6.6

$$Q = C(p + im\omega q) \qquad P = (p - im\omega q)C$$

$$\Rightarrow \dot{Q} = C(\dot{p} + im\omega \dot{q}) \qquad \dot{P} = (\dot{p} - im\omega \dot{q})C$$

Harmonic oscilator: $H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 q^2$

$$\Rightarrow \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q, \quad \dot{q} = \frac{\partial H}{\partial p} = p/m$$

$$\dot{Q} = C(-m\omega^2 q + im\omega(p/m)) = i\omega C(im\omega q + p) = i\omega Q$$

$$\dot{P} = C(-m\omega^2 q - im\omega(p/m)) = -i\omega C(-im\omega q + p) = -i\omega P$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q}, \quad \dot{Q} = \frac{\partial \tilde{H}}{\partial P}$$

$$\tilde{H}_1 = -\int \dot{P}\, dQ = -\int(-i\omega P)\, dQ = i\omega P Q$$

$$\tilde{H}_2 = \int \dot{Q}\, dP = \int i\omega Q\, dP = i\omega P Q$$

$$\Rightarrow \underline{\underline{\tilde{H} = i\omega P Q}}$$

$$\tilde{H}(P, Q, t) = H(p, q, t) + \frac{\partial}{\partial t} F(q, Q, t)$$

$F = F(q, Q) \Rightarrow \tilde{H} = H$

or $S = S(q, P) \Rightarrow \tilde{H} = H$

$$\tilde{H}(P,Q) = H(p,q)$$

$$\tilde{H}(P,Q) = i\omega C^2 (p - im\omega q)(p + im\omega q)$$

$$= i\omega C^2 (p^2 + m^2\omega^2 q^2)$$

$$H(p,q) = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

$$\Rightarrow \frac{1}{2m} = i m\omega C^2 \Rightarrow \underline{C = (2m^2\omega i)^{-1/2}}$$

$$\oint (p\,dq - P\,dQ) = 0$$

$$\Rightarrow \oint (p\,dq + Q\,dP) = 0$$

<u>case II</u>

$$\Rightarrow p = \frac{\partial S}{\partial q}, \quad Q = \frac{\partial S}{\partial P}$$

$$p = \frac{\partial S}{\partial q} = im\omega q + \frac{P}{C} \Rightarrow S_1 = \frac{1}{2} im\omega q^2 + \frac{Pq}{C}$$

$$Q = \frac{\partial S}{\partial P} = C(p + im\omega q) = \overbrace{C(p - im\omega q)}^{P} + 2C im\omega q$$

$$\Rightarrow S_2 = \int [P + 2C im\omega q]\,dP = \frac{1}{2}P^2 + 2C im\omega q P$$

$$\Rightarrow \underline{S = \frac{1}{2}P^2 + qP/C + q^2/4C^2}$$

$$\dot{Q} = \partial\tilde{H}/\partial P = i\omega Q \quad \Rightarrow \quad \underline{Q = Q_0 e^{i\omega + \varphi_1}}$$

$$\dot{P} = -\partial\tilde{H}/\partial Q = -i\omega P \quad \Rightarrow \quad \underline{P = P_0 e^{-i\omega + \varphi_2}}$$

2.) (6.7)

a) q, P are independent variables (case 2)

Phase vol. conservation:

$$\underbrace{\oint (p\,dq - P\,dQ) = \oint (p\,dq + Q\,dP)}_{\text{after Legendre transformation}} = 0$$

Set equal to:

$$\oint (p\,dq + Q\,dP) = \oint dS_0 = 0$$

$$= \oint \left(\frac{\partial S_0}{\partial q} dq + \frac{\partial S_0}{\partial P} dP \right)$$

$$\boxed{p = \frac{\partial S_0}{\partial q} = P} \quad \boxed{Q = \frac{\partial S_0}{\partial P} = q}$$

b) Let $S' = S_0 + H\,dt = \sum_i q_i P_i + H\,dt$

$$\frac{\partial S'}{\partial q} = p = P + \frac{\partial H}{\partial q} dt$$

$$P = p - dt \frac{\partial H}{\partial q} = p + dt \left(\frac{dp}{dt} \right)$$

$$\boxed{P = p(t + dt)}$$

$$\frac{\partial S'}{\partial P} = Q = q + \frac{\partial H}{\partial P} dt$$

$$= q + dt \frac{dq}{dt}$$

Phys 200B (Theoretical Mechanics), Problem Set II

Fetter & Walecka, problem #6.8.

Done by Munirov V. R.

Throughout this problem we deal with cartesian coordinates q_i and function (we will use the dummy summation convention in this problem):

$$S_0(\mathbf{q}, \mathbf{P}) = \sum_i q_i P_i \equiv q_i P_i.$$

a) Infinitesimal translation in space

We are given the generating function

$$F(\mathbf{q}, \mathbf{P}) = S_0 + \mathbf{P} d\mathbf{r} = S_0 + P_j dr_j.$$

Since F is type II generating function the following is true:

$$\begin{aligned} Q_i &= \frac{\partial F}{\partial P_i}, \\ p_i &= \frac{\partial F}{\partial q_i}. \end{aligned}$$

Hence

$$\begin{aligned} Q_i &= \frac{\partial S_0}{\partial P_i} + \frac{\partial}{\partial P_i}(P_j dr_j) = \\ &= q_i + dr_j \delta_{ij} = q_i + dr_j, \\ p_i &= \frac{\partial S_0}{\partial q_i} + \frac{\partial}{\partial q_i}(P_j dr_j) = P_i. \end{aligned}$$

Thus we just proved that

$$\begin{aligned} Q_i &= q_i + dr_i, \\ P_i &= p_i. \end{aligned}$$

Or in vector notations

$$\begin{aligned} \mathbf{R} &= \mathbf{r} + d\mathbf{r}, \\ \mathbf{P} &= \mathbf{p}. \end{aligned}$$

Therefore we see that F generates infinitesimal translation in space, QED.

b) Infinitesimal rotation

In this part we have the generating function

$$F(\mathbf{q}, \mathbf{P}) = S_0 + \hat{\mathbf{n}}\mathbf{L}d\varphi.$$

Here $\hat{\mathbf{n}}$ is a unit vector in the direction of rotation, while $\mathbf{L} = [\mathbf{r} \times \mathbf{P}]$ is angular momentum. Since F is type II generating function we again have

$$Q_i = \frac{\partial F}{\partial P_i},$$
$$p_i = \frac{\partial F}{\partial q_i}.$$

Hence

$$Q_i = q_i + d\varphi \frac{\partial}{\partial P_i} (\hat{\mathbf{n}} [\mathbf{r} \times \mathbf{P}]),$$
$$p_i = P_i + d\varphi \frac{\partial}{\partial q_i} (\hat{\mathbf{n}} [\mathbf{r} \times \mathbf{P}]).$$

Now let us consider the derivative $\frac{\partial}{\partial q_i} (\hat{\mathbf{n}} [\mathbf{r} \times \mathbf{P}])$.

$$\frac{\partial}{\partial q_i} (\hat{\mathbf{n}} [\mathbf{r} \times \mathbf{P}]) = \frac{\partial}{\partial q_i} (n_l \varepsilon_{ljk} q_j P_k) = n_l \varepsilon_{ljk} \frac{\partial q_j}{\partial q_i} P_k =$$
$$= n_l \varepsilon_{ljk} \delta_{ij} P_k = n_l \varepsilon_{lik} P_k = \varepsilon_{ikl} P_k n_l = [\mathbf{P} \times \hat{\mathbf{n}}]_i,$$

where ε_{ljk} is the Levi-Civita symbol. Analogously, we can show that $\frac{\partial}{\partial q_i} (\hat{\mathbf{n}} [\mathbf{r} \times \mathbf{P}]) = [\hat{\mathbf{n}} \times \mathbf{r}]_i$. Thus we just proved that

$$Q_i = q_i + d\varphi [\hat{\mathbf{n}} \times \mathbf{r}]_i,$$
$$P_i = p_i + d\varphi [\hat{\mathbf{n}} \times \mathbf{p}]_i.$$

We changed $\mathbf{P}$ to $\mathbf{p}$ in the last formula, we can do this because this substitution introduces error of the order $O(d\varphi^2)$, but we are only interested in the terms up to $O(d\varphi)$.

If we write canonical transformations in vector form we get

$$\mathbf{R} = \mathbf{r} + d\varphi [\hat{\mathbf{n}} \times \mathbf{r}],$$
$$\mathbf{P} = \mathbf{p} + d\varphi [\hat{\mathbf{n}} \times \mathbf{p}].$$

Therefore we see that in this case F generates infinitesimal rotation, QED.

6.8) Prove $S_0 + \vec{p}\cdot d\vec{r}$ generates infinitesimal translations $d\vec{r}$ and $S_0 + \hat{n}\cdot\vec{L}\, d\phi$ generates infinitesimal rotations $d\phi$

As before, for $S_0 + \vec{p}\cdot d\vec{r}$

$$Q_\sigma = \frac{\partial S}{\partial P_\sigma} = q + d\hat{q}_\sigma$$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + 0$$

Which is the equations describing translation

Similarly for $S + \hat{n}\cdot\vec{L}\, d\phi$

$$Q_\sigma = \frac{\partial S}{\partial P_\sigma} = q + \frac{\partial L_\sigma}{\partial P_\sigma} d\phi = q + \cancel{p}\, q\, d\phi$$

$$P_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + \frac{\partial \vec{L}}{\partial q_\sigma} d\phi = P_\sigma + p \times \hat{q}\, d\phi$$

Which are the equations describing rotation

(6.17) $S(q_1, q_2, \ldots q_n, P_1, \ldots P_n, t) = \sum_\sigma q_\sigma P_\sigma + \varepsilon\, G(q_1, q_2, \ldots q_n, P_1, \ldots, P_n, t)$

a) Show resulting transform gives

$$P_\sigma = p_\sigma - \varepsilon \frac{\partial G}{\partial q_\sigma} + O(\varepsilon^2)$$
$$Q_\sigma = q_\sigma + \varepsilon \frac{\partial G}{\partial p_\sigma} + O(\varepsilon^2)$$

$$p_\sigma = P_\sigma + \varepsilon \frac{\partial G}{\partial Q_\sigma} \quad P_\sigma = p_\sigma - \varepsilon \frac{\partial G}{\partial Q_\sigma}$$

$$Q_\sigma = q_\sigma + \varepsilon \frac{\partial G}{\partial P_\sigma} = q_\sigma + \varepsilon \frac{\partial G}{\partial p_\sigma} \frac{\partial p_\sigma}{\partial P_\sigma} = q_\sigma + \varepsilon \frac{\partial G}{\partial p_\sigma}\left(1 + \varepsilon \frac{\partial G}{\partial Q_\sigma \partial P_\sigma}\right)$$

$$\boxed{Q_\sigma = q_\sigma + \varepsilon \frac{\partial G}{\partial p} + O(\varepsilon^2)}$$

$$P_\sigma = p_\sigma - \varepsilon \frac{\partial G}{\partial Q_\sigma} = p_\sigma - \varepsilon \frac{\partial G}{\partial q_\sigma} \frac{\partial q_\sigma}{\partial Q_\sigma} = p_\sigma - \varepsilon \frac{\partial G}{\partial q_\sigma}\left(1 - \varepsilon \frac{\partial^2 G}{\partial Q_\sigma \partial p_\sigma}\right)$$

$$\boxed{P_\sigma = p_\sigma - \varepsilon \frac{\partial G}{\partial q_\sigma} + O(\varepsilon^2)}$$

b) $F[q_1, \ldots, q_n, p_1, \ldots p_n]$ transforms to $F + dF$

when $dF = \varepsilon [F, G]_{PB}$

$$= \varepsilon \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right)$$

Clearly,

$$F \to F + \sum_\sigma \frac{\partial F}{\partial q_\sigma} dq + \sum_\sigma \frac{\partial F}{\partial p_\sigma} dp_\sigma$$

$$dq_\sigma = \varepsilon \frac{\partial G}{\partial p_\sigma}$$

$$dp_\sigma = -\varepsilon \frac{\partial G}{\partial q_\sigma}$$

So

$$F \to F + \sum_\sigma \left(\varepsilon \frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \varepsilon \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right)$$

$$= F + \varepsilon [F, G]_{PB}$$

QED

c) by B, $H \to H + \varepsilon [H, G]$

if G is a constant of motion, $[H, G] = 0$.

Since $\frac{dG}{dt} = 0 = \sum_\sigma \left(\frac{\partial G}{\partial p_\sigma} \dot{p}_\sigma + \frac{\partial G}{\partial q_\sigma} \dot{q}_\sigma \right) = \sum_\sigma \left(\frac{\partial G}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} + \frac{\partial G}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} \right)$

$$= [H, G]$$

So $H \to H$ QED

If G is total linear or angular momentum, H must be translationally or rotationally symmetric respectively (recalling 6.8)

$\boxed{\text{Phys 200 B, HW 2, \#6}}$: Find the freq. of a 3D HO w/ unequal spring constants using action angle variables.

① START WITH THE 3-D Hamiltonian:

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{2m} + \frac{1}{2}\left[K_1 q_1^2 + K_2 q_2^2 + K_3 q_3^2\right] \rightarrow \boxed{\text{leave}}$$

· Jacobi: Let's choose the momentum coords to be the $\int$'s of the motion: ③

and make a transformation where $(q,p) \rightarrow (\theta, I)$ via the

$\underbrace{\text{generating fn}}_{\text{abbreviated action}}$ $S(q,I) \rightarrow$ to cyclic coords $p = \frac{\partial S}{\partial q}$, $\theta = \frac{dS}{\partial I}$. → θ is coord. conjugate to I, the action variable.

· Then Hamilton's eqns show us that:

That is, we choose "angle-like" variables since the coords ↑ linearly w/o bound.

$$\dot{p} = -\frac{\partial H}{\partial q} \longrightarrow \dot{I} = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow I \text{ const}$$

$\boxed{\text{leave}} \leftarrow$

$$\dot{q} = \frac{\partial H}{\partial p} \longrightarrow \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I) \Rightarrow \theta = \omega(I)t + \theta_0$$

↳ fund. freq. of osc. → what we are looking for.

② The H-J eqn is then:

$$H\left(q, \frac{\partial S}{\partial q}\right) = \frac{1}{2m}\left[\left(\frac{\partial S_1}{\partial q_1}\right)^2 + \left(\frac{\partial S_2}{\partial q_2}\right)^2 + \left(\frac{\partial S_3}{\partial q_3}\right)^2\right] + \frac{1}{2}\left[K_1 q_1^2 + K_2 q_2^2 + K_3 q_3^2\right] = E = \text{const}$$

→ The H is separable: $H = f(1) + f(2) + f(3) = E$, $\left| f(1) = \frac{1}{2m}(p^2) + \frac{1}{2}K_1 q_1^2 \right|$, $f(2) = $ etc

→ each term independently is constant. $= E_1 = \text{const}$

and $E_1 + E_2 + E_3 = E$ → total E

③ NOW WE'LL FIND THE ACTION, I, WHICH IS AN $\int$ OF THE ORBIT, OR THE AREA IN PS TAKEN BY 1 PERIOD OF THE MOTION. → the "new" momentum! The action is a constant of the motion since the shape of the orbit it describes (a torus) is invariant.

$$I \equiv \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}(\text{PS AREA})$$

→ FOR EA. DOF WE GET AN ACTION VARIABLE: I_1, I_2, I_3

WE KNOW THAT:

→ $p = \pm\left[2m\left(E - \frac{1}{2}Kq^2\right)\right]^{1/2}$ → will take ⊕ since when we $\int$ around one cycle in PS $p\,dq$ is always > 0.

TO DESCRIBE A CYCLIC MOTION, θ WILL BE chosen so it relates to q by:

→ $q \equiv \left[\frac{2E}{K}\right]^{1/2} \sin\theta$ ← definition

→ $dq = \left[\frac{2E}{K}\right]^{1/2} \cos\theta \, d\theta$

Then $I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\oint \left[2m\left(E - \frac{1}{2}kq^2\right)\right]^{1/2} dq$

subst. for q & dq (see expansion below)

$$I = \frac{2E}{\pi}\left[\frac{m}{k}\right]^{1/2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$\Rightarrow \quad I = E\left(\frac{m}{k}\right)^{1/2} \quad \text{or} \quad E = I\left(\frac{k}{m}\right)^{1/2}$$

(4) Separability implies that the total Hamiltonian is additive:

$$H(I_1, I_2, I_3) = E_1 + E_2 + E_3 = I_1\left(\frac{k_1}{m}\right)^{1/2} + I_2\left(\frac{k_2}{m}\right)^{1/2} + I_3\left(\frac{k_3}{m}\right)^{1/2}$$

From the def. of ω above,

$$\omega(I) = \frac{\partial H(I)}{\partial I} = \left(\frac{k_1}{m}\right)^{1/2}, \quad \omega(I) = \text{fundamental freq. of osc.}$$

Then,

$$\boxed{\omega_1 = \left(\frac{k_1}{m}\right)^{1/2}, \quad \omega_2 = \left(\frac{k_2}{m}\right)^{1/2}, \quad \omega_3 = \left(\frac{k_3}{m}\right)^{1/2}}$$

So the idea is that this method allows us to compute frequencies (periods) of the individual indep. motions w/o solving the complete multi-dim mechanical problem.

$$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\oint \overbrace{(2m)^{1/2}\left(E - \frac{1}{2}kq^2\right)^{1/2}}^{p} dq = \frac{1}{2\pi}\oint (2m)^{1/2}\left(E - \frac{1}{2}k\overbrace{\left(\frac{2E}{k}\right)\sin^2\theta}^{q^2}\right)^{1/2}\overbrace{\left(\frac{2E}{k}\right)^{1/2}\cos\theta\, d\theta}^{dq}$$

$$= \frac{1}{2\pi}\left(\frac{4mE}{k}\right)^{1/2}\int_{-\pi/2}^{\pi/2} E^{1/2}\left(1 - \sin^2\theta\right)^{1/2}\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\left(\frac{4mE}{k}\right)^{1/2}\int_{-\pi/2}^{\pi/2} E^{1/2}\left(\underbrace{1 - \frac{1-\cos 2\theta}{2}}_{\frac{1+\cos 2\theta}{2} = \cos^2\theta}\right)^{1/2}\cos\theta\, d\theta$$

$$= \frac{2E}{\pi}\left(\frac{m}{k}\right)^{1/2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta = \frac{2E}{\pi}\left(\frac{m}{k}\right)^{1/2}\frac{\pi}{2} = E\left(\frac{m}{k}\right)^{1/2} = I$$

7.) $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$

For a slowly varying parameter $w(\tau_{long})$ where $\tau_{long} \gg \frac{2\pi}{w}$ $\quad \tau_{short} = \frac{2\pi}{w}$

solutions look like

$a \sin(wt)$

$L = \frac{1}{2} a^2 m (w)^2 \sin^2(wt) - \frac{1}{2} K a^2 \sin^2(wt)$

$L = \frac{1}{2} a^2 \sin^2(wt) \left[m w^2 - K \right]$

$$S = \int_0^{\tau_{long}} L(a, w, t) \, dt$$

$$\bar{S} = \frac{1}{\tau_{short}} \int_0^{\tau_{short}} S \, dt \qquad \bar{L} = \frac{a^2}{4} [m w^2 - K]$$

$$\bar{S} = \int_0^{\tau_{long}} \bar{L}(a, \Phi_t) \, dt \qquad \Phi_t = \frac{d\Phi}{dt} = w$$

Extremize the action

$$\delta \bar{S} = 0 = \int_0^{\tau_{long}} \left[\frac{\partial \bar{L}}{\partial a} \delta a + \underbrace{\frac{\partial \bar{L}}{\partial (\Phi_t)} \delta \Phi_t}_{\text{integrate by Parts}} \right] dt$$

$$\int_0^{\tau} \frac{\partial \bar{L}}{\partial (\Phi_t)} \delta \left(\frac{d\Phi}{dt} \right) = \int_0^{\tau} \frac{\partial \bar{L}}{\partial (\Phi_t)} \frac{d}{dt} (\delta \Phi)$$

$$= \cancel{\frac{\partial \bar{L}}{\partial (\Phi_t)} \delta \Phi \Big|_0^{\tau}}^{\,0} - \int_0^{\tau} \frac{d}{dt} \frac{\partial \bar{L}}{\partial (\Phi_t)} \delta \Phi \, dt$$

$$\delta \bar{S} = \int_0^{\tau} \left[\frac{\partial \bar{L}}{\partial a} \delta a - \frac{d}{dt} \frac{\partial L}{\partial \phi_t} \delta \phi \right] dt = 0$$

$$\frac{\partial \bar{L}}{\partial a} = 0 = a/2 \left[m w^2 - k \right] \implies w^2 = k/m$$

$$\frac{d}{dt} \frac{\partial L}{\partial \phi_t} = 0 \qquad \frac{d}{dt} \left[\frac{a^2 m w}{2} \right] = 0 \implies \frac{a^2 m w}{2} = \text{const.}$$

Relationship to WKB Approximation

Recall solution to WKB: $\ddot{x} + \frac{Q(t)}{\varepsilon^2} x = 0$

$$x(t) \sim \frac{1}{Q^{1/4}} e^{i/\varepsilon \int \sqrt{Q}\, dt}$$

For H.O. oscillator with slowly changing $w(\tau)$ where $\tau = \varepsilon t$ $\varepsilon \ll 1$

$$\frac{\partial^2 x}{\partial t^2} + w^2(\tau) x = 0 \implies \varepsilon^2 \frac{\partial^2 x}{\partial \tau^2} + w^2(\tau) x = 0$$

$$\frac{\partial^2 x}{\partial \tau^2} + \frac{w^2(\tau)}{\varepsilon^2} x = 0$$

So $Q = w^2$ $\qquad x(t) \sim \frac{1}{\sqrt{w}} e^{i w t/\varepsilon}$

Amplitude $\sim w^{-1/2}$

Back to Adiabatic H.O.

If $a^2 w = \text{const.}$

$$a_0^2 w_0^2 = a^2 w^2$$

$$\boxed{a = a \sqrt{\frac{w_0}{w}} \qquad a \sim w^{-1/2}}$$