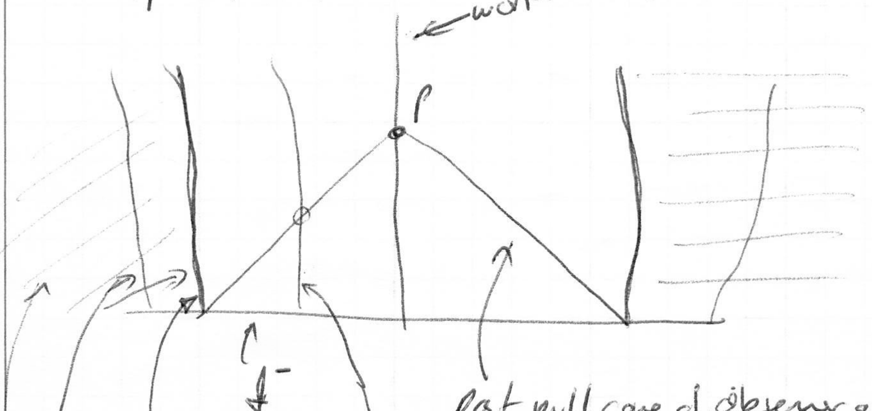
Horizons

~~Bear~~ deSitter future & past infinities are spacelike (contrast with Minkowski's timelike).
This gives rise to both particle & event horizons

Particle Horizon: defined for an observer at some event p in spacetime:



past null cone of observer at p

worldline of particle that may be observed at p (crosses past null cone at p).

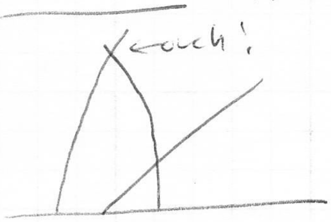
— not — (does not cross past null cone).

unobservable particles (at p)

particle horizon

~~So~~ The particle horizon separates the region of spacetime occupied by particles that may have been seen at p from those that can not be seen at p.

Particle Horizons are defined with respect to a "congruence of world-lines". Problem is

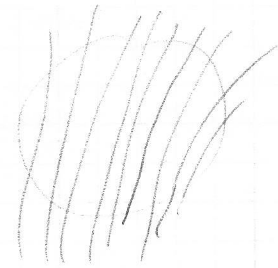


→ so we wouldn't be able to separate space into two pieces → no "horizon".

~~So we~~

Congruence is a set of ~~lines~~ curves such that each point p (in some ~~open~~ open set $U \subset \mathcal{M}$) is in exactly one ~~line~~ curve.

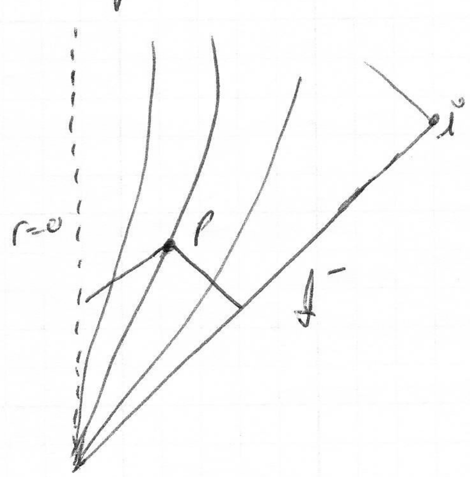
~~image~~



By definition, curves in a congruence do not cross.

<u>Examples:</u>

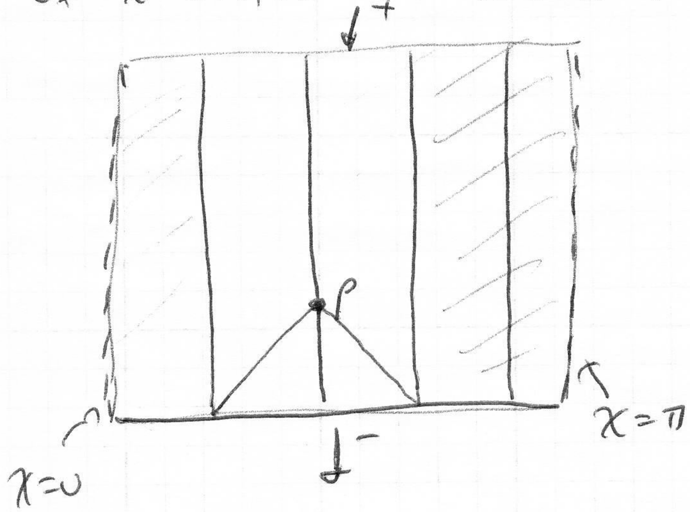
(i) There are no particle horizons in Minkowski space



every timelike geodesic crosses the past light cone of p.

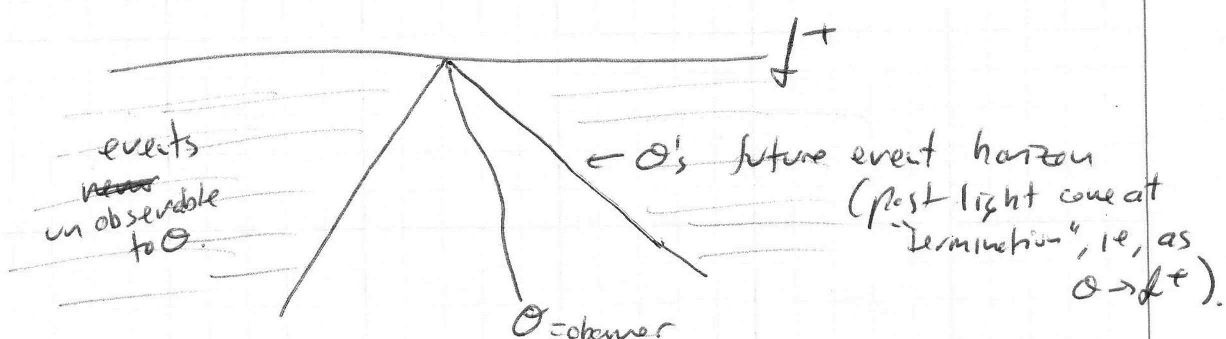
More generally, this is true if $\mathcal{I}^-$ is null.

(ii) de-Sitter does have particle horizons. Consider the congruence of $\chi = $ constant in the Penrose diagram

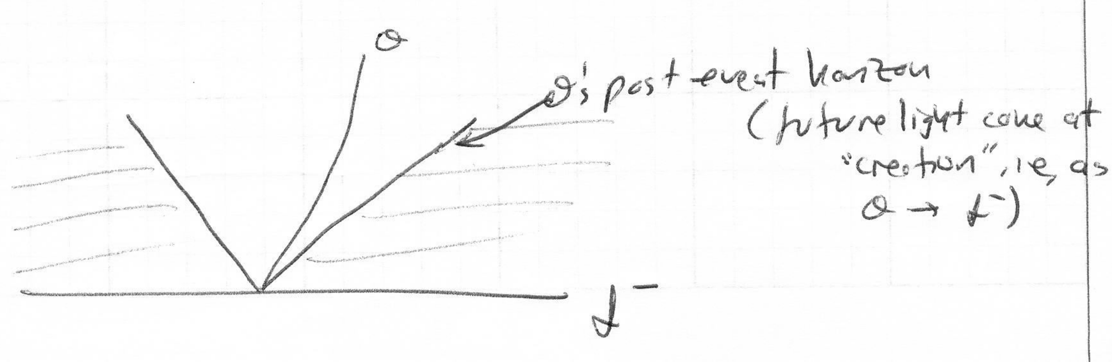


Event Horizons: ~~while~~ Particle horizon tells us which ~~observers~~ 'particles' may have been seen at p_0. We may ask instead ~~of~~ which particles may influence p at all throughout its whole history. That is, if the space time is expanding faster than the speed of light then if some observers far away from us, light sent to us will never reach us. We want to characterize this situation with an "event horizon" separating those events that can never influence us from those that can. ~~Clearl~~

Clearly, at any event p_0 the events inside its past light cone are observable, while those outside are not. The ~~future~~ <u>future event horizon</u> is the limiting light cone of an observer as it goes into future infinity, $\mathscr{I}^+$.

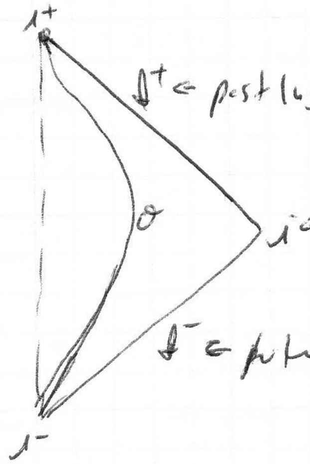


Similarly, <u>past event horizon</u> is defined to separate events that O will be able to influence in its history from those it won't:



Examples: (i) Minkowski space time.

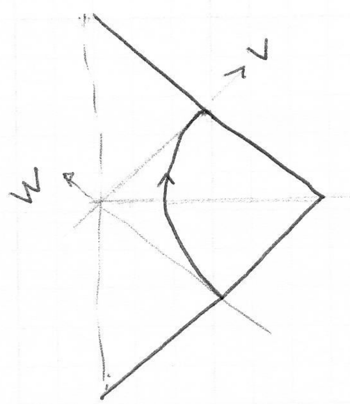
If O is a geodesic (free falling) observer $\Rightarrow$ no event horizons



$\mathcal{I}^+$ = past light cone of O at i^+

$\mathcal{I}^-$ = future light cone of O at i^-

(ii) Uniformly accelerated 'observer' in Minkowski space-time



picture is $r^2 - t^2 = a^2$

has both future and past event horizons.

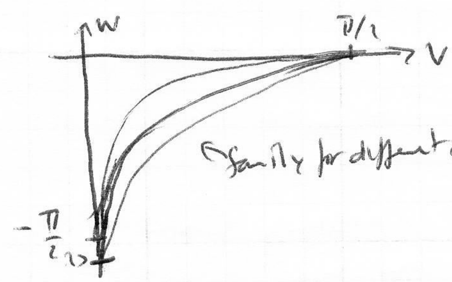
[Work it out: Recall $ds^2 = \frac{1}{\omega^2} ds_E^2$, see above,

and uniformly accelerated $\rightarrow$ $r^2 - t^2 = a^2$ or $vw = -a^2$

$\Rightarrow$ $\operatorname{tg} W \operatorname{tg} V = -a^2$ $\Rightarrow$ $\operatorname{tg}\left(\frac{1}{2}(T+R)\right) \operatorname{tg}\left(\frac{1}{2}(T-R)\right) = -a^2$

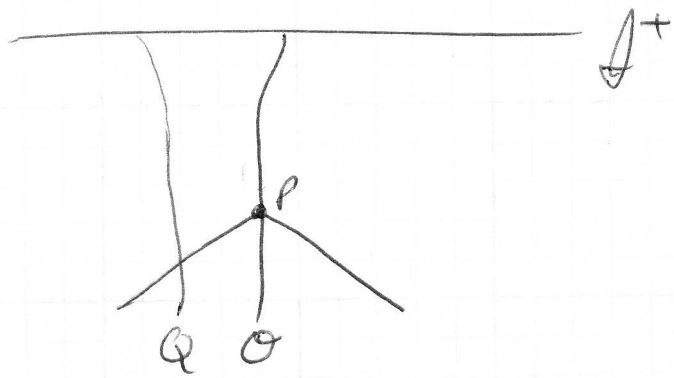
Here $ds_E^2 = -dT^2 + dR^2 + \sin^2 R \, d\Omega^2$ $\quad 0 \le R \le \pi \quad |T| + R < \pi$

Now $\operatorname{tg} W \operatorname{tg} V = -a^2$ is easy to draw

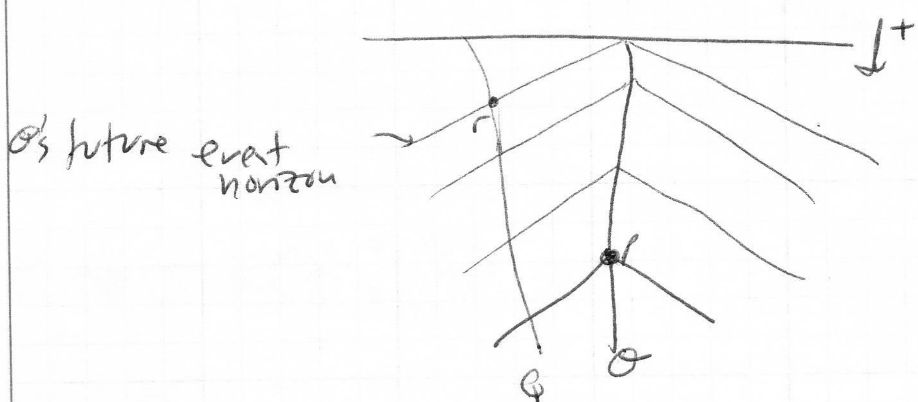


family for different a's.]

Consider (in de-Sitter space, or any space with $\mathscr{I}^+$ spacelike) an observer $\mathcal{O}$ and a particle worldline Q. Suppose Q intersects the past light cone of event p on $\mathcal{O}$:



$\Rightarrow$ Q is observable to $\mathcal{O}$ at any time after p:



But note, there is a point r on Q that lies on $\mathcal{O}$'s future event horizon $\Rightarrow$ Events on Q after r are NOT observable to $\mathcal{O}$.

Since r is seen at $\mathscr{I}^+$, it takes ∞ proper time from any event on $\mathcal{O}$ until observation of r on $\mathcal{O}$.

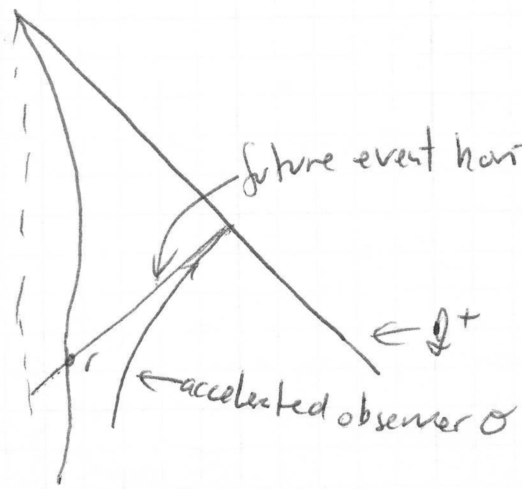
On Q, of course, it takes finite proper time from any past event to r.

$\boxed{\text{It takes an infinite time in } \mathcal{O} \text{ to see a finite part of } Q\text{'s history}}$

(eg, $\mathcal{O}$ observes infinite redshift of light from Q as it approaches r).

Likewise, Q will see ~~in~~finite history of $\mathcal{O}$ in infinite time.

Even in Minkowski space if we have non-geodesic observers:



which seems perfectly logical (~~as~~ redshifted light from ~~the~~ accelerated light source; light from r appears ∞ redshifted as $\mathcal{O} \to \mathscr{I}^+$).

anti-de Sitter space

($R < 0$ case) we now will have $\Lambda = \frac{1}{4} R < 0$.

Cosider hyperboloid

$$-U^2 - W^2 + X^2 + Y^2 + Z^2 = -\alpha^2$$

~~[illegible]~~ in flat R^5 with $--+++$ signature

$$ds^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$$

(compare signs with de-Sitter! both w^2 & a^2 (& dw^2) flipped).

Let

$$U = \alpha \sin t' \cosh \rho$$

$$W = \alpha \cos t' \cosh \rho$$

$$\left.\begin{aligned} x &= \alpha \sinh\rho \sin\theta \cos\phi \\ y &= \alpha \sinh\rho \sin\theta \sin\phi \end{aligned}\right\} \text{spherical coordinates in } R^3 \text{ with radius } \alpha \sinh\rho$$

$$z = \alpha \sinh\rho \cos\theta$$

This defines a map from the Hyperboloid, H^4, to R^5

$$\varphi : H^4 \to R^5$$

with induced metric $\varphi^* g$ (pullback of g).

Then $$ds^2 = \alpha^2 \left[-\cosh^2\rho \, dt'^2 + d\rho^2 + \sinh^2\rho (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Exercise: Check this

$$\Big[\frac{1}{\alpha^2} ds^2 = -dt'^2 \{\cosh^2\rho (\cos^2 t' + \sin^2 t') + d\rho^2 [-\sinh^2\rho (\sin^2 t' + \cos^2 t') + \cosh^2\rho (\cos^2\theta + \sin^2\theta (s_\phi^2 + c_\phi^2)$$

$$+ \sinh^2\rho \, d\theta^2 [\sin^2\theta + \cos^2\theta (s_\phi^2 + c_\phi^2)] + \sin^2\theta \, d\phi^2 \Big] \checkmark$$

Note that with $\rho \geq 0$ a radius-like coordinate, the ~~spa~~ $t' =$ constant sections are R^3 (topologically).

But for ρ, θ, ϕ fixed, t' lines are periodic $t' \to t' + 2\pi$

$\to$ Space has closed timelike curves (a no-no). (maybe... see later, causality).

Another coordinate system:

$$U = \alpha \sin t$$

$$V = \alpha \cos t \cosh r$$

$$x = \alpha \cos t \sinh r \sin\theta \cos\varphi$$
$$y = \alpha \cos t \sinh r \sin\theta \sin\varphi$$
$$z = \alpha \cos t \sinh r \cos\theta$$

Now $\varphi^* g$ is

$$\Big[\frac{1}{\alpha^2} ds^2 = \left(-\cos^2 t - \sin^2 t\left(\cosh^2 r - \sinh^2 r(\cos^2\theta + \sin^2\theta)\right)\right) dt^2$$
$$+ \left(\cos^2 t\left(-\sinh^2 r + \cosh^2 r(-)\right)\right) dr^2 + \cos^2 t \sinh^2 r \,(\sin^2\theta + \cos^2\theta)\, d\theta^2 -$$

$$\frac{1}{\alpha^2} ds^2 = -dt^2 + \cos^2 t \left[dr^2 + \sinh^2 r \, d\Omega_2^2 \right]$$

As we'll see this system has simple geodesics: $(r, \theta, \varphi) =$ constant. So these lines are orthogonal to $t =$ constant surface.

But note that at $t = \pm\frac{1}{2}\pi$ there are singularities. Clearly these are only coordinate singularities, but this frame can only be used for one piece of the space.

So the space described so far is one with topology $S^1 \times R^3$.

We take de-Sitter space to be the universal covering space of this, meaning, take $t' \in [-\infty, \infty]$ and keep the metric as above (the embedding no longer makes sense).

Structure at infinity and Penrose diagram: Let's define (similar to the de-Sitter case)

$$\cos\chi = \frac{1}{\cosh\rho}$$

[so

$$\cancel{d\rho^2} = \sinh\rho \, d\rho = \frac{\sin\chi}{\cos^2\chi} d\chi$$

$$\Rightarrow (-1+\cosh^2\rho) d\rho^2 = \frac{\sin^2\chi}{\cos^4\chi} d\chi^2 \qquad -1 + \frac{1}{\cos^2\chi} = \frac{1-\cos^2\chi}{\cos^2\chi} = \tan^2\chi$$

$$\Rightarrow d\rho^2 = \frac{\cos^2\chi}{\sin^2\chi} \frac{\sin^2\chi}{\cos^4\chi} d\chi^2 = \frac{1}{\cos^2\chi} d\chi^2$$

and

$$ds^2 = \alpha^2 \left[-\frac{1}{\cos^2\chi} dt'^2 + \frac{1}{\cos^2\chi} d\chi^2 + \tan^2\chi \, d\Omega_2^2 \right]$$]

which has $\chi \in [0, \frac{\pi}{2})$ and

$$ds^2 = \frac{\alpha^2}{\cos^2\chi} \left[-dt'^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2 \right] = \frac{\alpha^2}{\cos^2\chi} d\bar{s}^2$$

recognizing again the metric of Einstein-static universe.

Note that with $t' \in (-\infty, \infty)$ but $\chi \in [0, \frac{\pi}{2}]$ anti-de Sitter is conformally related to half of the Einstein-static universe (the $\chi \in [\frac{\pi}{2}, \pi]$ is missing).

Geodesics in anti de Sitter (not for class)

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Find geodesics? Start $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu})$

$\Gamma_{\rho\rho\rho} = 0$

$\Gamma_{tt\rho} = \Gamma_{t\rho t} = -\frac{1}{2}(\cosh^2\rho)_{,\rho} = -\cosh\rho\sinh\rho \qquad \Rightarrow \Gamma^t{}_{t\rho} = \Gamma^t{}_{\rho t} = \frac{\sinh\rho}{\cosh\rho}$

$\Gamma_{\rho tt} = \cosh\rho\sinh\rho \qquad \Rightarrow \Gamma^\rho{}_{tt} = \cosh\rho\sinh\rho$

$\Gamma_{\theta\theta\rho} = \Gamma_{\theta\rho\theta} = \frac{1}{2}(\sinh^2\rho)_{,\rho} = \cosh\rho\sinh\rho \qquad \Rightarrow \Gamma^\theta{}_{\theta\rho} = \Gamma^\theta{}_{\rho\theta} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\theta\theta} = -\cosh\rho\sinh\rho \qquad \Rightarrow \Gamma^\rho{}_{\theta\theta} = -\cosh\rho\sinh\rho$

Ignore ϕ: always look at $\phi = $ const plane (could have done that w/ θ?).
Then

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

To be sure, let's keep ϕ:

$\Gamma_{\phi\phi\rho} = \Gamma_{\phi\rho\phi} = \frac{1}{2}\sin^2\theta\; 2\sinh\rho\cosh\rho = \sin^2\theta\sinh\rho\cosh\rho \qquad \Gamma^\phi{}_{\rho\phi} = \Gamma^\phi{}_{\phi\rho} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho \qquad \Gamma^\rho{}_{\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho$

$\Gamma_{\phi\phi\theta} = \sin\theta\cos\theta\sinh^2\rho \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

$\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta\sinh^2\rho \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$

$\cancel{\frac{d^2t}{d\tau^2} +}$ Conserved quantities $\quad g_{t\mu}\frac{dx^\mu}{d\tau} = -\cosh^2\rho\frac{dt}{d\tau} = T$

but it works if $\Phi = 0$ see below $\leftarrow$ not true, since $g_{\phi\phi}$ depends on θ $\rightarrow$ $\boxed{g_{\theta\mu}\frac{dx^\mu}{d\tau} = \sinh^2\rho\frac{d\theta}{d\tau} = \Theta}$?? $\quad g_{\phi\mu}\frac{dx^\mu}{d\tau} = \sin^2\theta\sinh^2\rho\frac{d\phi}{d\tau} = \Phi$

So

$$\frac{d^2\rho}{d\tau^2} + \cosh\rho\sinh\rho\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\theta}{d\tau}\right)^2 - \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] = 0$$

$$\frac{d^2\rho}{d\tau^2} + \frac{\sinh\rho}{\cosh^3\rho}T^2 - \frac{\cosh\rho}{\sinh^3\rho}\Theta^2 - \frac{\cosh\rho}{\sin^2\theta\sinh^3\rho}\Phi^2 = 0$$

This equation has a 1^{st} integral that is easy to find. But, even easier, use τ = proper time so

$$g_{\mu\nu} U^\mu U^\nu = -1$$

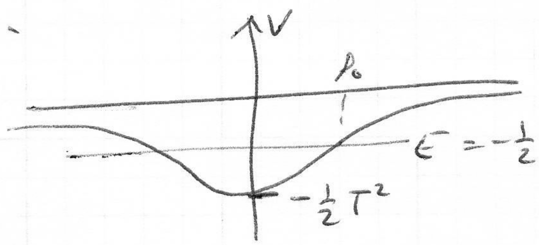
or

$$\left(\frac{d\rho}{d\tau}\right)^2 - \frac{T^2}{\cosh^2\rho} + \frac{\Theta^2}{\sinh^2\rho} + \frac{\Phi^2}{\sinh^2\rho \sin^2\theta} = -1$$

Look for solutions with $\Theta = \Phi = 0$. Then

$$\frac{d\rho}{d\tau} = \sqrt{\frac{T^2}{\cosh^2\rho} - 1} \qquad (\star)$$

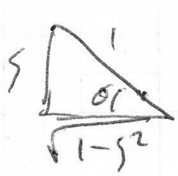
This is like motion in a potential $-\frac{T^2}{2\cosh^2\rho}$ with total energy $-\frac{1}{2}$.



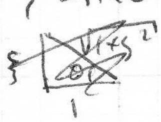
And clearly there are "bound state" solutions, with turning points at $\cosh^2\rho_0 = T^2$ or $\rho_0 = \operatorname{arccosh} T$. Now, it is easy to integrate $(\star)$

$$\int \frac{d\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}} = \int \frac{\cosh\rho \, d\rho}{\sqrt{T^2 - \cosh^2\rho}} = \int \frac{d\sinh\rho}{\sqrt{T^2 - 1 - \sinh^2\rho}}$$

Let $\sinh\rho = \sqrt{T^2-1}\,\xi \Rightarrow \; = \int \frac{d\xi}{\sqrt{1-\xi^2}}$



$\xi = \sin\theta$



$$\Rightarrow \int \frac{\cos\theta \, d\theta}{\cos\theta} = \theta = \arcsin\xi = \operatorname{arctg}\frac{\xi}{\sqrt{1-\xi^2}}$$

$$= \arcsin\left(\frac{\sinh\rho}{\sqrt{T^2-1}}\right)$$

or $\operatorname{arctg}\left(\frac{\sinh\rho}{\sqrt{T^2-1-\sinh^2\rho}}\right) = \operatorname{arctg}\left(\frac{\tanh\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}}\right)$

Then $t(\tau)$ is obtained from

$$\frac{dt}{d\tau} = -\frac{T}{\cosh^2\rho}$$

For this we need

$$\boxed{\sin\tau = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

or $\quad (T^2-1)\sin^2\tau = \sinh^2\rho = \cosh^2\rho - 1$

so

$$\frac{dt}{d\tau} = -\frac{T}{1+(T^2-1)\sin^2\tau}$$

We need

$$\int \frac{d\tau}{1+k^2\sin^2\tau} = \frac{\mathrm{tg}^{-1}\left[\sqrt{1+k^2}\,\mathrm{tg}\,\tau\right]}{\sqrt{1+k^2}} \qquad \text{(mathematica)}$$

so

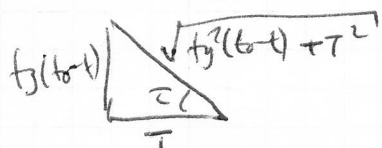
$$-\frac{(t-t_0)}{T} = \frac{1}{\sqrt{1+(T^2-1)}}\,\mathrm{arctg}\left[T\,\mathrm{tg}\,\tau\right]$$

or

$$\boxed{-\mathrm{tg}(t-t_0) = T\,\mathrm{tg}\,\tau}$$

(The sign is because ~~$T(o)$ is proper distance, not proper time~~).

We can also obtain the trajectory. Since $\mathrm{tg}\,\tau = \frac{1}{T}\mathrm{tg}(t_0-t)$



$$\Rightarrow \sin\tau = \frac{\mathrm{tg}(t_0-t)}{\sqrt{\mathrm{tg}^2(t_0-t)+T^2}} = \frac{1}{\sqrt{1+T^2\,\mathrm{ctg}^2(t_0-t)}}$$

so

$$\boxed{\frac{1}{\sqrt{1+T^2\,\mathrm{ctg}^2(t_0-t)}} = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

In all these it's worth remembering $T = -\cosh\rho_0$

Check the θ piece (recall $g = g(\rho)$ so we were justified in using $g_{\sigma\mu}\frac{dx^\mu}{d\tau} = \text{constant}$)

Now

$$\frac{d^2\theta}{d\tau^2} + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

But if $\phi = \text{constant}$ ($\Phi = 0$) we have

$$\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} = 0$$

Now, check: $\frac{d\theta}{d\tau} = \frac{\Theta}{\sinh^2\rho}$ gives $\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = -2\frac{\cosh\rho}{\sinh^3\rho}\Theta\frac{d\rho}{d\tau}$

while the 2nd term is $2\frac{\cosh}{\sinh\rho}\frac{d\rho}{d\tau}\frac{\Theta}{\sinh^2\rho}$

so they cancel ✓

Connecting both coordinates systems:

in (t, r, θ, ϕ) system geodesics are $r, \theta, \phi = \text{constant}$ with $r = \rho_0$

Comparing both systems:

u: $\sin t' \cosh \rho = \sin t$

v: $\cos t' \cosh \rho = \cos t \cosh r$

z: $\boxed{\sinh \rho = \cos t \sinh r}$

u/v: $\boxed{\operatorname{tg} t' = \frac{1}{\cosh r} \operatorname{tg} t}$

(θ, φ remain the same)

Geodesics
$$\begin{cases} \sinh \rho = \sinh \rho_0 \sin \tau \\ \operatorname{tg}(t' - t_0) = \cosh \rho_0 \operatorname{tg} \tau \end{cases}$$

Going to other system:

$$\sinh \rho_0 \sin \tau = \cos t \sinh r$$

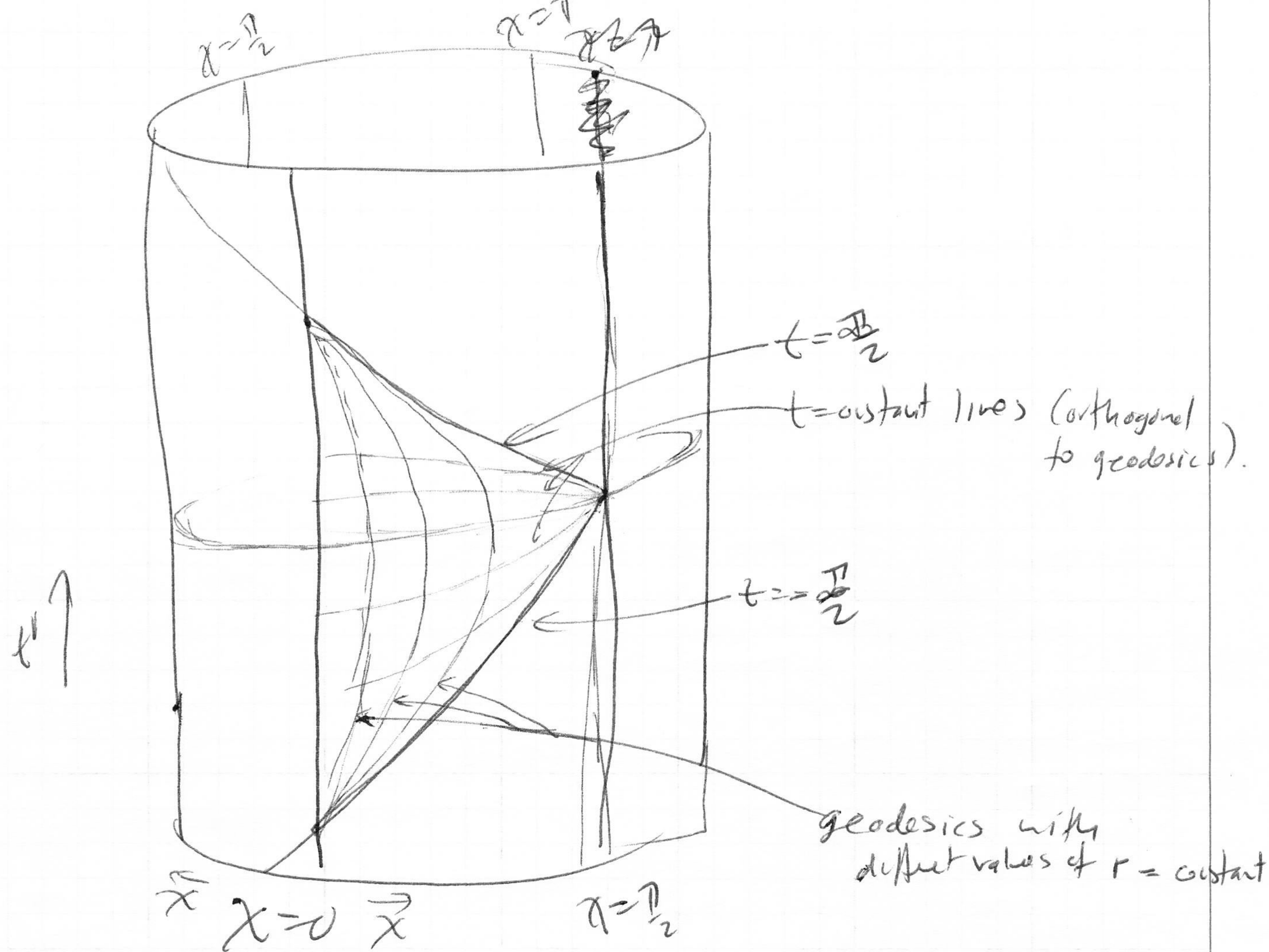
$\tau \to \tau + \frac{\pi}{2}$ $\quad$ ~~st~~ $\rho_0 = r$

and then

$$\operatorname{tg}(t' - t_0) = \cosh \rho_0 \operatorname{tg}\left(\tau + \frac{\pi}{2}\right) = \cosh \rho_0 \frac{\cos \tau}{-\sin \tau}$$

$$\operatorname{ctg}(t' - t_0) = -\frac{1}{\cosh \rho_0} \operatorname{tg} \tau$$

$\Rightarrow t_0 = \frac{\pi}{2}$ $\quad \Rightarrow$ it works



(The lines $t = \pm\frac{\pi}{2}$ are easy to understand. Since

$$\sin t = \sin t' \cosh\rho$$

we have

$$\pm 1 = \sin t' \cosh\rho = \sin t' \frac{1}{\cos\chi}$$

where we introduced the variable χ for the conformal mapping

$\Rightarrow$

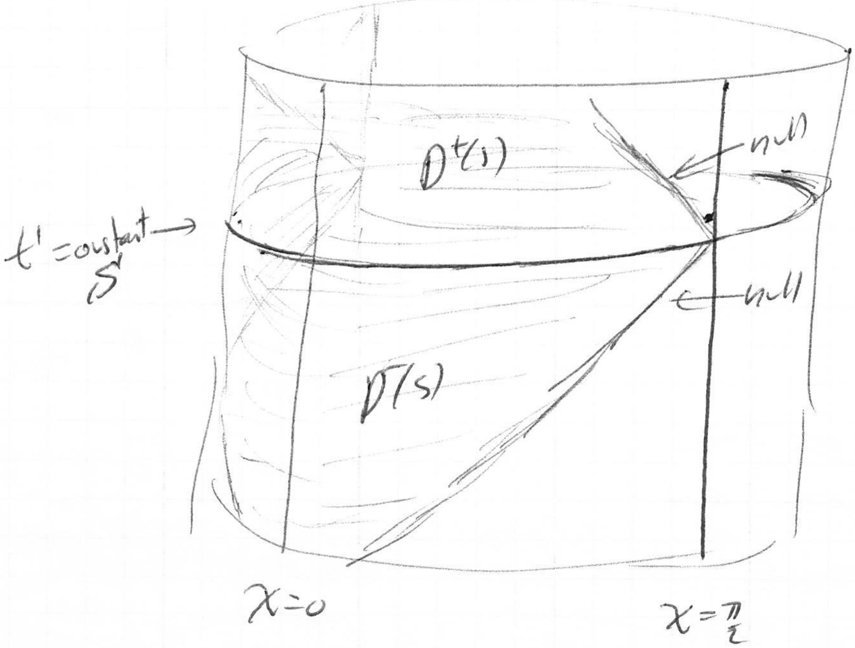
$$\cos\chi = \pm \sin t'$$

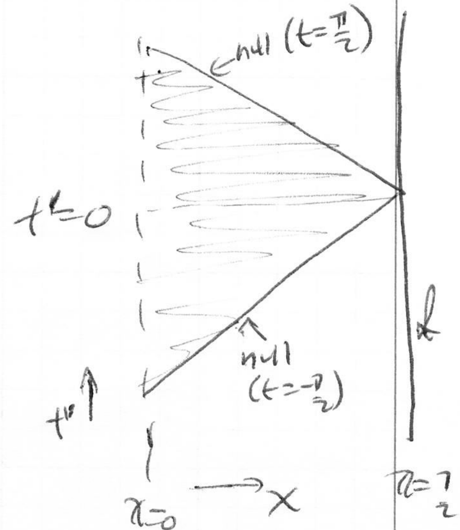
or $\chi = \frac{\pi}{2} \pm t'$).

Note that the apparent singularity in t, r, θ, φ coord's is related to convergence of geodesics.

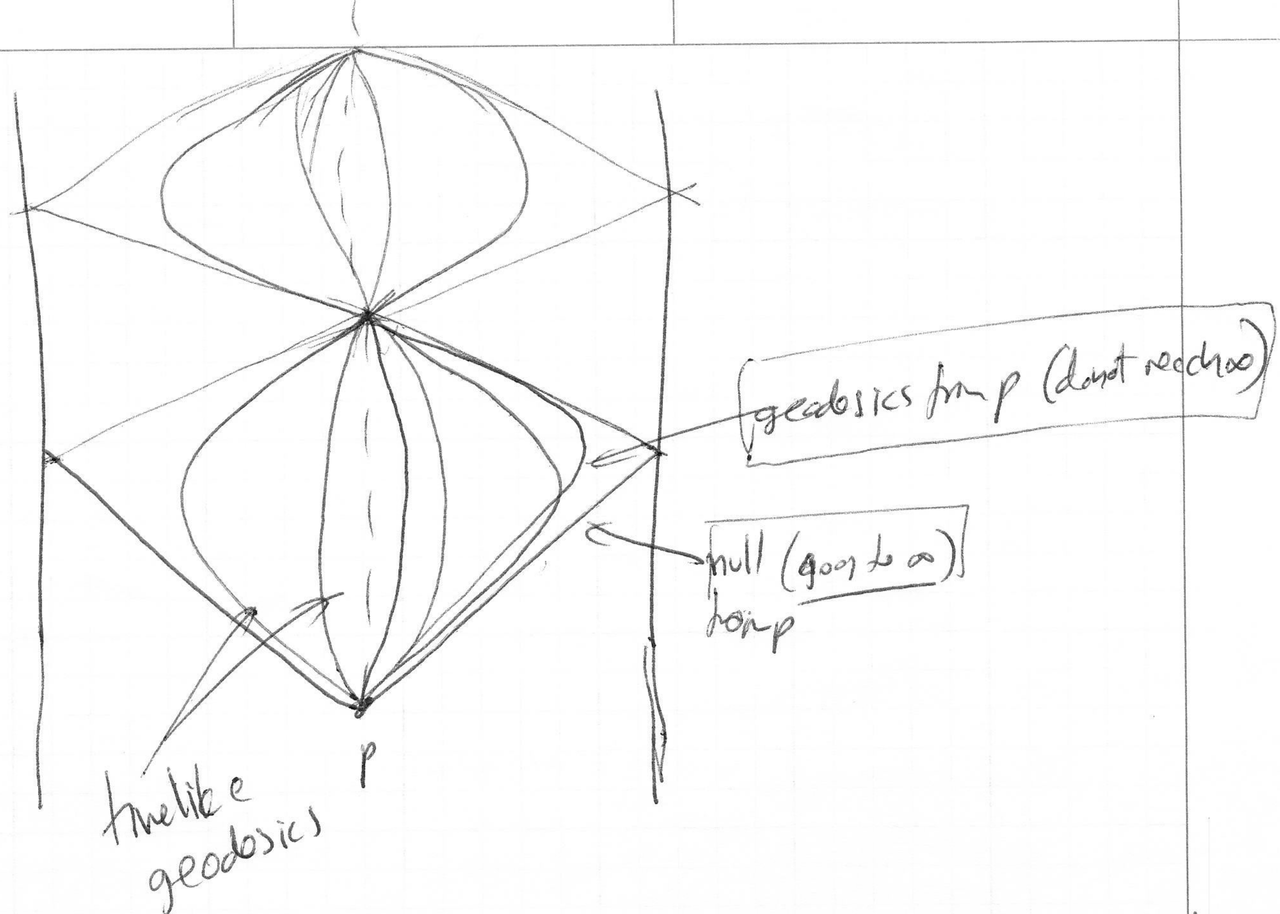
Causal structure of anti-de Sitter space:

$\boxed{\text{NO CAUCHY SURFACE}}$!



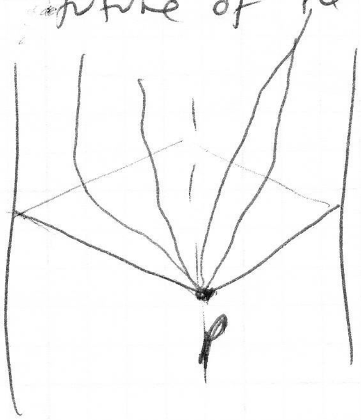


Evident ⇒ information flows in/out from boundary at ∞.

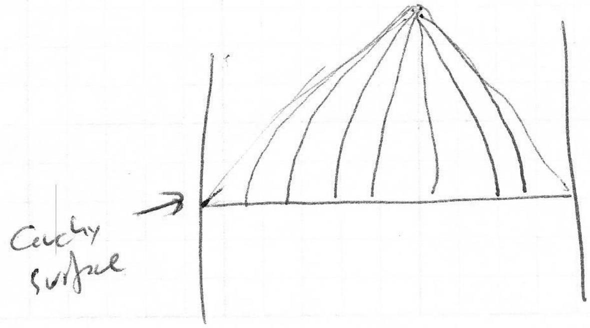


timelike geodesics from p are confined to infinite sequence of diamonds

But there are timelike curves (non-geodesic) from p that can reach any point future of the null-cone from p.



Also



every point in $D^+(S)$ can be reached by a unique geodesic from S, normal to S.