7.21 a) The $G(\tau)$ corresponding to this $\epsilon(\omega)$ is (eq. 7.110):

$$G(\tau) = \omega_p^2 e^{-\gamma\tau/2} \cdot \frac{\sin \nu_0 \tau}{\nu_0} \Theta(\tau)$$

where $\nu_0^2 = \omega_p^2 - \gamma^2/4$

The series expansion of the integral is:

$$\int G(\tau) E(t-\tau)\, d\tau = E(t) \int G(\tau)\, d\tau - \frac{\partial E}{\partial t} \int G(\tau) \cdot \tau\, d\tau + \frac{1}{2} \frac{\partial^2 E}{\partial t^2} \int G \cdot \tau^2 d\tau$$

where I used $\frac{\partial E}{\partial \tau} = -\frac{\partial E}{\partial t}$

These integrals are just Laplace Trans.

$$\frac{\omega_p^2}{\nu_0} \int e^{-\frac{\gamma\tau}{2}} \sin(\nu_0 \tau)\, d\tau = \omega_p^2 \frac{1}{(\gamma/2)^2 + \nu_0^2}$$

and we can use the property

$$\int e^{-st} f(t) \cdot t\, dt = \frac{\partial}{\partial s} \int e^{-st} f(t)\, dt$$

then: $$\int G(\tau)\, \tau\, d\tau = -\omega_p^2 \frac{2(\gamma/2)}{((\gamma/2)^2 + \nu_0^2)^2}$$

$$\int G(\tau)\, \tau^2 d\tau = \omega_p^2 \left[\frac{-4}{((\gamma/2)^2 + \nu_0^2)^2} + \frac{16(\gamma/2)^2}{((\gamma/2)^2 + \nu_0^2)^3} \right]$$

Then we can use $(\gamma/2)^2 + \nu_0^2 = \omega_0^2$ to obtain:

$$\boxed{\cancel{\vec{D} = \epsilon_0 \left[2\vec{E} + \frac{2\gamma}{\omega_p^2} \frac{\partial \vec{E}}{\partial t} + \left(\frac{2\gamma^2}{\omega_p^4} - \frac{2}{\omega_p^2} \right) \frac{\partial^2 E}{\partial t^2} \right]}}$$

$$\cancel{b)\ \varepsilon(\omega) = \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right]}$$

$$\boxed{\vec{D}(x,t) \simeq \varepsilon_0\left[\left(1 + \frac{\omega_p^2}{\omega_0^2}\right)\vec{E} + \frac{2\gamma\omega_p^2}{\omega_0^4}\frac{dE}{dt} + \left(\frac{2\gamma^2\omega_p^2}{\omega_0^6} - \frac{2\omega_p^2}{\omega_0^4}\right) \cdot \frac{d^2\vec{E}}{dt^2}\right]}$$

b) $$\varepsilon(\omega) = \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right]$$

$$= \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2}\left(1 + \frac{\omega^2 - i\gamma\omega}{\omega_0^2} + \frac{(\omega^2 - i\gamma\omega)^2}{2\omega_0^4} + \mathcal{O}(\omega^3)\right)\right]$$

$$\simeq \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2} - i\frac{\omega_p^2\gamma}{\omega_0^4}\omega + \left(\frac{\omega_p^2}{\omega_0^4} - \frac{\gamma^2\omega_p^2}{2\omega_0^6}\right)\omega^2 + \mathcal{O}(\omega^3)\right]$$

$$\boxed{\varepsilon\left(i\frac{\partial}{\partial t}\right) \simeq \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2} + \frac{\omega_p^2\gamma}{\omega_0^4} \cdot \frac{\partial}{\partial t} + \left(\frac{\gamma^2\omega_p^2}{2\omega_0^6} - \frac{\omega_p^2}{\omega_0^4}\right)\frac{\partial^2}{\partial t^2} + \dots\right]}$$

which apart from some factors of 2 that I've lost, or found, is the same as the integral expansion above.

7.22 a) ~~Re Ed...~~

$$\mathrm{Re}\,\varepsilon/\cancel{\varepsilon_0} = \varepsilon_0 + \frac{2\lambda}{\pi} P\int_0^\infty \frac{\omega'\cdot(\theta(\omega'-\omega_1) - \theta(\omega'-\omega_2))}{\omega'^2-\omega^2}\,d\omega'$$

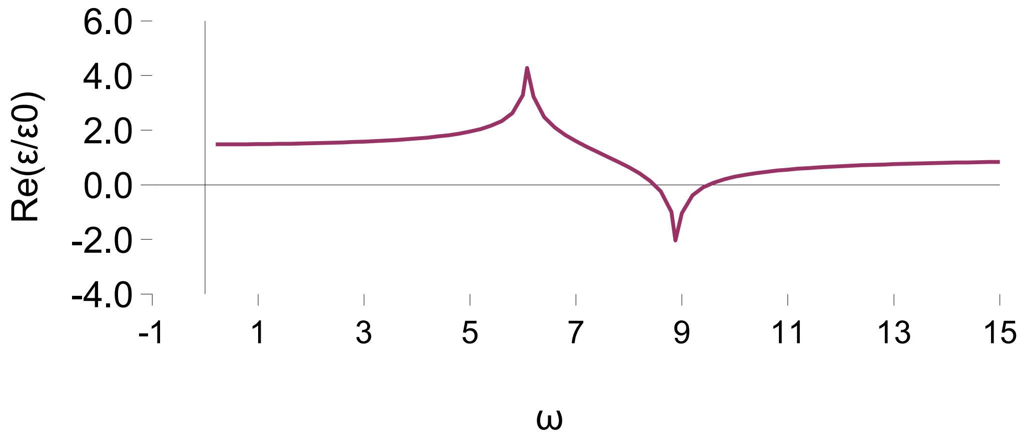
$$= \varepsilon_0 + \frac{2\lambda}{\pi} P\int_{\omega_1}^{\omega_2} \frac{\omega'}{\omega'^2-\omega^2}\,d\omega'$$

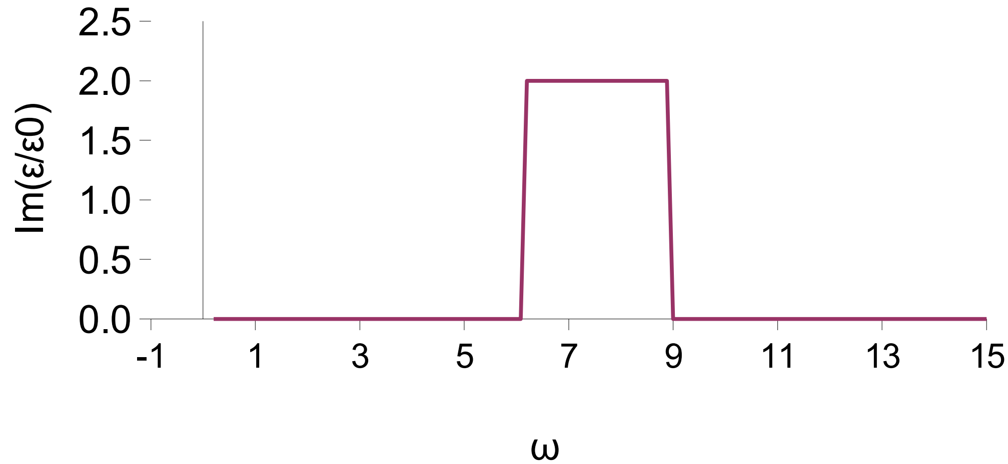
$$= \varepsilon_0 + \frac{2\lambda}{\pi}\left(\int_{\omega_1}^{\omega-\varepsilon}\frac{\omega'}{\omega'^2-\omega^2}\,d\omega' + \int_{\omega+\varepsilon}^{\omega_2}\frac{\omega'}{\omega'^2-\omega^2}\,d\omega'\right)$$

$$= \varepsilon_0 + \frac{\lambda}{\pi}\Big(\log(\omega_2^2-\omega^2) - \log((\omega+\varepsilon)^2-\omega^2)$$
$$+ \log((\omega-\varepsilon)^2-\omega^2) - \log(\omega_1^2-\omega^2)\Big)$$

$$= \boxed{\varepsilon_0 + \frac{\lambda}{\pi}\log\left(\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right)},$$ when $\varepsilon \to 0$ (for Principal Part)

b) See attached solution.





This looks quite similar to the results from the harmonic model. The difference is that the flat top of the imaginary part of the dielectric constant leads to an extended region of anomalous dispersion in the real part. Also the unrealistically steep walls of the imaginary part leads to infinite peaks in the real part.

(b) Using $\Im(\epsilon/\epsilon_0)=\dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ we have:

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty \frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}d\omega'$$

Let us use partial fraction decomposition to break up the fraction:

$$\frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}=\frac{A}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}+\frac{B}{(\omega'^2-\omega^2)}$$

$$(\omega'^2-\omega^2)A-\omega'^2+((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)B=0$$

This must be true for all ω' so we can set $\omega'=\omega$ to pick out B:

$$B = \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Plug this back in and solve for A:

$$A = \frac{\omega_0^4 - \omega'^2\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

So that finally we have:

$$\frac{\omega'^2}{((\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2)(\omega'^2 - \omega^2)} = \frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}\left[\frac{\omega_0^4 - \omega'^2\omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2} + \frac{\omega^2}{\omega'^2 - \omega^2}\right]$$

We can further decompose the first fraction in the brackets:

$$\frac{\omega_0^4 - \omega'^2\omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2} = \frac{A}{(\omega_0^2 - \omega'^2 + i\gamma\omega')} + \frac{B}{(\omega_0^2 - \omega'^2 - i\gamma\omega')}$$

Multiply out, choose $B = \omega_0^2 - A$, and solve for A and B. This leads to the expansion:

$$\frac{\omega_0^4 - \omega'^2\omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2} = \left[\frac{\omega_0^2}{2} - \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2 - \omega'^2 + i\gamma\omega')} + \left[\frac{\omega_0^2}{2} + \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2 - \omega'^2 - i\gamma\omega')}$$

Plugging this back in, the whole integrand in completely expanded form becomes:

$$\frac{\omega'^2}{((\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2)(\omega'^2 - \omega^2)} = \frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}$$
$$\times\left[\left[\frac{\omega_0^2}{2} - \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2 - \omega'^2 + i\gamma\omega')} + \left[\frac{\omega_0^2}{2} + \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2 - \omega'^2 - i\gamma\omega')} + \frac{\omega^2}{\omega'^2 - \omega^2}\right]$$

This may seem like a lot of work that just leads to a less useful form, but we must do this to solve the integral analytically.

$$\Re(\epsilon(\omega)/\epsilon_0) = 1 + \frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}[-\frac{\omega_0^2}{2}P\int_0^\infty \frac{1}{(\omega'^2 - \omega_0^2 - i\gamma\omega')}d\omega'$$
$$+\frac{(\omega_0^2 - \omega^2)}{2i\gamma}P\int_0^\infty \frac{\omega'}{\omega'^2 - \omega_0^2 - i\gamma\omega'}d\omega' - \frac{\omega_0^2}{2}P\int_0^\infty \frac{1}{\omega'^2 - \omega_0^2 + i\gamma\omega'}d\omega'$$
$$-\frac{(\omega_0^2 - \omega^2)}{2i\gamma}P\int_0^\infty \frac{\omega'}{\omega'^2 - \omega_0^2 + i\gamma\omega'}d\omega' + \omega^2 P\int_0^\infty \frac{1}{\omega'^2 - \omega^2}d\omega']$$

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\Bigg[-\frac{\omega_0^2}{2}\left[\frac{1}{2\sqrt{\omega_0^2+i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2+i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2+i\gamma\omega'}}\right)+\frac{1}{2\sqrt{\omega_0^2-i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2-i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2-i\gamma\omega'}}\right)\right]_0^\infty + \frac{(\omega_0^2-\omega^2)}{2i\gamma}\frac{1}{2}\left[\ln\left(\omega'^2-\omega_0^2-i\gamma\omega'\right)-\ln\left(\omega'^2-\omega_0^2+i\gamma\omega'\right)\right]_0^\infty\Bigg]$$

Now we must be careful and expand out the logarithm of a complex number according to:

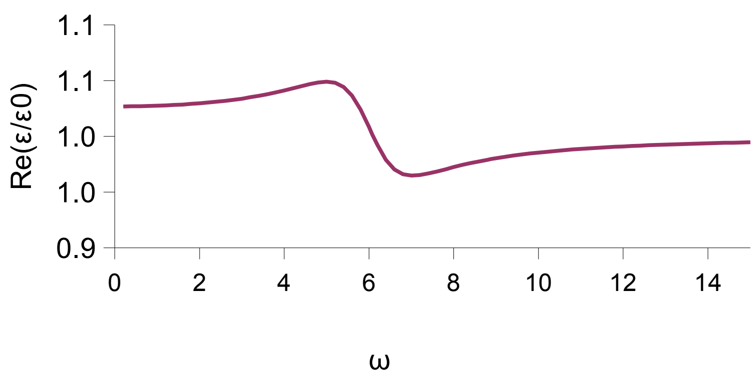
$$\ln(z)=\ln(|z|)+i\,Arg(z)$$

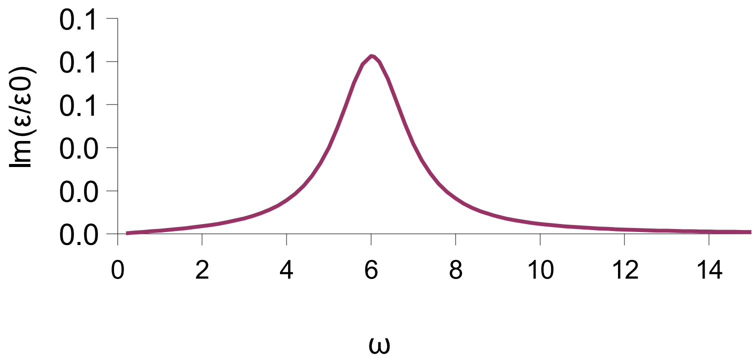
while evaluating case by case to make sure the answer ends up in the right quadrant of the complex plane. Several integrals drop out.

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\frac{(\omega_0^2-\omega^2)}{2i\gamma}\frac{1}{2}\left[i\left(\pi+\tan^{-1}\left(\frac{-\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right)-i\left(-\pi+\tan^{-1}\left(\frac{\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right)\right]_0^\infty$$

$$\boxed{\Re(\epsilon(\omega)/\epsilon_0)=1+\lambda\frac{(\omega_0^2-\omega^2)}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}}$$

This is just the result you get if you use the harmonic model with one resonant frequency and split up the dielectric constant into real and imaginary parts, where $\lambda=(Ne^2f_0)/(\epsilon_0 m)$





7.23 If we rewind the derivation to 7.116, we can see that adding a conductance simply adds a pole at $\omega = 0$, which appears as an additional $\delta(\omega)$ in 7.117.

In other words, if $\varepsilon(\omega) = \tilde{\varepsilon}(\omega) + i\frac{\sigma}{\omega}$, then

$$\int \frac{\varepsilon(\omega) - \varepsilon_0}{\omega' - \omega - i\delta} = P\int \frac{\tilde{\varepsilon} - \varepsilon_0}{\omega' - \omega} d\omega' + \int (\tilde{\varepsilon} - \varepsilon_0)\,\delta(\omega' - \omega)\, d\omega'$$

$$+ P\int \frac{i\sigma\, d\omega'}{\omega(\omega' - \omega)}$$

$$+ i\sigma\left[\int \left(\frac{\delta(\omega')}{\omega' - \omega} + \frac{\delta(\omega' - \omega)}{\omega'}\right) d\omega'\right]$$

$$= P\int \frac{\varepsilon(\omega') - \varepsilon_0}{\omega' - \omega} d\omega' + \int (\varepsilon(\omega') - \varepsilon_0)\,\delta(\omega' - \omega)\, d\omega'$$

$$+ \int \frac{i\sigma\,\delta(\omega')}{\omega' - \omega} d\omega'$$

$$= P\int \frac{\varepsilon(\omega') - \varepsilon_0}{\omega' - \omega} d\omega + \varepsilon(\omega') - \varepsilon_0 - i\frac{\sigma}{\omega}$$

Plugging this into 7.116 just gives

$$\varepsilon(\omega)/\varepsilon_0 = 1 + \frac{1}{i\pi} P\int \frac{\left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right]}{\omega' - \omega} d\omega - i\frac{\sigma}{\omega}$$

The Real part of the equation is unchanged and the imaginary part will have an extra $-\frac{\sigma}{\omega}$ on the RHS.

7.26 a) $\rho(x,t) = Ze\,\delta^3(\vec{x}-\vec{v}t)$

$$\tilde{\rho}(\vec{k},\omega) = \frac{Ze}{(2\pi)^4}\int d^3x\,dt\; e^{i(\vec{k}\cdot\vec{x}-\omega t)}\;\delta^3(\vec{x}-\vec{v}t)$$

$$= \frac{Ze}{(2\pi)^4}\int dt\; e^{i(\vec{k}\cdot\vec{v}-\omega)t}$$

$$= \frac{Ze}{(2\pi)^3}\,\delta(\vec{k}\cdot\vec{v}-\omega)$$

b) $\nabla\cdot\vec{D}(x,\omega) = \rho(x,\omega)$

$$\Longrightarrow\; +ik\,\tilde{D}(k,\omega) = \tilde{\rho}(k,\omega)$$

$$\tilde{D} = \varepsilon(\omega)\tilde{E}$$

$$\vec{E} = -\vec{\nabla}\varphi$$

$$\Rightarrow\; \vec{E} = -ik\,\varphi$$

$$\longrightarrow\; i\vec{k}\cdot\left(\varepsilon(\omega)\cdot(-ik)\tilde{\varphi}\right) = \tilde{\rho}(\vec{k},\vec{\omega})$$

$$\tilde{\varphi} = \frac{\tilde{\rho}}{k^2\varepsilon(\omega)}$$

c) Next page

7.26 c) ~~Ex~~ Substitute F-transforms:

$$\frac{dW}{dt} = \int d^3x \int d^3k\, d^3k_2\, d\omega\, d\omega_2 \cdot \sigma(k,\omega)\vec{E}(k,\omega)\cdot\vec{E}(k_2,\omega_2) \cdot \exp\left[i(\omega+\omega_2)t + i(\vec{k}+\vec{k}_2)\cdot x\right].$$

take d^3x integral and dk_2, over delta-fcn:

$$\frac{dW}{dt} = (2\pi)^3\int d^3k\, d\omega\, d\omega_2\; \sigma(k,\omega)\vec{E}(k,\omega)\cdot E(-k,\omega_2)\, e^{i(\omega+\omega_2)t}$$

Then use $\vec{E} = -i\vec{k}\varphi = -i\vec{k}\frac{\varrho}{\varepsilon k^2}$

$$= -i\frac{ze\,\hat{n}}{(2\pi)^3 k\,\varepsilon(\omega)}\delta(\vec{k}\cdot\vec{v}-\omega)$$

$$\frac{dW}{dt} = \frac{z^2e^2}{(2\pi)^3}\int d^3k\, d\omega\, d\omega_2\; \frac{\sigma(\omega,k)\cdot\delta(\vec{k}\cdot\vec{v}-\omega)\,\delta(\vec{k}\cdot\vec{v}-\omega_2)}{-k^2\,\varepsilon^*(\omega)\,\varepsilon(\omega_2)}$$

We can take the integral over ω_2, then swap $k\cdot v \leftrightarrow \omega$ interchangeably thanks to $\delta(k\cdot v-\omega)$

$$\frac{dW}{dt} = \frac{z^2e^2}{(2\pi)^3}\int d^3k\, d\omega\; \frac{\sigma(\omega,k)\,\delta(k\cdot v-\omega)}{-k^2\,\varepsilon(\omega)\,\varepsilon(-\omega)}$$

$$\varepsilon(\omega) = \varepsilon_0 + i\frac{\sigma}{\omega}$$

$$\frac{1}{\varepsilon(\omega)} = \frac{1}{\varepsilon_0+i\sigma/\omega} = \frac{\varepsilon_0 - i\sigma/\omega}{\varepsilon_0^2+\sigma^2/\omega^2} \Rightarrow \mathrm{Im}\left[\frac{1}{\varepsilon(\omega)}\right] = \frac{-\sigma}{|\varepsilon|^2\cdot\omega}$$

$$\frac{dW}{dt} = -\frac{z^2e^2}{(2\pi)^3}2\int\frac{d^3k}{k^2}\int_0^\infty d\omega\;\mathrm{Im}\left[\frac{1}{\varepsilon(\omega)}\right]\cdot\omega\,\delta(k\cdot v-\omega)$$

7.28 a) For circular polarization E_x, E_y must differ by phase of $\pi/2$ (see 7.20)

Assuming only wave-behavior in $\hat{z}$, we can write $\vec{E}$ as:

$$\vec{E} = \left[E_0(x,y)(\hat{x} \pm i\hat{y}) + f(x,y)\hat{z} \right] e^{ikz-\omega t}$$

To determine f, enforce $\nabla \cdot E = 0$:

$$\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} + f \cdot (ik) = 0$$

$$\rightarrow \boxed{f(x,y) = \frac{1}{k}\left(i \frac{\partial E_0}{\partial x} \mp \frac{\partial E_0}{\partial y} \right)}$$

b) Because the wave varies slowly in $\hat{x}, \hat{y}$ $\frac{1}{k}\frac{\partial E}{\partial x,y}$ are small numbers

so

$$-i\omega \vec{B} = \nabla \times \vec{E}$$

$$\vec{B} = \frac{i}{\omega} \cdot \left[\nabla \times \left(E_0 (\hat{x} \pm i\hat{y}) e^{ikz-\omega t} \right) \right]$$

$$\vec{B} = \frac{i}{\omega}\Big[-ik \cdot (\pm i) E_0 \hat{x} + (ik) E_0 \hat{y} + (\pm i)\frac{\partial E_0}{\partial x}\hat{z} - \frac{\partial E_0}{\partial y}\hat{z} \Big]$$

$$= \pm i \frac{k}{\omega}\left[E_0 \hat{x} \pm ik E_0 \hat{y} + \left(i \frac{\partial E_0}{\partial x} \mp \frac{\partial E_0}{\partial y} \right) \hat{z} \right]$$

$$= \boxed{\pm i \sqrt{\mu\varepsilon}\, \vec{E}}$$

7.30 This problem will use the following facts:

1) $\int d^3k\, \tilde{f}(k) = \int d^3x\, f(x)$. ~~i.e. the integrals of~~ a

actually, might not be used ↗ i.e the integral of a fcn over all space is the same as F-trans integrated over all k-space.

2) $\mathcal{F}\left[\frac{1}{k}\right] = \frac{4\pi}{r^2}$

$$\int e^{i k\cdot r - \varepsilon k} \frac{1}{k}\, dk = 2\pi \int k \exp(ikr\cos\theta - \varepsilon k)\, dk\, d\cos\theta)$$

$$= 4\pi \int_0^\infty dk \cdot \frac{1}{r} \exp(-\varepsilon k) \sin(kr)$$

$$= \frac{4\pi}{r} \cdot \frac{r}{\varepsilon^2 + r^2} = \frac{4\pi}{r^2}, \quad \varepsilon \to 0$$

where we inserted $e^{-\varepsilon r}$ term to make integral converge.

Then, the ~~energy~~ number of photons in mode k is

$$n_k = \frac{\frac{\varepsilon_0}{2}\left[E(k)E^*(k) + c^2 B(k)B^*(k)\right]}{\hbar c k}$$

$$N_k = \int n_k\, dk = \cancel{\frac{\varepsilon_0}{(4\pi)^3 \hbar c} \int dx_1 \int dx_2\, \mathcal{F}^{-1}(E)(x_1)\, \mathcal{F}^{-1}(E^*)(x_1 - x_2)\, \mathcal{F}^{-1}(x_2 - x)}$$

$$= \int d^3x\, \mathcal{F}^{-1}(n_k) \rightarrow \text{double convolution}$$

$$= \int d^3x \int d^3x_1\, d^3x_2\, \mathcal{F}^{-1}(E)(x_1)\, \mathcal{F}^{-1}(E^*)(x_2 - x_1) \cdot \mathcal{F}^{-1}\left(\frac{1}{k}\right)(x - x_2)$$

+ same for B-terms

$$N_k = \frac{\varepsilon_0}{\hbar c 4\pi^2}\int d^3x\, d^3x_1\, d^3x_2 \quad \frac{E(x_1)E(x_1-x_2)}{|x_2-x|^2} + \frac{c^2B(x_1)B(x_1-x_2)}{|x_2-x|^2}$$

Alternatively, we can swap some variables around to write this as:

$$N_k = \frac{\varepsilon_0}{\hbar c 4\pi^2}\int d^3x\, d^3x_1\, d^3x_2 \quad \frac{E(x_1)\,E(x_2)}{|\vec{x}-\vec{x}_1-\vec{x}_2|^2} + \frac{c^2B(x_1)\,B(x_2)}{|x-x_1-x_2|^2}$$

This is not quite the answer ~~is~~ the book is looking for, but it's close and I haven't figured out the whole thing yet.