7.16 Plane waves propagate in a homogeneous, nonpermeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ϵ_{ij}, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i$ $(i = 1, 2, 3)$, where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

a) Show that plane waves with frequency ω and wave vector $\vec{k}$ must satisfy

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

This is in fact the general Maxwell wave equation, and does not depend on the details of the dielectric tensor. This may be derived from the curl equations, using $\vec{\nabla} \to i\vec{k}$ and $\partial/\partial_t \to -i\omega$. In a source-free region, the Ampère-Maxwell and Faraday laws give

$$i\vec{k} \times \vec{H} = -i\omega\vec{D}, \qquad i\vec{k} \times \vec{E} - i\omega\vec{B} = 0$$

Taking $i\vec{k}$ cross Faraday's law, and using $\vec{B} = \mu_0 \vec{H}$ gives

$$i\vec{k} \times (i\vec{k} \times \vec{E}) - i\mu_0\omega(i\vec{k} \times \vec{H}) = 0$$

It is then straightforward to substitute in Ampère's law in the second term to arrive at

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

b) Show that for a given wave vector $\vec{k} = k\hat{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component of $\hat{n}$ along the ith principal axis.

Letting $\vec{k} = k\hat{n}$, and using the BAC–CAB rule, we find

$$\hat{n}(\hat{n} \cdot \vec{E}) - \vec{E} + \mu_0 v^2 \vec{D} = 0$$

By working with the principle axes, this equation may be entirely written in terms of $\vec{E}$. Introducing the real symmetric matrices

$$A_{ij} = n_i n_j - \delta_{ij}, \qquad W_{ij} = \delta_{ij}\mu_0\epsilon_j = \delta_{ij}/v_j^2$$

we arrive at a generalized eigenvalue problem

$$\mathbf{A}\vec{E} = -v^2\mathbf{W}\vec{E} \qquad \text{or} \qquad (\mathbf{A} + v^2\mathbf{W})\vec{E} = 0 \tag{8}$$

The velocities of propagation are then the eigenvalues of this problem, and may be obtained by solving the secular equation

$$\begin{aligned} 0 = \det(\mathbf{A} + v^2\mathbf{W}) &= v^6 \frac{n_1^2 + n_2^2 + n_3^2}{v_1^2 v_2^2 v_3^2} - v^4 \left(\frac{n_2^2 + n_3^2}{v_2^2 v_3^2} + \frac{n_1^2 + n_3^2}{v_1^2 v_3^2} + \frac{n_1^2 + n_2^2}{v_1^2 v_2^2} \right) \\ &\quad + v^2 \left(\frac{n_1^2}{v_1^2} + \frac{n_2^2}{v_2^2} + \frac{n_3^2}{v_3^2} \right) \\ &= \frac{v^2}{v_1^2 v_2^2 v_3^2} \Big[n_1^2 (v^2 - v_2^2)(v^2 - v_3^2) + n_2^2 (v^2 - v_1^2)(v^2 - v_3^2) \\ &\qquad + n_3^2 (v^2 - v_1^2)(v^2 - v_2^2) \Big] \end{aligned}$$

Other than a trivial solution, $v = 0$ (which does not correspond to a propagating mode), we find two velocities, v_a and v_b, corresponding to the two roots of the quadratic equation for v^2 in the square brackets. In fact, taking the equation in brackets and dividing out by the product $\Pi_i (v^2 - v_i^2)$ immediately gives the Fresnel equation

$$\sum_i \frac{n_i^2}{v^2 - v_i^2} = 0$$

c) Show that $\vec{D}_a \cdot \vec{D}_b = 0$, where $\vec{D}_a$, $\vec{D}_b$ are the displacements associated with the two modes of propagation.

Here we may use standard linear algebra techniques related to the orthogonality of eigenvectors. Considering first the generalized eigenvalue problem (8), we take distinct eigenvalues v_a and v_b. Then the corresponding eigenvectors satisfy the equations

$$(\mathbf{A} + v_a^2 \mathbf{W})\vec{E}_a = 0, \qquad (\mathbf{A} + v_b^2 \mathbf{W})\vec{E}_b = 0$$

Left-multiplying the first equation by $\vec{E}_b$ and the second by $\vec{E}_a$ gives

$$\vec{E}_b \mathbf{A} \vec{E}_a + v_a^2 \vec{E}_b \mathbf{W} \vec{E}_a = 0, \qquad \vec{E}_a \mathbf{A} \vec{E}_b + v_b^2 \vec{E}_a \mathbf{W} \vec{E}_b = 0$$

Since $\mathbf{A}$ and $\mathbf{W}$ are symmetric (real Hermitian), we may transpose the first equation and subtract it from the second. The result is

$$(v_b^2 - v_a^2)\vec{E}_a \mathbf{W} \vec{E}_b = 0$$

which implies $\vec{E}_a \mathbf{W} \vec{E}_b = 0$, since $v_a \neq v_b$ (in the case that $v_a = v_b$, we may instead Gram-Schmidt orthogonalize to make the eigenvectors orthogonal). Finally, since $\mathbf{W}$ is μ_0 times the dielectric matrix $\mathbf{\Sigma} = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$, and since $\vec{D} = \mathbf{\Sigma}\vec{E}$, we may equivalently rewrite this orthogonality (with respect to the 'measure' or 'metric' $\mathbf{W}$) as

$$\vec{E}_a \cdot \vec{D}_b = 0 \qquad \text{or} \qquad \vec{E}_b \cdot \vec{D}_a = 0$$

However, we can in fact learn more than this. Since the matrix $\mathbf{A} = \hat{n} \otimes \hat{n} - I$ is not arbitrary, it satisfies the (almost) projection condition $\mathbf{A}^2 = -\mathbf{A}$. As a result

$$\vec{D}_a \cdot \vec{D}_b = \vec{E}_a \mathbf{\Sigma}^2 \vec{E}_b = \frac{1}{\mu_0^2} \vec{E}_a \mathbf{W}^2 \vec{E}_b = \frac{1}{\mu_0^2 v_a^2 v_b^2} \vec{E}_a \mathbf{A}^2 \vec{E}_b = -\frac{1}{\mu_0^2 v_a^2 v_b^2} \vec{E}_a \mathbf{A} \vec{E}_b$$

But since $\mathbf{A}\vec{E}_b = -v_b^2 \mathbf{W} \vec{E}_b$, we obtain

$$\vec{D}_a \cdot \vec{D}_b = \frac{1}{\mu_0^2 v_a^2} \vec{E}_a \mathbf{W} \vec{E}_b = \frac{1}{\mu_0 v_a^2} \vec{E}_a \cdot \vec{D}_b = 0$$

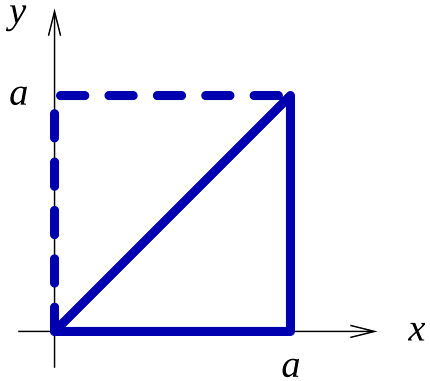
(Note, however, that in general $\vec{E}_a \cdot \vec{E}_b \neq 0$.)

8.5 A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$.

a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

In general, to solve a problem like this, we need to consider the Dirichlet or Neumann problem for a boundary without any 'standard' (ie rectangular or circular) symmetry. In particular, this means there is no natural coordinate system to use for the two-dimensional Helmholtz equation $[\nabla_t^2 + \gamma^2]\psi = 0$ that both allows for separation of variables *and* respects the symmetry of the boundary surface (which would allow a simple specification of the boundary data). A general problem of this form (with no simple boundary symmetry) is quite unpleasant to solve.

In this case, however, we can think of the triangle as 'half' of a square.



In particular, the key step to this problem is to note that the triangle may be obtained from the square by imposing reflection symmetry along the $x = y$ diagonal. This symmetry is a $\mathbb{Z}_2$ reflection on the coordinates of the form

$$\mathbb{Z}_2 : \quad x \to y, \;\; y \to x$$

Eigenfunctions $\psi(x, y)$ can then be classified as either $\mathbb{Z}_2$-even or $\mathbb{Z}_2$-odd

$$\mathbb{Z}_2 : \quad \psi(x, y) \to \pm\psi(y, x)$$

The odd functions vanish along the diagonal, so they automatically satisfy Dirichlet conditions $\psi(x = y) = 0$ on the diagonal. Similarly, the even functions have vanishing normal derivative on the diagonal and hence automatically satisfy Neumann conditions. We will use this fact to construct TM and TE modes for the triangle.

We begin with the TM modes. Using rectangular coordinates, it is natural to write solutions of the Helmholtz equation $[\partial_x^2 + \partial_y^2 + \gamma^2]\psi = 0$ as $\psi \sim e^{i(k_x x + k_y y)}$ where $k_x^2 + k_y^2 = \gamma^2$. This means we may expand the eigenfuctions in terms of sines and cosines. For TM modes satisfying the Dirichlet condition $\psi_S = 0$, we start with eigenfunctions on the square

$$\psi \sim \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

which automatically satisfy the boundary conditions on the four walls of the square. This gives

$$\gamma_{mn} = \frac{\pi}{a}\sqrt{m^2 + n^2}$$

so the cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu_0 \epsilon_0} a}\sqrt{m^2 + n^2} = \frac{\pi c}{a}\sqrt{m^2 + n^2} \tag{1}$$

In order to satisfy the Dirichlet condition on the diagonal, we take the $\mathbb{Z}_2$-odd combination

(TM) $$\psi_{mn} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} - \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a}$$

It is simple to verify that $\psi(x, 0) = \psi(a, y) = \psi(x, x) = 0$, so that all boundary conditions on the triangle are indeed satisfied. The cutoff frequencies are given by (1). Note here that the $\mathbb{Z}_2$ projection removes the $m = n$ modes and also antisymmetrizes m with n. As a result, the integer labels m and n may be taken to satisfy the condition $m > n > 0$.

The analysis for TE modes is similar. However, for Neumann conditions, we take cosine combinations as well as a $\mathbb{Z}_2$-even eigenfunction. This gives

(TE) $$\psi_{mn} = \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{a} + \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{a}$$

with identical cutoff frequencies as in (1). This time, however, the labels m and n may be taken to satisfy $m \geq n \geq 0$ (except $m = n = 0$ is not allowed).

b) For the lowest modes of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

The attenuation coefficients are determined by power and power loss. We begin with TM modes. For the power, we need to compute

$$\int_A |\psi|^2 \, da = \int_A \left[\sin k_m x \sin k_n y - \sin k_n x \sin k_m y\right]^2 da \tag{2}$$

It is perhaps easiest to compute this by integrating over the square and then dividing by two for the triangle. This is because the integration separates into x and y integrals, and we may use orthogonality

$$\int_0^a \sin k_i x \sin k_j x \, dx = \frac{a}{2}\delta_{i,j} \qquad \left(\text{where } k_j = \frac{j\pi}{a}\right)$$

This gives

$$\int_A |\psi|^2 \, da = \frac{1}{2} \times 2\left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$$

The factor of $1/2$ is for the triangle, while the factor of 2 is because two non-vanishing terms arise when squaring the integrand in (2). (Recall that $m \neq n$ for TM modes.) This gives an expression for the power

$$\begin{aligned} P &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 \, da \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\frac{A}{2} \end{aligned}$$

where $A = a^2/2$ is the area of the triangle. Calculating the power loss involves integrating a normal derivative

$$\oint_C \left|\frac{\partial \psi}{\partial n}\right|^2 dl$$

We break this into three parts: along $y = 0$, along $x = a$ and along the diagonal $x = y$. Along the $y = 0$ wall, we have $\hat{n} = \hat{y}$ and

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=0} = \frac{\pi}{a}\left[n \sin k_m x - m \sin k_n x\right]$$

As a result

$$\int_0^a \left|\frac{\partial \psi}{\partial y}\right|^2 dx = \left(\frac{\pi}{a}\right)^2 \frac{a}{2}(m^2 + n^2) = \frac{\pi^2}{2a}(m^2 + n^2) \tag{3}$$

A similar calculation, or use of symmetry, will result in an identical expression for the integral along the $x = a$ wall. For the diagonal, we use $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ to compute

$$\begin{aligned} \frac{\partial \psi}{\partial n} = \frac{1}{\sqrt{2}}\left[\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y}\right]_{y=x} &= \sqrt{2}\frac{\pi}{a}\left[m \cos k_m x \sin k_n x - n \cos k_n x \sin k_m x\right] \\ &= \frac{\sqrt{2}}{2}\frac{\pi}{a}\left[(m-n)\sin k_{m+n} x - (m+n)\sin k_{m-n} x\right] \end{aligned}$$

This gives

$$\int_0^{\sqrt{a}} \left|\frac{\partial\psi}{\partial n}\right|^2 dl = \sqrt{2}\int_0^a \left|\frac{\partial\psi}{\partial n}\right|^2 dx = \sqrt{2}\frac{1}{2}\left(\frac{\pi}{a}\right)^2 \frac{a}{2}\left[(m-n)^2+(m+n)^2\right]$$
$$= \sqrt{2}\frac{\pi^2}{2a}(m^2+n^2)$$

Combining this diagonal with (3) for the sides, we obtain

$$\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl = C\frac{\pi^2}{2a^2}(m^2+n^2) = \frac{C}{2}\gamma_{mn}^2$$

where $C = a + a + \sqrt{2}a$ is the circumference of the triangle. This gives a TM mode power loss of

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{1}{\mu^2\omega_{mn}^2}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl$$
$$= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{1}{\mu^2\omega_{mn}^2}\frac{C}{2}\gamma_{mn}^2 = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{\epsilon}{\mu}\frac{C}{2}$$

The attenuation coefficient is thus

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{C}{2A}$$

so that the geometrical factor $\xi_{mn} = 1$ is trivial. Note that the energy loss calculation along the diagonal of the triangle gives the same result as along the square edges. As a result, the geometrical factor $\xi_{mn} = 1$ is independent of whether the waveguide is square or right triangular. This is why the triangular TM result is identical to the square TM result, at least up to the ratios $C/A = 2(2+\sqrt{2})/a \approx 6.83/a$ for the triangle and $C/A = 4/a$ for the square.

The power loss for the TE modes is somewhat harder to deal with because of the possibility of special cases. Consider

$$\psi = \cos k_m x \cos k_n y + \cos k_n x \cos k_m y \tag{4}$$

where $m \geq n \geq 0$. If $n = 0$, we end up with

$$\psi = \cos k_m x + \cos k_m y \qquad (m > 0)$$

In this case

$$\int_A |\psi|^2 da = \frac{1}{2}\int_0^a dx \int_0^a dy\left[\cos k_m x + \cos k_m y\right]^2 = \frac{1}{2}\times 2(\tfrac{1}{2}a^2) = \frac{a^2}{2} = A$$

while the perimeter integrals are

$$\int_0^a dx\big|\psi(y=0)\big|^2 = \int_0^a dx\big[1+\cos k_m x\big]^2 = a(1+\tfrac{1}{2}) = \frac{3a}{2}$$
$$\sqrt{2}\int_0^a dx\big|\psi(y=x)\big|^2 = \sqrt{2}\int_0^a dx\big[2\cos k_m x\big]^2 = 4\sqrt{2}(\tfrac{1}{2}a) = 2\sqrt{2}a$$

which gives

$$\oint_C \big|\psi\big|^2 dl = (3+2\sqrt{2})a$$

and

$$\int_0^a dx\big|\hat{n}\times\vec{\nabla}_t\psi\big|^2 = \int_0^a dx\big|\hat{y}\times\vec{\nabla}_t\psi\big|^2 = \int_0^a dx\big|-\hat{z}\partial_x\psi\big|^2_{y=0}$$
$$= \int_0^a dx\frac{\pi^2}{a^2}m^2\big|\sin k_m x\big|^2 = \frac{\pi^2}{2a}m^2$$
$$\sqrt{2}\int_0^a dx\big|\hat{n}\times\vec{\nabla}_t\psi\big|^2_{y=x} = \sqrt{2}\int_0^a dx\big|\tfrac{1}{\sqrt{2}}\hat{z}(\partial_y+\partial_x)\psi\big|^2_{y=x}$$
$$= \frac{\sqrt{2}}{2}\int_0^a dx\frac{\pi^2}{a^2}m^2\big|2\sin k_m x\big|^2 = \sqrt{2}\frac{\pi^2}{a}m^2$$

which gives

$$\oint_C \big|\hat{n}\times\vec{\nabla}_t\psi\big|^2 dl = (1+\sqrt{2})\frac{\pi^2}{a}m^2 = (1+\sqrt{2})a\gamma_{m0}^2$$

Using

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 da$$

and

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_C\left[\frac{1}{\gamma_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)|\hat{n}\times\vec{\nabla}_t\psi|^2 + \frac{\omega_{mn}^2}{\omega^2}|\psi|^2\right]dl$$

with the above integrals gives an attenuation coefficient

$$\begin{aligned}
\beta_{m0} &= -\frac{1}{2P}\frac{dP}{dz}\\
&= \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)^{-1/2}\left[(1+\sqrt{2})\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)+\frac{\omega_{m0}^2}{\omega^2}(3+2\sqrt{2})\right]\frac{a}{A}\\
&= \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)^{-1/2}\left[\frac{1+\sqrt{2}}{2+\sqrt{2}}+\frac{\omega_{m0}^2}{\omega^2}\right]\frac{C}{2A}
\end{aligned}$$

where $C = (2+\sqrt{2})a$ and $A = a^2/2$. Here the geometrical factors are

$$\xi_{m0} = \frac{1+\sqrt{2}}{2+\sqrt{2}}, \qquad \eta_{m0} = 1 \qquad (m > n = 0)$$

For the rectangular waveguide, one has instead

$$\xi_{m0} = \frac{a}{a+b} \to \frac{1}{2}, \qquad \eta_{m0} = \frac{2b}{a+b} \to 1 \qquad \text{when } b \to a$$

This is different because the power loss calculation is no longer universal, giving different coefficients along the diagonal as along the square edges. The remaining TE cases to consider are modes (4) where $m = n > 0$ and $m > n > 0$. Here we simply state the results. For $m = n > 0$ we have

$$\psi = \cos k_m x \cos k_m y$$

(we have removed an unimportant factor of two) so that

$$\begin{aligned}
\int_A |\psi|^2 da &= \frac{a^2}{8} = \frac{A}{4} \\
\oint_C |\psi|^2 dl &= \left(1 + \frac{3\sqrt{2}}{8}\right) a \\
\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= \left(1 + \frac{\sqrt{2}}{4}\right) \frac{\pi^2}{a} m^2 = \left(\frac{1}{2} + \frac{\sqrt{2}}{8}\right) a \gamma_{mm}^2
\end{aligned}$$

This gives

$$\xi_{mm} = \frac{4+\sqrt{2}}{4+2\sqrt{2}}, \qquad \eta_{mm} = 1 \qquad (m = n > 0)$$

On the other hand, for the general case $m > n > 0$ we find

$$\begin{aligned}
\int_A |\psi|^2 da &= \frac{a^2}{4} = \frac{A}{2} \\
\oint_C |\psi|^2 dl &= (2+\sqrt{2})a = C \\
\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= (2+\sqrt{2}) \frac{\pi^2}{2a} (m^2 + n^2) = \frac{C}{2} \gamma_{mn}^2
\end{aligned}$$

which yields

$$\xi_{mn} = 1, \qquad \eta_{mn} = 1 \qquad (m > n > 0)$$

In all cases, $\eta_{mn} = 1$, which is the same for the triangle or the square waveguide. For ξ_{mn}, the factor is essentially a geometric combination of contributions along the perimeter of either 1 or 1/2 depending on the particular mode and its degeneracies.