Physics 176/276 - Quantitative Molecular Biology
Problem Set #2
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1 Random Walk and Arcsine Law

Consider a one-dimensional symmetric random walk. At each step, i, the position of the random walker is increased or decreased by one, $X_i = \pm 1$, with equal probability. The position of the walker after the nth step is therefore given by $S_n = X_1 + ... + X_n$. A typical path of a random walk can be illustrated as a graph with the number of steps on the abscissa and the actual position on the ordinate, see Fig. 1. All paths start at zero, unless noted differently.

a) What is the probability, u_{2n} that after $2n$ steps the random walk is exactly at its starting point, i.e. $S_{2n} = 0$?

b) Show that $u_{2n-2} = \gamma(n) u_{2n}$ and determine the proportionality factor $\gamma(n)$.

c) Calculate the number of paths, $N_{n,x}$ from the origin to the point (n, x), *i.e.* the number of paths which are at position x after n steps. What is the corresponding probability?

Consider two points A and B as in Fig. 1. A' shall be obtained by reflecting A with respect to the x-axis. The reflection principle states that the number of paths from A to B which touch or cross the x-axis equals the number of paths from A' to B, see Fig. 1.

d) The ballot theorem states that the probability that a path of length n from the point (0,0) to (n, x) never touches or crosses the x-axis ($S_1 > 0, ...S_n > 0$) is given by $\frac{x}{n}$.

(i) Explain why the number of paths from (0,0) to (n, x) above the x-axis is equal to the number of paths from $(1, 1)$ to (n, x) above the x-axis.

(ii) Use the reflection principle to explain why the number of such paths is equal to $N_{n-1,x-1} - N_{n-1,x+1}$. Employ the result from **c)** to simplify this expression and prove the ballot theorem.

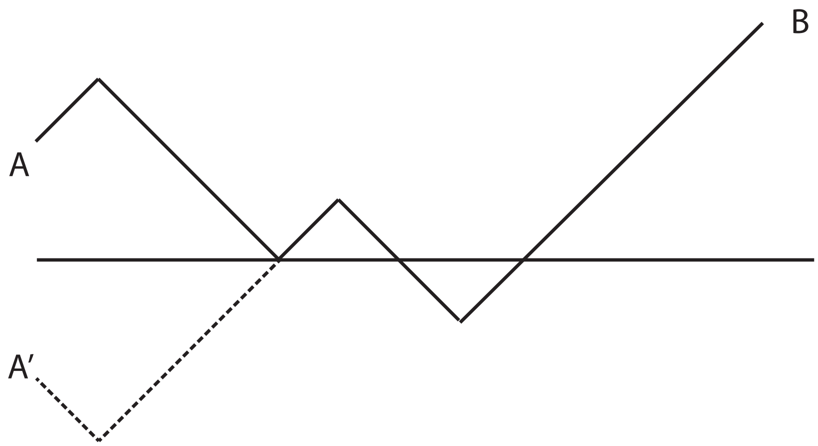

Figure 1: Illustration of the reflection principle.

e) The probability that no return to the origin occurs up to $2n$ is equivalent to u_{2n}, *i.e.* $P\{S_1 \neq 0, .., S_{2n} \neq 0\} = P\{S_{2n} = 0\} = u_{2n}$. In **(i)-(ii)** this result shall be proven.

(i) Explain why the statement above is equivalent to $P\{S_1 > 0, .., S_{2n} > 0\} = \frac{1}{2} u_{2n}$.

(ii) Explain why $P\{S_1 > 0, .., S_{2n} > 0\} = \sum_{r=1} P\{S_1 > 0, .., S_{2n-1} > 0, S_{2n} = 2r\}$ holds. Use the ballot theorem to evaluate the sum and finish the proof. Hint: The expression simplifies due to a telescoping sum.

(iii) Explain why $P\{S_1 \geq 0, .., S_{2n} \geq 0\} = u_{2n}$ holds. Use that the first step must be positive and then that staying above or touching the axis $x = 1$ is equivalent to staying above the axis $x = 0$.

f) The quantity f_{2n} is the probability that the random walker reaches its starting point for the first time after $2n$ steps. Use the result of **e)** to explain why this probability is given by $f_{2n} = u_{2n-2} - u_{2n}$.

g) Use previous results to show that $f_{2n} = \beta(n) u_{2n}$ holds and determine the proportionality factor $\beta(n)$. Employ **a)** to further evaluate the expression.

h) Consider a random walk with $2n$ steps. Express the probability, $\alpha_{2k,2n}$ that S_i be positive for exactly $2k$ steps in terms of u_{2k} and u_{2n-2k}. Values where $S_i = 0$ are counted as positive/negative if S_{i-1} was positive/negative. Hint: Draw the path of the random walk as a graph with the number of steps on the abscissa and S on the ordinate. Reshuffle the path by joining first all the positive segments and joining then the negative segments. Employ results in (e).

j) Calculate $\alpha_{2k,2n}$ for $n = 10$ and $k \in \{0, 1, ..., 10\}$.

k) Use Stirling's formula to approximate $\alpha_{2k,2n}$. Sketch the result together with the exact numbers calculated in **j)**.

l) Now, we can derive the quantity $P = \sum_{k<xn} \alpha_{2k,2n}$. Interpret this probability and calculate it by approximating the sum with an integral.