9.1 A common textbook example of a radiating system (see Problem 9.2) is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not in the form of (9.1).

a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\vec{x}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (9.1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

For a rotating set of charges, where the rotation is along the z axis, the charge density may be written as

$$\rho = \rho(r, \theta, \phi - \omega_0 t)$$

where ω_0 is the angular frequency of rotation. Using this, we first examine the time-dependent multipole moments

$$\begin{aligned} q_{lm}(t) &= \int r^l Y_{lm}^*(\theta, \phi) \rho(r, \theta, \phi - \omega_0 t) d^3x \\ &= \int r^l Y_{lm}^*(\theta, \phi' + \omega_0 t) \rho(r, \theta, \phi') d^3x \end{aligned}$$

where in the second line we have made the substitution $\phi = \phi' + \omega_0 t$. We now note that the azimuthal behavior of the spherical harmonics goes as

$$Y_{lm}(\theta, \phi) \sim e^{im\phi}$$

Hence

$$Y_{lm}(\theta, \phi' + \omega_0 t) = Y_{lm}(\theta, \phi') e^{im\omega_0 t}$$

This allows us to isolate the time dependence of $q_{lm}(t)$ as

$$q_{lm}(t) = \overline{q}_{lm} e^{-im\omega_0 t} \tag{8}$$

where $\overline{q}_{lm}$ is the static multipole moment calculated in the body frame [ie with $\rho(r, \theta, \phi)$]. This expression is almost of the analogous form as (9.1), in the sense that the harmonic time dependence is given by a complex exponential. One interesting difference, however, is that (9.1) involves a pure frequency ω of the form

$$\rho(\vec{x}, t) = \rho(\vec{x}) e^{-i\omega t}$$

while (8) involves a different frequency

$$\omega_m = m\omega_0$$

for each different value of m. This demonstrates that a rotating set of charges generally radiates at the fundamental frequency ω_0 as well as all higher harmonics.

Another important difference, however, is that $m < 0$ components of (8) appear to have a negative frequency. This is somewhat artificial, since the harmonic prescription we are using is to take the real part of the complex time-dependent quantities. In particular

$$\Re(e^{-im\omega t}) = \Re(e^{+im\omega t}) = \cos(m\omega t)$$

This indicates that modes $q_{lm}(t)$ and $q_{l,-m}(t)$ radiate at the same frequency $m\omega_0$. To avoid negative frequencies, we may use the identity

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y^*_{lm}(\theta, \phi)$$

to show that $\overline{q}_{l,-m} = (-1)^m \overline{q}^*_{l,m}$. This allows us to rewrite (8) as

$$q_{lm}(t) = \begin{cases} \overline{q}_{lm} e^{-im\omega_0 t} & m > 0 \\ \overline{q}_{l0} & m = 0 \\ (-1)^m [\overline{q}_{l|m|} e^{-i|m|\omega_0 t}]^* & m < 0 \end{cases}$$

Note that the $m = 0$ term has zero frequency, and hence does not radiate. At this stage, we still have not specified what to do with the $m < 0$ multipoles. We note, however, that since ultimately we only care about the real parts of these complex expressions, it does not matter much whether we take a complex conjugate or not. Hence we can drop the complex conjugate in the $m < 0$ expression above. In this case, both $q_{lm}(t)$ and $q_{l,-m}(t)$ can be expressed using $\overline{q}_{l|m|}$, at least up to a possible minus sign. To see how to deal with this sign, we note that $q_{lm}(t)$ is essentially the coefficient of $Y_{lm}(\theta, \phi)$ in the spherical harmonic expansion. The product $q_{lm}(t)Y_{lm}(\theta, \phi)$ then has a simple $m \to -m$ behavior

$$q_{l,-m}(t)Y_{l,-m}(\theta, \phi) = [q_{lm}(t)Y_{lm}(\theta, \phi)]^*$$

Linearly superposing the $+m$ and $-m$ moments then gives

$$q_{lm}(t)Y_{lm}(\theta, \phi) + q_{l,-m}(t)Y_{l,-m}(\theta, \phi) = \Re[2\overline{q}_{lm}Y_{lm}(\theta, \phi)e^{-im\omega_0 t}]$$

This demonstrates that, when summing over all multipoles for radiation, it is sufficient to sum over the positive frequency modes only while including an extra factor of two. In particular, we can take

$$q_{lm}^{\text{eff}} = \begin{cases} 2\overline{q}_{lm} & m > 0 \\ \overline{q}_{l0} & m = 0 \end{cases} \qquad \text{with frequencies } m\omega_0 \tag{9}$$

b) Consider a charge density $\rho(\vec{x}, t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\vec{x}, t) = \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty} \Re[2\rho_n(\vec{x}\,)e^{-in\omega_0 t}]$$

where

$$\rho_n(\vec{x}\,) = \frac{1}{T}\int_0^T \rho(\vec{x}, t)e^{in\omega_0 t}dt$$

This shows explicitly how to establish connection with (9.1).

Recall that the complex Fourier series in the time variable t may be written as

$$\rho(\vec{x}, t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{x}\,)e^{-in\omega_0 t}$$
$$\rho_n(\vec{x}\,) = \frac{1}{T}\int_0^T \rho(\vec{x}, t)e^{in\omega_0 t}dt$$

Assuming that $\rho(\vec{x}, t)$ is real (as it ought to be) we note that

$$\rho_{-n}(\vec{x}\,) = \rho_n(\vec{x}\,)^*$$

Hence

$$\begin{aligned}
\rho(\vec{x},t) &= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}[\rho_n(\vec{x}\,)e^{-in\omega_0 t} + \rho_{-n}(\vec{x}\,)e^{in\omega_0 t}] \\
&= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}[\rho_n(\vec{x}\,)e^{-in\omega_0 t} + (\rho_n(\vec{x}\,)e^{-in\omega_0 t})^*] \\
&= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}\Re[2\rho_n(\vec{x}\,)e^{-in\omega_0 t}]
\end{aligned}$$

Taking the real part of a complex time harmonic quantity is of course what we want to make connection to (9.1). In particular, we show that the periodic in time charge distribution $\rho(\vec{x},t)$ may be treated as a collection of harmonic charge densities

$$\rho_n^{\rm eff}(\vec{x}\,) = \begin{cases} 2\rho_n(\vec{x}\,) & n>0 \\ \rho_0(\vec{x}\,) & n=0 \end{cases} \qquad \text{with frequencies } n\omega_0 \tag{10}$$

Of course, $\rho_0(\vec{x}\,)$ is static and does not radiate. Note the similarity in form between this and (9).

c) For a single charge q rotating about the origin in the x-y plane in a circle of radius R at constant angular speed ω_0, calculate the $l=0$ and $l=1$ multipole moments by the methods of parts a and b and compare. In method b express the charge density $\rho_n(\vec{x}\,)$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

For a single rotating charge q, the time dependent charge density may be written in spherical coordinates as

$$\rho(\vec{x},t) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

We start with the method of part a. Here we calculate the body-centric multipole moments

$$\begin{aligned}
\overline{q}_{lm} &= \int r^l Y_{lm}^*(\theta,\phi)\overline{\rho}(r,\theta,\phi)r^2 dr\,d\cos\theta\,d\phi \\
&= qR^l Y_{lm}^*(\pi/2,0) \\
&= qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)
\end{aligned} \tag{11}$$

The $l=0$ and $l=1$ moments are

$$\overline{q}_{00} = \sqrt{\frac{1}{4\pi}}q, \qquad \overline{q}_{11} = -\sqrt{\frac{3}{8\pi}}qR$$

so that, according to (9), we have

$$q_{00}^{\rm eff} = \sqrt{\frac{1}{4\pi}}q, \qquad q_{11}^{\rm eff} = -\sqrt{\frac{3}{2\pi}}qR$$

For the method of part b, we start by calculating the n-th Fourier mode

$$\begin{aligned}\rho_n(\vec{x}\,) &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \rho(\vec{x},t)e^{in\omega_0 t}dt \\ &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)e^{in\omega_0 t}dt \\ &= \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}\end{aligned}$$

The multipole moments calculated from $\rho_n(\vec{x})$ are

$$\begin{aligned}q_{lm}[\rho_n] &= \int r^l Y_{lm}^*(\theta,\phi)\rho_n(r,\theta,\phi)r^2 dr\,d\cos\theta\,d\phi \\ &= \frac{q}{2\pi R^2}\int r^l Y_{lm}^*(\theta,\phi)\delta(r-R)\delta(\cos\theta)e^{in\phi}r^2 dr\,d\cos\theta\,d\phi \\ &= qR^l\frac{1}{2\pi}\int_0^{2\pi} Y_{lm}^*(\pi/2,\phi)e^{in\phi}d\phi \\ &= qR^l\delta_{mn}Y_{lm}^*(\pi/2,0) \\ &= qR^l\delta_{mn}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\end{aligned}$$

Note that the moments calculated from $\rho_n(\vec{x}\,)$ have $m=n$, but otherwise agree with (11). Since the effective charge density $\rho_n(\vec{x}\,)$ is doubled for $n>0$ according to (10), the effective moments $q_{lm}[\rho_n]$ are doubled as well. This effective doubling is consistent across parts a and b.

Finally, we note from (11) that all higher multipoles are present, so long as $P_l^m(0)$ is non-vanishing. By parity, this happens whenever $l+m$ is even. Thus the l-th multipole will radiate at frequences $l\omega_0, (l-2)\omega_0, (l-4)\omega_0, \ldots$.

9.3 Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

In the long wavelength limit, the electric dipole approximation ought to be reasonable. In this case, we may first work out the multipole expansion of the source, and then extract the dipole term. For this problem, the source is essentially a harmonically ($e^{-i\omega t}$) varying version of the electrostatic problem with hemispheres at opposite potential. The long wavelength limit is also equivalent to the low frequency limit. Thus it is valid to think of the source as a quasi-static object. Using azimuthal symmetry, the potential then admits an expansion in Legendre polynomials

$$\Phi(r,\theta) = \sum_l \alpha_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

where

$$\alpha_l = \frac{2l+1}{2} \int_{-1}^{1} \Phi(R, \cos\theta) P_l(\cos\theta) d\cos\theta$$

For hemispheres at opposite potential $\pm V$ (times $e^{-i\omega t}$, which is to be understood), the expansion coefficients are

$$\alpha_l = (2l+1)V \int_0^1 P_l(x)\, dx \qquad \text{odd } l \text{ only}$$

The dipole term is dominant, and it is easy to see that $\alpha_1 = \frac{3}{2}V$. This gives rise to a dipole potential of the form

$$\Phi_{l=1} = \frac{3}{2} V \left(\frac{R}{r}\right)^2 P_1(\cos\theta) = \frac{3}{2} V R^2 \frac{z}{r^3}$$

Comparing this with the dipole expression

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

allows us to read off an electric dipole moment

$$\vec{p} = 4\pi\epsilon_0 (\tfrac{3}{2} V R^2 \hat{z}) = 6\pi\epsilon_0 V R^2 \hat{z}$$

Working in the radiation zone, this electric dipole gives

$$\vec{H} = \frac{ck^2}{4\pi} (\hat{r}\times\vec{p}) \frac{e^{ikr}}{r} = -\frac{ck^2}{4\pi} 6\pi\epsilon_0 V R^2 \frac{e^{ikr}}{r} \sin\theta\hat{\phi} = -\frac{3}{2} Z_0^{-1} V (kR)^2 \frac{e^{ikr}}{r} \sin\theta\hat{\phi}$$

and

$$\vec{E} = -Z_0 \hat{r}\times\vec{H} = -\frac{3}{2} V (kR)^2 \frac{e^{ikr}}{r} \sin\theta\hat{\theta}$$

The angular distribution of dipole radiation gives

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\vec{p}|^2 \sin^2\theta = \frac{c^2 Z_0}{32\pi^2} k^4 36\pi^2 \epsilon_0^2 V^2 R^4 \sin^2\theta = \frac{9}{8} Z_0^{-1} V^2 (kR)^4 \sin^2\theta$$

and the total radiated power is

$$P = 3\pi Z_0^{-1} V^2 (kR)^4$$

4. Problem 9.5 **10 Points**

a): For $\mathbf{A}(\mathbf{x})$, copy Eqns. 9.13-9.16 of Jackson. For $\Phi(\mathbf{x})$, write the analogue of Eq. 9.30 for $\Phi(\mathbf{x})$,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{1}{r} - \mathrm{i}k\right)\int \rho(\mathbf{x}')\hat{\mathbf{n}}\cdot\mathbf{x}' d^3x' = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{1}{r} - \mathrm{i}k\right)\hat{\mathbf{n}}\cdot\mathbf{p}$$

b):

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} = -\frac{\mathrm{i}\mu_0\omega}{4\pi}\nabla\times\left(\mathbf{p}\frac{\exp(\mathrm{i}kr)}{r}\right) \\
&= -\frac{\mathrm{i}\mu_0\omega}{4\pi}\left[\left(\nabla\frac{\exp(\mathrm{i}kr)}{r}\right)\times\mathbf{p} + \frac{\exp(\mathrm{i}kr)}{r}(\nabla\times\mathbf{p})\right] = \\
&= -\frac{\mathrm{i}\mu_0\omega}{4\pi}\left(\hat{\mathbf{n}}\frac{\exp(\mathrm{i}kr)}{r}\left[\mathrm{i}k - \frac{1}{r^2}\right]\right)\times\mathbf{p} \\
&= \frac{ck^2\mu_0}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\left[1 - \frac{1}{\mathrm{i}kr}\right](\hat{\mathbf{n}}\times\mathbf{p}) \qquad (5)
\end{aligned}$$

One way to obtain $\mathbf{E}$ is

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{A} - \nabla\Phi \\
&= \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left\{k^2\mathbf{p} + \hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{p}\left(\left(\frac{1}{r} - \mathrm{i}k\right)^2 + \frac{1}{r^2}\right)\right] - \hat{\theta}\left[\hat{\theta}\cdot\mathbf{p}\left(\frac{1}{r} - \mathrm{i}k\right)\left(\frac{1}{r}\right)\right] - \hat{\phi}\left[\hat{\phi}\cdot\mathbf{p}\left(\frac{1}{r} - \mathrm{i}k\right)\left(\frac{1}{r}\right)\right]\right\}
\end{aligned}$$

where we have used $\partial_\theta(\hat{\mathbf{r}}\cdot\mathbf{p}) = \mathbf{p}\cdot\hat{\theta}$ and $\partial_\phi(\hat{\mathbf{r}}\cdot\mathbf{p}) = (\mathbf{p}\cdot\hat{\phi})\sin\theta$. The result simplifies to

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}k^2\left\{\mathbf{p} - p_r\hat{\mathbf{r}}\right\} - \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r^2}\left(\frac{1}{r} - \mathrm{i}k\right)\left\{\hat{\theta}p_\theta + \hat{\phi}p_\phi - 2\hat{\mathbf{r}}p_r\right\}$$

Noting that $\hat{\mathbf{r}} = \hat{\mathbf{n}}$, the first curly bracket equals $(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}$ and the second $\mathbf{p} - 3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})$ we find the final result,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{p})\frac{\exp(\mathrm{i}kr)}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right)\exp(\mathrm{i}kr)\right\} \qquad (6)$$

9.11 Three charges are located along the z axis, a charge $+2q$ at the origin, and charges $-q$ at $z = \pm a\cos\omega t$. Determine the lowest nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated. Assume that $ka \ll 1$.

We start by specifying the charge and current densities

$$\begin{aligned}\rho &= q[2\delta(z) - \delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)]\delta(x)\delta(y)\\ \vec{J} &= \hat{z}qa\omega\sin\omega t[\delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)]\delta(x)\delta(y)\end{aligned} \tag{6}$$

It should be clear that these moving charges do not directly correspond to time harmonic sources of the form

$$\rho e^{-i\omega t}, \qquad \vec{J}e^{-i\omega t}$$

Thus we must use some of the techniques discussed in problem 9.1 in Fourier decomposing the source charge and current distributions. Essentially we find it easiest to take the approach of 9.1a, which is to compute the time-dependent multipole moments first before Fourier decomposing in frequency.

Assuming that $ka \ll 1$, we may directly compute the first few multipole moments. Working with Cartesian tensors, we have

$$\vec{p}(t) = \int \vec{x}\,\rho\,d^3x = -q(a\cos\omega t - a\cos\omega t) = 0$$

and

$$\vec{m}(t) = \frac{1}{2}\int \vec{x}\times\vec{J}\,d^3x = 0$$

In fact, all magnetic multipole moments vanish since the charges are undergoing linear motion. The electric quadrupole moment is non-vanishing, however

$$Q_{ij}(t) = \int (3x_ix_j - r^2\delta_{ij})\rho(t)\,d^3x = -qa^2\cos^2\omega t(3\delta_{i3}\delta_{j3} - \delta_{ij})$$

The non-vanishing moments are then

$$Q_{33}(t) = -2Q_{11}(t) = -2Q_{22}(t) = -4qa^2\cos^2\omega t$$

Note that this may be written as

$$Q_{33}(t) = -2qa^2[1 + \cos(2\omega t)] = \Re[-2qa^2(1 + e^{-2i\omega t})]$$

Since the zero frequency term does not radiate, this indicates that we may assume a *harmonic* quadrupole moment

$$Q_{33} = -2Q_{11} = -2Q_{22} = -2qa^2 \tag{7}$$

which oscillates with angular frequency 2ω. The angular distribution of radiation is then given by

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2} |Q_{33}|^2 \sin^2\theta\cos^2\theta = \frac{Z_0 q^2}{128\pi^2}(ck)^2(ka)^4 \sin^2\theta\cos^2\theta$$

Using $ck = 2\omega$ (since the harmonic frequency is 2ω), we find

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2\omega^2}{32\pi^2}(ka)^4 \sin^2\theta\cos^2\theta \tag{8}$$

Integrating this over the solid angle gives a total power

$$P = \frac{Z_0 q^2 \omega^2}{60\pi}(ka)^4 \tag{9}$$

Alternatively, we may apply the multipole expansion formalism to write down all multipole coefficients. Using the $ka \ll 1$ approximation, these expansion coefficients are given by

$$\begin{aligned} a_E(l,m) &\approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}Q_{lm} \\ a_M(l,m) &\approx \frac{ik^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}M_{lm} \end{aligned} \tag{10}$$

where

$$Q_{lm} = \int r^l Y_{lm}^* \rho \, d^3x, \qquad M_{lm} = -\frac{1}{l+1}\int r^l Y_{lm}^* \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, d^3x$$

To proceed, we convert the charge and current densities (6) to spherical coordinates

$$\begin{aligned} \rho &= \frac{q}{2\pi r^2}\big(\delta(r) - \delta(r - a\cos\omega t)[\delta(\cos\theta - 1) + \delta(\cos\theta+1)]\big) \\ \vec{J} &= \hat{r}\frac{qa\omega}{2\pi r^2}\sin\omega t\, \delta(r - a\cos\omega t)[\delta(\cos\theta-1) + \delta(\cos\theta+1)] \end{aligned}$$

For the magnetic multipoles, we see that since $\vec{J} \sim \hat{r}$ the cross product vanishes, $\vec{r}\times\vec{J} = 0$. Thus all magnetic multipoles vanish

$$a_M(l,m) = 0$$

We are thus left with the electric multipoles

$$\begin{aligned} Q_{lm}(t) &= \frac{q}{2\pi}\int r^l Y_{lm}^*\big(\delta(r) - \delta(r - a\cos\omega t) \\ &\qquad\qquad \times [\delta(\cos\theta - 1) + \delta(\cos\theta+1)]\big) dr\, d\cos\theta\, d\phi \\ &= q\delta_{m,0}\left(2\delta_{l,0}Y_{00}^* - [Y_{l0}^*(0,0) + Y_{l0}^*(\pi,0)](a\cos\omega t)^l\right) \end{aligned}$$

Using

$$Y_{00} = \sqrt{\frac{1}{4\pi}}, \qquad [Y_{l0}(0,0) + Y_{l0}(\pi,0)] = \sqrt{\frac{2l+1}{4\pi}}[P_l(1) + P_l(-1)]$$

then gives

$$Q_{l0}(t) = -2q\sqrt{\frac{2l+1}{4\pi}}(a\cos\omega t)^l \qquad l = 2, 4, 6, \ldots$$

A Fourier decomposition gives both positive and negative frequency modes

$$\begin{aligned} Q_{l0}(t) &= -2q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \left(e^{i\omega t} + e^{-i\omega t}\right)^l \\ &= -2q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \sum_{k=0}^{l} \binom{l}{k} e^{i(l-2k)\omega t} \qquad l = 2, 4, 6, \ldots \end{aligned}$$

However, since the real part of $e^{\pm in\omega t}$ does not care about the sign of $in\omega t$ we may group such terms together to eliminate negative frequencies

$$Q_{l0}(t) = -4q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \Re\left[\frac{1}{2}\binom{l}{l/2} + \sum_{n=2,4,\ldots,l}\binom{l}{(l-n)/2}e^{-in\omega t}\right]$$

where $l = 2, 4, 6, \ldots$. Note that the zero frequency mode does not radiate, and hence may be ignored. In general, the l-th mode radiates at frequencies $l\omega, (l-2)\omega, (l-4)\omega, \ldots$.

The lowest non-vanishing moment is the electric quadrupole moment

$$Q_{20} = -qa^2\sqrt{\frac{5}{4\pi}}$$

and its harmonic frequency is 2ω. Note, in particular, that this agrees with (7) when converted to a Cartesian tensor. Using (10), this yields a multipole coefficient

$$a_E(2,0) = iqk(ck)(ka)^2\sqrt{\frac{1}{120\pi}} \tag{11}$$

We now turn to the angular distribution of the radiation. For a pure multipole of order (l,m), the angular distribution of radiated power is

$$\begin{aligned} \frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2 l(l+1)}|a(l,m)|^2\Big[&\tfrac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 \\ &+ \tfrac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2|Y_{lm}|^2\Big] \end{aligned}$$

This simplifies considerably for $m = 0$

$$\frac{dP(l,0)}{d\Omega} = \frac{Z_0}{2k^2}|a(l,0)|^2|Y_{l,1}|^2$$

Since the lowest multipole is the electric quadrupole, we substitute in $l = 2$ and $m = 0$ to obtain

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}|a_E(2,0)|^2\frac{15}{8\pi}\sin^2\theta\cos^2\theta \tag{12}$$

Using (11) then gives

$$\frac{dP}{d\Omega} = \frac{Z_0q^2}{128\pi^2}(ck)^2(ka)^4\sin^2\theta\cos^2\theta = \frac{Z_0q^2\omega^2}{32\pi^2}(ka)^4\sin^2\theta\cos^2\theta$$

Note that we have used $ck = 2\omega$, since the harmonic frequency is 2ω. The total radiated power is given by

$$P = \frac{Z_0}{2k^2}\sum_{l,m}[|a_E(l,m)|^2 + |a_M(l,m)|^2]$$

For the electric quadrupole, this gives

$$P = \frac{Z_0}{2k^2}|a_E(2,0)|^2 = \frac{Z_0q^2\omega^2}{60\pi}(ka)^4 \tag{13}$$

The next non-vanishing multipole would be $l = 4$, which radiates at frequencies 2ω and 4ω. However, this will be subdominant, so long as $ka \ll 1$. Note that the angular distribution (12) and the total radiated power (13) agree with those found earlier, namely (8) and (9).