1. The fields ψ_λ, ψ_μ satisfy

$$(\nabla^2 + \gamma_\lambda^2)\psi_\lambda = 0$$

$$(\nabla^2 + \gamma_\mu^2)\psi_\mu = 0$$

cross multiplying ψ's and subtracting gives:

$$(\gamma_\mu^2 - \gamma_\lambda^2)\psi_\lambda\psi_\mu = \psi_\mu \nabla^2 \psi_\lambda - \psi_\lambda \nabla^2 \psi_\mu$$

Integrating:

$$(\gamma_\mu^2 - \gamma_\lambda^2)\int da\, \psi_\lambda \psi_\mu = \int da \left[\psi_\mu \nabla^2 \psi_\lambda - \psi_\lambda \nabla^2 \psi_\mu\right]$$

Green's Thm

$$= \oint_C dl \left[\psi_\mu \frac{\partial \psi_\lambda}{\partial n} - \psi_\lambda \frac{\partial \psi_\mu}{\partial n}\right]$$

The RHS vanishes both for TM $(\psi|_S = 0)$ and TE $(\frac{\partial}{\partial n}\psi|_S = 0)$ Boundary Cond.

so, provided $\gamma_\mu^2 \neq \gamma_\lambda^2$, $\boxed{\int \psi_\mu \psi_\lambda \, da = 0}$

b) The transverse components of the fields are

TM: $E_t = \frac{ik}{\gamma^2}\vec{\nabla}_t E_z$, $H_t = \frac{1}{Z}\hat{z} \times \vec{E}_t$, $Z =$

TE: $E_t = -\frac{i\mu\omega}{\gamma^2}\hat{z} \times \vec{\nabla}_t H_z$, $H_t = \frac{1}{Z}\hat{z} \times \vec{E}_t$, $Z =$

$$\int E_{z\lambda} E_{t\mu}\, da = 0$$

For 2 TM modes

$$\int E_{t\lambda} E_{t\mu}\, da = -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \nabla_t E_{z\lambda} \cdot \nabla_t E_{z\mu}\, da$$

$$= -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \left[-\oint_S \cancelto{0}{E_{z\lambda} \frac{\partial E_{\mu}}{\partial n} dl} - \int_A E_{z\lambda} \nabla^2 E_{z\mu}\, da \right]$$

Using $\nabla_t^2 E_{z\mu} = -\gamma_\mu^2 E_{z\mu}$

$$\int E_{t\lambda} E_{t\mu}\, da = -\frac{k^2}{\gamma_\lambda^2} \int E_{z\lambda} E_{z\mu} = 0$$

given normalization

$$= -\frac{k^2}{\gamma_\lambda^2} \left(\frac{-\gamma_\lambda^2}{k_\lambda^2} \delta_{\lambda\mu} \right) = \boxed{\delta_{\lambda\mu}}$$

For TE modes

$$\int E_{t\mu} E_{t\lambda}\, da = -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int (\hat{z} \times \nabla H_{z\mu}) \cdot (\hat{z} \times \nabla_t H_{z\lambda})\, da$$

$$= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int \left[\nabla_t H_{z\lambda} \cdot \nabla_t H_{z\mu} - \cancelto{0}{(\hat{z} \cdot \nabla_t H)} \cancelto{0}{(\hat{z} \cdot \nabla_t H)} \right] da$$

$\hat{z} \perp \hat{z}$

as above,

$$= -\frac{\mu^2 \omega^2}{\gamma_\lambda^2} \int H_{z\lambda} H_{z\mu}\, da = -\frac{\mu^2 \omega^2}{\gamma_\lambda^2} \left(\frac{-\gamma_\lambda^2}{k^2} \delta_{\lambda\mu} \right)$$

$$= \boxed{\delta_{\lambda\mu}}$$

For TE & TM mode

$$\int E_{t\lambda} E_{t\mu}\, da = \frac{\mu\omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int \nabla_t E_{z\lambda} \cdot (\hat{z} \times \nabla_t H_{z\mu})\, da$$

$$= -\frac{\mu\omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int [\nabla_t E_{z\lambda} \times \nabla_t H_{z\mu}]\, da$$

$$= -\frac{\mu\omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int \nabla \times (E_{z\lambda} \nabla H_{z\mu})\, da$$

$$= -\frac{\mu\omega k}{\gamma_\mu^2 \gamma_\lambda^2} \oint E_{z\lambda} \nabla_t H_{z\mu}\, d\vec{l} = 0$$

So all TM Modes $\perp$ to TE modes.

for $\int H_{\lambda t} H_{\mu t}\, da$:

TE modes: and TM modes:

$$\int \left(\frac{\varepsilon\omega}{k} \hat{z} \times E_{z\lambda}\right)\left(\frac{\varepsilon\omega}{k} \hat{z} \times E_{z\mu}\right) da$$

$$= \frac{1}{Z_\lambda Z_\mu} \int (\hat{z} \times E_{t\lambda})(\hat{z} \times E_{t\mu})\, da$$

$$= \frac{1}{Z_\lambda Z_\mu} \int E_{\lambda t} \cdot E_{\mu t} - \cancelto{0}{(\hat{z} \cdot E_{t\mu})}\cancelto{0}{(\hat{z} \cdot E_{t\lambda})}\, da$$

$$= \frac{1}{Z_\lambda Z_\mu} \int \underset{\uparrow \text{ previous result}}{E_{\lambda t} E_{\mu t}}\, da = \boxed{\frac{1}{Z_\lambda Z_\mu} \cdot (\delta_{\lambda\mu})}$$

typo in exam

This result works for all types of modes because it only depends on $\int E_t E_t\, da$ which we showed were orthonormal previously

~~TE & TM modes~~

for $\quad \frac{1}{2}\int (\vec{E}_{t\lambda} \times H_{t\mu})\cdot\hat{z}\, da$

$$= \frac{1}{2Z_\mu}\int E_{t\lambda} \times (\hat{z} \times E_{t\mu})\cdot\hat{z}\, da$$

$$= \frac{1}{2Z_\mu}\int E_{t\lambda}\cdot E_{t\mu} - \cancelto{0}{(\hat{z}\cdot E_{t\mu})(\hat{z}\cdot E_{t\lambda})}\, da$$

$$= \boxed{\frac{1}{2Z_\mu}\,\delta_{\lambda\mu}}$$

2. For TE modes we have the relations:

$$\nabla^2 H_z + \gamma_1^2 H_z = 0 \qquad |x| < a$$
$$\nabla^2 H_z + \gamma_2^2 H_z = 0 \qquad |x| \geq a$$

where $\gamma_1^2 = \omega^2 \mu \varepsilon_1 - k^2 = \frac{n_1^2}{c^2} \omega^2 - k^2$

$\gamma_2^2 = \omega^2 \mu \varepsilon_2 - k^2 = \frac{n_2^2}{c^2} \omega^2 - k^2$

For the fields to be confined, must decay outside waveguide, so $\gamma_2^2 < 0$

We can match B.C. with either
$H_x \sim \sin(\gamma_1 x)$ (odd)
or $H_x \sim \cos(\gamma_1 x)$ (even)

The even modes are: (where $i\alpha = \gamma_2$)

$$H_x = \begin{cases} \frac{ik}{\gamma_1} \cdot H_1 \cos(\gamma_1 x) & |x| < a \\ \frac{ik}{\alpha} H^+ e^{-\alpha x} & x > a \\ \frac{-ik}{\alpha} H^- e^{\alpha x} & x < -a \end{cases}$$

$$H_z = \begin{cases} H_1 \sin(\gamma_1 x) & |x| < a \\ H^+ e^{-\alpha x} & x > a \\ H^- e^{+\alpha x} & x < -a \end{cases}$$

H_x, H_z continuous at $x = \pm a$:

$\cancel{H_1 \cos(\gamma_1 a) = \frac{\gamma_1}{\alpha} H_2 e^{-\alpha a}}$

$$H_1 \cos(\gamma_1 a) = \frac{\gamma_1}{\alpha} H_2 e^{-\alpha a} \qquad (1)$$

$$H_1 \cos(\gamma_1 a) = -\frac{\gamma_1}{\alpha} H_3 e^{-\alpha a} \qquad (2)$$

$$H_1 \sin(\gamma_1 a) = H_2 e^{-\alpha a} \qquad (3)$$

$$-H_1 \sin(\gamma_1 a) = H_3 e^{-\alpha a} \qquad (4)$$

Taking either (3)/(1) or (4)/(2) :

$$\underline{\tan(\gamma_1 a) = \alpha/\gamma_1} \quad \text{, even modes}$$

For the odd modes, sin & cosines are switched in the Fields, Doing the same steps then will give

$$\underline{\cot(\gamma_1 a) = -\alpha/\gamma_1} \quad \text{, odd modes}$$

These can be combined as

$$\tan\left(\gamma_1 x - p\frac{\pi}{2}\right) = \alpha/\gamma_1 \quad \text{, with } p \text{ even or odd int.}$$

$$\frac{\alpha}{\gamma_1} = \left[\frac{k^2 - n_2^2\frac{\omega^2}{c^2}}{n_1^2\frac{\omega^2}{c^2} - k^2}\right]^{1/2} =$$

$$= \left[\frac{(n_1^2 - n_2^2)\omega^2/c^2}{n_1^2\omega^2/c^2 - k^2} - 1\right]^{1/2} = \left[\frac{(n_1^2 - n_2^2)\omega^2/c^2}{\gamma_1^2} - 1\right]^{1/2}$$

but γ is wave vector component in $\hat{x}$, so

$$\gamma_1 = \frac{\omega^2 n_1^2}{c^2}\sin^2\theta$$

Then we have

$$\tan\left(\frac{\omega n_1}{c} a \sin\theta - \frac{p\pi}{2}\right) = \sqrt{\frac{n_1^2 - n_2^2}{n_1^2 \sin^2\theta} - 1}$$

The substitutions

$$V = ka\sqrt{\Delta}, \quad \xi = \frac{\sin\theta}{\sqrt{\Delta}}, \quad \Delta = \frac{n_1^2 - n_2^2}{n_1^2}$$

give $\boxed{\tan\left(V\xi - \frac{p\pi}{2}\right) = \sqrt{\frac{1}{\xi^2} - 1}}$

b) squaring both sides and rearranging gives:

$$1 + \tan^2\left(V\xi - p\frac{\pi}{2}\right) = 1/\xi^2$$

$$\Rightarrow \sec^2\left(V\xi - p\pi/2\right) = 1/\xi^2$$

$$\Rightarrow 1/\cos(--) = 1/\xi$$

$$\cancel{\Rightarrow 1 + \frac{1}{2}\left(V\xi - p\pi/2\right)^2 = \xi}$$

$$\cancel{1 + \frac{1}{2}V^2\xi^2 - V\frac{p\pi}{2}\xi + \left(p^2\pi^2/4\right) = 0}$$

$$\cancel{\xi^0 = \frac{1 + p^2\pi^2/4}{Vp\pi/2} \approx \frac{2}{Vp\pi} + \frac{p\pi}{2V}}$$

Now shift $\cos(x) \Rightarrow \sin(\pi/2 - x)$

$$\rightarrow \sin\left(\frac{(p+1)\pi}{2} - V\xi\right) = \xi$$

$$(p+1)\pi/2 - V\xi^0 = \xi^0$$

$$\underline{\xi^0 = \frac{\pi}{2} \cdot \frac{p+1}{V+1}}$$

Next term is $\sin x \sim x - \frac{x^3}{6}$

so $\xi^{(1)} = -\frac{1}{6}(\xi^0)^2 = -\frac{\pi^3 (p+1)^3}{48 (V+1)^3}$

$$\rightarrow \boxed{\xi \sim \frac{\pi (p+1)}{2 V+1} \cdot \left(1 - \frac{\pi^2}{24}\frac{(p+1)^2}{(V+1)^2}\right)}$$

We see that indeed $\xi \ll 1$, when $p \ll 1$ & $V \gg 1$, so the approximation is consistent.

c) For even mode

$$E_y = \begin{cases} -\frac{ik}{\gamma_1} z H_1 \cos(\gamma_1 x) & |x| < a \\ (ik/\alpha) z H^+ e^{-\alpha x} & x > a \\ -(ik/\alpha) z H^- e^{\alpha x} & x < -a \end{cases}$$

$W_e = 1/4\, \varepsilon\, |E_y|^2$

$W_m = \frac{\mu}{4}\left(|H_x|^2 + |H_z|^2\right)$

$$W_{e,\,core} = \frac{1}{4} \varepsilon H_1^2 \left(\frac{z^2 k^2}{\gamma_1^2}\right) \int_{-a}^{a} dx \cos^2(\gamma_1 x)$$

$$= \frac{1}{4} \varepsilon H_1^2 \left(\frac{z^2 k^2}{\gamma_1^2}\right) \left(\gamma_1 a + \sin(\gamma_1 a)\cos(\gamma_1 a)\right)$$

$$= \boxed{\frac{1}{4} \varepsilon H_1^2 \left(\frac{z^2 k^2}{\gamma^2}\right) \left(a + 2 \sin(2\gamma_1 a)\right)}$$

$$W_{e,clad} = \varepsilon/4 \left(\frac{z^2 k^2}{\alpha^2}\right) H^2 \left[\int_a^{\infty} e^{-2\alpha x} dx + \int_{-\infty}^{-a} e^{2\alpha x} dx\right]$$

$$= \varepsilon/4 \left(\frac{z^2 k^2}{\alpha^2}\right) \cdot 2 (H^+)^2 \int_a^{\infty} e^{-2\alpha x} dx$$

$$= \varepsilon/4 \cdot (H^+)^2 \left(\frac{z^2 k^2}{\alpha^2}\right) \cdot \left(\frac{1}{\alpha} e^{-2\alpha a}\right)$$

Use B.C. $\frac{\gamma_1}{\alpha} H^+ e^{-\alpha a} = H_1 \cos(\gamma_1 a) \rightarrow$

$$= \boxed{\varepsilon/4 \left(\frac{\alpha^2}{\gamma_1^2}\right) H_1^2 \cos^2(\gamma_1 a) \left(\frac{z^2 k^2}{\alpha^2}\right)}$$