give you this question as a homework problem! It is a value quite a bit smaller than black holes expected to form from normal stellar processes. I think the answer comes out to be around the mass of a mountain. While no known stellar processes could form such small black holes, Stephen Hawking proposed that the Big Bang itself could create a large number of these **primordial black holes**. These then could be floating around in space. Could you detect them? The Schwarzschild radius of the Earth is only 0.88 centimeter and these would be much smaller. Their gravity would be much less than a planet or moon or even an asteroid. So could you see it? Well, look again at the temperature formula. What happens as $M \to 0$? The temperature of the black hole increases; in fact it increases without limit! If you plug in a guess at the mass of such a primordial black hole into the Hawking Temperature formula you find that they are radiating at a temperature of billions of degrees K! Thus the shine very bright! However, they are small and can't be seen very far away until the very end when they explode in a burst of very high energy gamma rays.

We can be more precise. Consider the last second of the primordial black hole's life. Plugging $\tau = 1$ second into the lifetime formula you get a mass of about 10^6 kg which (using $E = Mc^2$) is about 10^{23} Joules. Thus the power output during the last second is about 10^{23} Joules/sec which can be compared to the solar luminosity of 4×10^{26} J/s. Thus this is a few thousand times dimmer than the Sun, about the energy of a very small star. This could be seen if it was fairly near. The energy spectrum would not be that of a star but would be peaked in the gamma rays. Thus the signature of these small primordial black holes is an intense burst of gamma rays that last a very short time. When gamma ray bursts were discovered people considered this as a possible source, but the energy spectrum was wrong. For evaporating black holes we expect the energy of the emitted radiation to become more energic near the end since the temperature continually rises. The non-detection of these bursts mean that primordial black holes that have lifetimes equal to the age of the Universe can't exist. However, primordial black holes that are more massive and therefore live longer are still a possibility and could even be the dark matter. There has been much theoretical work done on how such holes could be created in the early universe and much experimental work done on trying to detect them. So far, there have been no detections, but many mass ranges have been ruled out.

Note the final fate of an evaporating black hole is still not really known. All of Hawking's calculations break down when the quantum mechanical wavelength of the particles near the size of the black hole horizon $\lambda \sim r_S$. As the black holes evaporate and get smaller we have to eventually reach this limit. At that point we would need a real theory of quantum gravity, which does not yet exist. String theory is a candidate, but so far they can't calculate anything using string theory. We can estimate when the breakdown occurs using the above condition

$$\lambda \sim r_S.$$

Using

$$E = h\nu = hc/\lambda = Mc^2,$$

we find

$$\lambda = \frac{h}{Mc},$$

and the above condition becomes

$$\frac{h}{Mc} = \frac{GM}{c^2},$$

or $M^2 = hc/G$, or

$$M_{planck} = \sqrt{hc/G} \approx 5.5 \times 10^{-5}\text{gm},$$

where we introduced the standard name of this mass: **the Planck mass**. At this mass, and at the scales corresponding to this mass, we expect quantum effects to become so important that standard General Relativity breaks down. This is the scale string theory and any theory of quantum gravity works at.

16. Entropy and Black Holes; Observing Real Black Holes

16.1. Observations of Black Holes; powerpoint slide presentation

16.2. Entropy and Black Holes

It turns out that because black holes have a temperature, they also have an entropy. In fact the entropy of the black hole is enormous even though a black hole is described by only three numbers and therefore seems to be a very simple state! Remember entropy is a measure of the disorder in a system; it is equal to k times the log of the number of quantum states of the system.

A clue to this is the Hawking area theorem: the area of the horizon of a black hole cannot decrease. That is

$$dA/dt \geq 0.$$

It turns out this is the same statement as the second law of thermodynamics, that the entropy in an isolated system cannot decrease. We won't prove this, but can get a flavor of it by fooling around with equations we already know.

Consider the area of a black hole horizon $A = 4\pi r_S^2 = 16\pi G^2 M^2/c^4$, where I put the c's back in. Taking the derivative of the above equation we get

$$dA = \frac{32\pi M G^2}{c^4} dM,$$

or

$$dM = \frac{c^4 dA}{32\pi G^2 M}.$$

The total energy of the black hole is $E = Mc^2$, so $dE = c^2 dM$, and the Hawking temperature of the black hole is $T = hc^3/(8\pi kGM)$, so we can write

$$dE = \left(\frac{hc^3}{8\pi kGM}\right)\left(\frac{kc^3 dA}{4hG}\right),$$

or

$$dE = Td\left(\frac{kAc^3}{4hG}\right).$$

This all follows simply from the Hawking temperature formula, but Hawking and Bekenstein noticed that this equation is the same as a famous equation from thermodynamics.

$$dE = TdS,$$

where S is the thermodynamic entropy of the system. Thus they identified the entropy of the black hole as

$$S = \frac{A}{4}\left(\frac{kc^3}{hG}\right).$$

Thus in this identification the entropy is proportional to the surface area of the black hole. What is totally cool about this identification, is the Hawking area theorem that says the area of a black hole can never decrease becomes the **second law of thermodynamics** which says that the entropy of a system can never decrease. This thermodynamic identification also implies that black holes should radiate as blackbodies, which they do. In fact, the surface area of a black hole does decrease as it radiates, but then one needs to take into account the entropy of the radiation. Overall the entropy of the entire system, black hole plus Hawking radiation does not decrease. The thermodynamics works out great. It is a beautiful connection that Hawking and Bekenstein and others made, and fits together perfectly.

String theorists recently had one of their few successes, when they managed to calculate the entropy of a black hole using string theory and got the Hawking/Bekenstein answer. It was a post-diction not a prediction, but it gave great encouragement to the string theory community and lead to several new and deep understandings of the relation between black holes and thermodynamics, including information loss in black holes.

The string theorists were able to count the quantum states of a black hole by finding a dual system, that is, a set of equations that described a black hole but using a specific quantum field theory rather than general relativity. The black hole was dual to a gas of hot gluon-like particles, in which the quantum states could be counted. It came out to be just the Hawking/Bekenstein answer. The spectrum of radiation coming from the gas of hot gluons could also be counted using the QFT and it was Hawking answer. One should note that this calculation was not for a normal spherical black hole, but for a certainly limiting case of a very particular model that does not exist in the real world, that is a maximally charged black hole with 4 supersymmetries), but still this was a great success for string theory and helps greatly in our understanding of what black holes really are.

Finally, there are interesting new developments that are very related to these issues. They go by the name "holographic" principle in string theory. The basic puzzle is why the entropy of a black hole is proportional to its surface area. Normally we think of entropy as an extensive quantity; if you double the volume of a box of gas, you double the entropy of that system. Here however, the entropy does not go as the volume. This is weird since you really think that the number of quantum states that a system has should depend upon its volume size, not the area of the bounding surface. The holographic idea is to generalize this principle to all systems: the number of degrees of freedom of any system should be limited by a bounding surface area, and scale as surface area, not volume. For a black hole it might make some kind of sense, since we saw everything kind of gets hung up at the horizon and never makes it in (as observed from far away). But why would this be true in general? There are several deep string theory reasons for these speculations and they are the topics of current research.

An interesting way to see some of this is consider paving the horizon of a black hole with little "Planck areas", little squares of size Planck length. How many such squares are there on the horizon of a black hole? Well, the Planck length is $l_{pl} = \sqrt{hG/c^3}$, where of course this is just an order of magnitude quantity (there could easily be factor of π or 2, etc.). The Planck area then is $A_{pl} = hG/c^3 = 2.6 \times 10^{-70}$ m^2. The number of Planck areas on the horizon is then just the ratio of the horizon area, $A = 4\pi r_S^2$ to the Planck area. $N = A/(hG/c^3) = Ac^3/(hG)$. This is remarkably close to the entropy of the black hole $S = \frac{A}{4}(kc^3/(hG))$. If one decided that each Planck area was capable of holding one bit of information (e.g. 1 or 0) and redefined the Planck size to be twice as big as above, then the number of bits that a BH horizon could hold would be $N_{bits} = \frac{A}{4}(c^3/(hG))$ and the number of states would $N_{\rm states} = 2^{N_{bits}}$. Finally using information theory to calculate the entropy of an N bit system one finds $S = k\log_2(N_{\rm states}) = \frac{A}{4}(kc^3/(hG))$, exactly the entropy of the black hole. What does this mean? I'm not sure, but it could be saying something deep about the nature of spacetime and information. It is the basic observation that gave rise to the idea of the holographic principle. We will have to see how this holographic principle plays out over the next few years.

17. Gravitational Waves

17.1. Introduction

Newton's law of gravity has a problem. Consider two masses M_1 and M_2 separated by a long distance. According to Newton, the force between them is $F = GM_1M_2/r^2$. Now suppose you move the first mass closer. Newton's law predicts that the force on the 2nd mass changes, and in fact changes instantly. But that can't happen. Things cannot communicate faster than the speed of light without violating causality. Newton's thus law violates causality. This is a problem, which luckily Einstein's theory of GR fixes. When the first mass is moved, the metric of spacetime is changed, and the solution to Einstein's field equations shows that this change in metric propagates outward at the speed of light. It is actually very similar to electricity and magnetism, where if one moves a charge it causes a change electric field that propagates outward at c. This is called electromagnetic radiation. Thus one expects that GR has something like gravitational radiation, aka gravity waves in it.

In order find these, we can proceed in the same way as you would in E&M. In E&M you write down Maxwell's equations and set all the sources, e.g. charges and currents to zero. This gives the wave equation $(\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2)\vec{E} = 0$, which has solution $E_i = A_i \exp(i\omega t - i\vec{k}\cdot\vec{x})$; that is a sine or cosine traveling wave moving in direction $\vec{k}$, with polarization A_i. Plugging this solution into the equation gives $\omega = \pm ck$. Since the speed of a wave is $v = \omega/k$, this implies electromagnetic radiation travels at the speed of $v = c$, the speed of light.

17.2. Linearized Weak Field GR

We would like to do the same thing with Einstein's field equations, but as we have seen these are substantially more complicated:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where we set the cosmological constant to zero. For gravitational radiation, we want to set all 16 terms in the stress-energy tensor $T_{\mu\nu}$ to zero, and then untangle the complicated differential equations implied in $G_{\mu\nu}$ to solve for the metric $g_{\mu\nu}$, where the line element we have been using is $ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2$. This is hard, but can be made easier if we employ the "weak field" limit. This is an example of perturbation theory, where we assume that whatever answer we get for $g_{\mu\nu}$ will be very close to the flat space Minkowski metric $\eta_{\mu\nu}$, where remember $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and all other elements of $\eta_{\mu\nu}$ are zero. Thus we start by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu} \ll 1$. We expand out the field equations, dropping any terms of order h^2 or higher. Remarkably this gives (in Lorentz gauge)

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu},$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$, is the "reverse trace weak field metric", and the trace is $h = h^{\mu}_{\mu} = h^{\mu\nu}\eta_{\mu\nu} = -h^{00} + h^{11} + h^{22} + h^{33}$.

17.3. Connection with Newton

Before using this equation for gravitational waves we can make a connection with Newton by noting that for normal slowly moving matter the only significant element of the stress-energy tensor is $T^{00} = \rho$, the mass density. Thus we can solve this equation using only the 00 component and setting $T^{11} = T^{22} = T^{33} = 0$. If we set up a mass density and look for a static solution, we can also set the time derivative to zero, giving

$$\nabla^2 \bar{h}^{00} = -16\pi G\rho.$$

This is almost exactly the equation for the Newtonian potential $\nabla^2\phi = 4\pi G\rho$, where in the Newtonian case for a spherical mass M, the gravitational potential would be $\phi = -GM/r$. Thus we can think of h as the gravitational potential with $\bar{h}^{00} = -4\phi$, $\bar{h}^{11} = \bar{h}^{22} = \bar{h}^{33} = 0$. Using the definition of the trace-reverse metric allows us to solve for the components of $h^{\mu\nu}$. For example, $\bar{h}^{00} = h^{00} - \frac{1}{2}\eta^{00}h = h^{00} + h/2$. Note trace of the tensor $\bar{h}^{\mu\nu}$ is just $\bar{h} = 4\phi$, implying $h = -4\phi$. Thus $h^{00} = \bar{h}^{00} - h/2 = -4\phi + 2\phi = -2\phi$. Likewise, $\bar{h}^{11} = h^{11} - h/2 = 0$, implying $h^{11} = -2\phi$. Likewise $h^{22} = h^{33} = -2\phi$. Plugging these values for $h^{\mu\nu}$ into the line element (metric) we get $ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)$, where we used our perturbation expansion, $g^{00} = -1 + h^{00}$, $g^{11} = 1 + h^{11}$, etc.

Finally, compare this to the Schwarzschild metric, for simplicity paying attention only to the time and radial components, $dr^2 \sim dx^2 + dy^2 + dz^2$. In the weak field limit one can Taylor expand the $(1 - 2GM/r)^{-1}$ term in the Schwarzschild metric to find to find $ds^2 \approx (-1 + 2GM/r)dt^2 + (1 + 2GM/r)dr^2 + ...$, precisely the weak field metric found above with ϕ substituted. Thus the connection between the Newtonian potential and the metric becomes more clear.

17.4. Gravitational Waves

Now set $T^{\mu\nu} = 0$ in the weak field Einstein equations. This gives

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\mu\nu} = 0,$$

again a wave equation for the 16 components of $\bar{h}^{\mu\nu}$. The solution is again sine and cosine traveling waves

$$\bar{h}^{\mu\nu} = A^{\mu\nu}\exp(i\omega t - i\vec{k}\cdot\vec{x}),$$

where now the "polarization", $A^{\mu\nu}$, of the gravity wave has more components. Again we see $\omega = \pm k$, or the speed of gravity waves is 1 (that is c). It is worth discussing the polarization since it differs from electromagnetism. Recall that an electromagnetic wave traveling in the z direction is transverse, meaning it can have an "x" component or a "y" component, but no "z" component. This is set by the gauge condition in electromagnetism. A gravity wave has more components and it's gauge condition means that it is quadrapolar. Traveling in the z direction it has two possible polarizations, one called + polarization with $A^{xx} = -A^{yy}$ and all other components zero, and the other, called $\times$, with $A^{xy} = A^{yx}$ and all other components zero (note I am calling A^{11} as A^{xx}, etc.) This polarization has a big effect on what gravity waves do.

For example, consider a gravity wave traveling in the z direction with + polarization. The spatial part of the metric will be something like: $ds^2 = (1 + h^{11})dx^2 + (1 + h^{22})dy^2 + (1 + h^{33})dz^2$, where we should substitute $h^{11} = A^{11}\sin(\omega t)$, $h^{22} = h^{yy} = A^{yy}\sin(\omega t)$, $h^{33} = A^{zz}\sin(\omega t)$, and we assume we are at $z = 0$, so we can set $\vec{k} \cdot \vec{x} = 0$. This gives: $ds^2 = (1 + h\sin(\omega t))dx^2 + (1 - h\sin(\omega t))dy^2 + dz^2 - ...dt^2$. This means that there will be no change in the z direction, and opposite sinusoidal motion in the x and y directions. What will this do?

Imagine a circular ring of test particles in the x-y plane. As the gravity wave comes through what will happen? What will the forces on the particles be? Zero of course! In GR gravity is not a force! In fact, the coordinate positions (x, y, z) will not change at all. But using the metric above we see that the distances between the particles will move! This is the effect of the gravity wave passing by. When the $\sin(\omega t)$ in the x-direction is maximum ($\omega t = \pi/2$), it will be at a minimum in the y-direction. Thus the circle of test particles will alternatively squeeze and stretch in the x and y directions. You can work out the $\times$ polarization also, it is similar except the squeezing and stretching at a 45^0 angle.

So to detect gravity waves all you have to do is measure the distances between test masses. They should move in the odd pattern just described when a gravity wave comes by. Do we expect gravity waves to exist? Yes of course. As we said at the beginning there has to be gravity waves whenever masses move, just as in electromagnetism there has to be electromagnetic waves whenever charges move. But be careful. A charge moving at uniform velocity does cause a change in the electric field to propagate outward, but does not radiate electromagnetic wave waves that carry energy. It takes an accelerating charge to create electromagnetic waves. Likewise it takes an accelerating movement of mass to create a gravitational wave. In fact, due to the quadrapolar nature of the polarization, it takes a quadrapolar motion to do it. For example, an expanding spherical shell of mass has no quadrapole moment and therefore does not emit gravitational waves. However, if two masses move back and forth (say on a spring) or orbit around each other (like two stars) then there is a quadrapole moment and there is gravitational radiation.

The formula for the strain (i.e. h, i.e. change in the metric) when two object of mass M orbit each other is roughly:

$$A \sim \frac{GMl_0^2\omega^2}{rc^4},$$

where l_0 is the rough size of the orbit, ω is the angular velocity of the orbit, and r is your distance from the system. We can make a good rule of thumb by noting that Schwarzschild radius $GM/c^2 \approx$

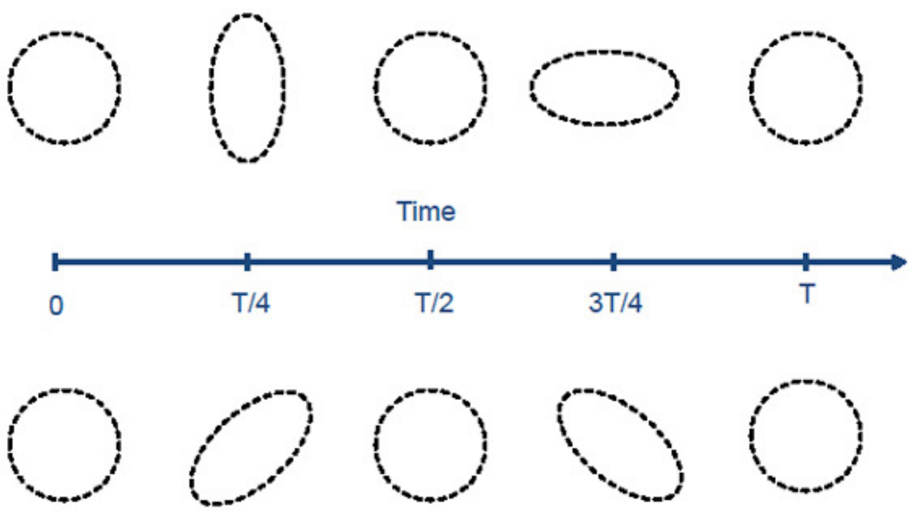

Fig. 17.— Figure for Chapter 17a: Effect of passing gravitational wave on ring of test particles

r_S, and the speed of orbit is roughly $v \approx l_0\omega$. so

$$A \approx \left(\frac{r_S}{r}\right)\left(\frac{v}{c}\right)^2 .$$

Consider then a barbell with two 100 kg masses separated by 1 m, spinning at 10 m/s. What is the size of the metric distortion caused by the resulting gravitational wave? Using $r_S = 3km(M/M_\odot)$,

$$h \sim A \sim (\frac{3000\text{m}}{1\text{m}})(\frac{100\text{kg}}{2 \times 10^{30}\text{kg}})(\frac{10m/s}{3 \times 10^8 m/s})^2 \sim 10^{-40}.$$

Recall this A is the fractional change in distance between two test particles sitting 1 m from the barbells. This is an absolutely tiny distortion, much much less than the radius of an atomic nucleus. So this is not measurable.

17.5. Detecting Gravitational Waves

Can one think of anything that would give rise to to a measurable gravitational wave? Well, in the equation above, we need to increase both the masses and speeds involved. How about two neutron stars (masses 1.4 $M_\odot$) orbiting very close to each other? In 1974, Hulse and Taylor discovered a binary pulsar, PSR 1913+16. This is system of two neutrons stars orbiting other with a period of 7.75 hours. The system is about 6.4 kpc (21,000 light years) away from us, so plugging these numbers in you find $A \sim 10^{-23}$, a much bigger value. But over one meter this is a change in distance of less than 100 millionth of the radius of a proton, unmeasurable with current equipment. Still the 1993 Nobel prize was given to Hulse and Taylor for their discovery of gravitational radiation using this system. Why?

Well the beauty of this system is that one of the neutron stars is a pulsar. It is rapidly spinning and beaming us with a period of $P = 0.059029997929613(7)$ seconds. This is an incredibly accurately measured period, and for a long time this system was more stable and accurate than any atomic clock here on earth. These pulses were modified every 7.75 hour by the Doppler effect indicating that the pulsar orbited another neutron star, and in addition a periastron advance was measured at 4.2 degrees per year (a GR effect larger than Mercury's perihelion advances in century). Using all this information almost everything about this system could be understood. It all checked out using Einstein gravity, Over time is was also possible to detect that the orbital period was slowing down at a rate of about 76.5 microseconds/year. Why does the orbital period change? Something must be bleeding energy from the system, at a rate of about 7.4×10^{24} Watts. What could this be? If one plots the orbital period over a period of years it falls in exactly the manner predicted by GR, but only if one includes the energy lost from gravitational radiation. Thus this was an indirect measurement of gravitation wave emission. To this day, this system (and others like it) are the only "detections" we have of gravitational radiation.

Is there hope to ever directly detect a gravitational wave? Well we need to be able to measure strains of around 10^{-23}. Of course if objects were closer than PSR 1913+16 or involved heavier objects (e.g. super massive black holes weighing billions of solar masses) then larger values of h might be produced. The current leading experiment is called LIGO, the Laser Interferometer Gravitational-Wave Observatory. It is two large Michelson interferometers each about 4 km long. One interferometer is in Lousiana and the other is in Washington state. Each interferometer has

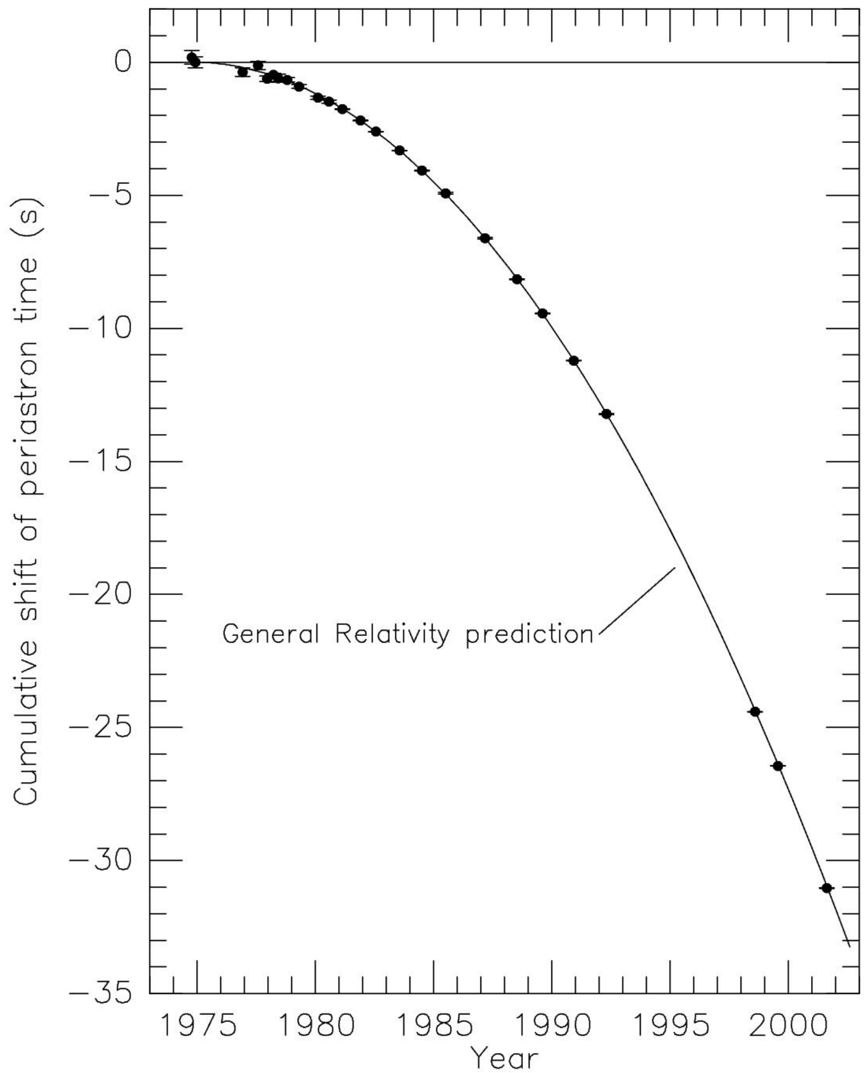

Fig. 18.— Figure for Chapter 17b: Orbital decay of PSR B1913+16.The data points indicate the observed change in the epoch of periastron with date while the parabola illustrates the theoretically expected change in epoch according to general relativity

large test masses carefully hung in vacuum pipes 4 km apart in an L shape. There are mirrors attached to the test masses and laser systems that carefully measure the distances between these test masses. If a gravity wave comes from outer space onto the detector, one arm of the interferometer will stretch while the other shrinks. This shrinking/stretching shape changing will reoccur with the frequency of the gravity wave. For instance, for the binary pulsar, the period will be 7.5 hours. However, LIGO cannot detect this kind of period; its sensitivity is only to waves with frequencies between 30 Hz and 7000 Hz. There are very few astronomical sources of strong gravity waves that emit at these frequencies, so in fact, there really is almost nothing that LIGO can detect. LIGO started in 2002 and still has not detected anything except noise. In fact, when trying to measure distances to a tiny fraction of the radius of the proton, there are innumerable sources of noise: thermal motion, ground motion, traffic, logging, ocean waves, etc, etc. etc. This is why there are two interferometers separated by about 3000 km. A gravity wave from space will hit both detectors, while noise sources will affect one but not the other. Only signals that appear in both detectors are being considered as real gravity waves. However, there is some hope for LIGO since they are upgrading it. This will make it sensitive to strains about 10 times smaller, and there are actual astronomical sources of gravity waves in the frequency band that might be detected.

Finally, there is an exciting proposed space mission called LISA, the Laser Interferometer Space Antenna, that if launched, would give huge numbers of detections. Here the interferometer arms would be around 5 million km! This long distance moves the range of frequency sensitivity down to between 10^{-5} Hz and 0.1 Hz, just in the range of binary pulsars, orbiting black holes and other likely astronomical phenomena. Here the interferometer is made of 3 free flying spacecraft orbiting the Sun (not the Earth!) and the lasers will be measuring the distances between the test masses made of pure gold and platinum to about 20 picometers, for a strain sensitivity of around 10^{-23} as required.