PHYSICS 161 Prof: Kim Griest
HOMEWORK 6 DUE: Wed 14 May, 2014

1. A robot worker is constructing a shell at radius $r = 3.5r_S$, around a $5M_\odot$ black hole. It drops a wrench, which falls to the already completed floor 10 meters below. (You may assume 10 coordinate meters below) How fast is that wrench going when it hits the floor? Do the calculation both for the shell worker frame and the Schwarzschild coordinate frame by performing the following steps.

a) Use the radial and time Schwarzschild geodesic equations to find the equation for radial infall for an object that starts at rest ($dr/d\tau = 0$) at $r = r_0$, and $\tau = 0$. [Hint: first plug in boundary conditions to find the energy E, then substitute in that energy and divide the r equation by the t equation to eliminate tau and find

$$v = dr/dt = -\left(1 - \frac{r_S}{r}\right)\left[\frac{\frac{r_S}{r} - \frac{r_S}{r_0}}{1 - \frac{r_S}{r_0}}\right]^{1/2}.$$

b) Use results from class in the above to switch from dt to dt_{shell} and from r to dr_{shell} to find:

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = -\left[\frac{\frac{r_S}{r} - \frac{r_S}{r_0}}{1 - \frac{r_S}{r_0}}\right]^{1/2}.$$

c) Evalute the above formulas for the case at hand. Be sure to put back the $c's$ and express the results for v and v_{shell} in meters/second.

d) If the wrench has mass 5 kg, how many joules of energy are released when it hits the floor? Would another worker robot hit by that wrench survive?

e) Estimate the gravitational acceleration g in the shell frame using $g_{\text{shell}} \approx (v_{\text{shell}}(r_0) - v_{\text{shell}}(r))/\Delta t_{\text{shell}}$, and estimating the time for the wrench to fall by approximating the average speed of the wrench as one-half the speed when it hits the floor.

f) Give the answer above in "gee"s, that is give your approximate acceleration in units of 9.8m/s^2. Could a person work in that environment?

The next problems are relevant to Cygnus X-1, a source of strong X-rays and one of the best black hole candidates. In these problems there is both gravitational time dilation and time dilation from relative motions ($\gamma = 1/\sqrt{(1 - v^2)}$. Think carefully which times and velocities should be used in each part.

2. Consider a clock in circular orbit in a Schwarzchild metric. Use the following method to find how much proper time elapses on the clock during one orbit?
(a) Integrate the geodesic equation from class for $d\phi/d\tau$ for one orbit to find the period.
(b) Differentiate the effective potential: $V_{eff} = (1 - 2GM/r)(m + l^2/(mr^2))$ to find the equation for circular orbits, and solve for l^2 as a function of r.
(c) Plug the result from (b) into (a) to find the answer. $\tau = (2\pi r/c)\sqrt{(2r/r_s - 3)}$, where $r_s = 2GM/c^2$.

3. This clock sends out a signal to a distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals? [Hint: you could transform the answer from the last problem, including both gravitational and relative motion time dilation but it is not easy to get it right this way; it is better to use a method from a previous homework problem; the answer is $\tau = (2\pi r/c)\sqrt{2r/r_s}$]

4. A second clock is located at rest at the same radius and next to the orbit of the first clock. (Rockets keep it there.) How much time elapses on it between successive passes of the orbiting clock? [This is yet another period! It is at rest w.r.t. clock at infinity, but moving w.r.t. orbiting clock.]

5. Evaluate the three orbital periods found above for the smallest stable orbit for a black hole like Cygnus X-1 ($M = 13M_\odot$). Give the answers in milliseconds. Which of the periods calculated above is most relevant to observers monitoring black hole candidates from a NASA satellite like RXTE (in orbit around Earth)?

6. When stuff falls into black holes, the minimal orbit period calculated above is expected to be roughly the shortest time on which one expects to see changes in the X-ray emission from the object. Why? If we found fluctuations much shorter than this period how might we try and explain it? (Just speculate; no right answer to this last question).