Redshift and Distances (a la Carroll).

FRW has no timelike killing vector (the metric depends explicitly on t). But there is a Killing tensor. Let $U^\mu = (1, \vec{0})$, that is, $\vec{U}$ is the 4-vector tangent to isotropic observers in comoving coordinates (ie, their 4-velocity). Then let

$$K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu)$$

where $g_{\mu\nu}$ is the FRW metric with scale factor a.

Then $\nabla_{(\sigma} K_{\mu\nu)} = 0$ (see next page for check of this).

Now, take V^μ to be a tangent to a particle trajectory $V^\mu = \frac{dx^\mu}{d\lambda}$. This is the 4-velocity for a massive particle, or the wave 4-vector for a massless particle.

Along the geodesic

$$K^2 \equiv K_{\mu\nu} V^\mu V^\nu$$

is constant. Then, for a massive particle $V_\mu V^\mu = -1$

$$\frac{K^2}{a^2} = V_\mu V^\mu + (U_\mu V^\mu)^2$$

$$= -1 + (V^0)^2$$

But $V_\mu V^\mu = -1 \Rightarrow (V^0)^2 - g_{ij} V^i V^j = 1$ so

$$|\vec{V}|^2 \equiv g_{ij} V^i V^j = \frac{K^2}{a^2}$$

For massless particles $V_\mu V^\mu = 0$ and $U_\mu V^\mu = -\omega$

so $$\frac{K^2}{a^2} = \omega^2 \quad \text{or} \quad \omega = \frac{K}{a}$$

Check the $K_{(\mu\nu;\sigma)} = 0$

$$K_{\mu\nu;\sigma} = K_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma} K_{\lambda\nu} - \Gamma^{\lambda}_{\nu\sigma} K_{\mu\lambda}$$

Check

$$K_{00;0} = \cancel{K_{00,0}} - 2\Gamma^{\lambda}_{00} K_{\lambda 0} = 0$$

$$K_{00;i} = \cancel{K_{00,i}} - 2\Gamma^{\lambda}_{0i} K_{\lambda 0} = 0 \qquad (K_{i0} = 0 = K_{00})$$

$$K_{i0;0} = \cancel{K_{i0,0}} - \Gamma^{\lambda}_{i0} \cancel{K_{\lambda 0}}^{0} - \cancel{\Gamma^{\lambda}_{00}}^{0} K_{i\lambda}$$

$$K_{ij;0} = K_{ij,0} - \Gamma^{\lambda}_{i0} K_{\lambda j} - \Gamma^{\lambda}_{j0} K_{\lambda i}$$

Here $K_{ij} = a^2 g_{ij} = a^4 h_{ij}$

where h_{ij} is the metric on the hypersurface of curvature k.

so $K_{ij,0} = 4\left(\frac{\dot a}{a}\right) K_{ij}$

Also $\Gamma^{\lambda}_{i0} K_{\lambda j} = \Gamma^{l}_{i0} K_{lj} = \frac{\dot a}{a} K_{ij}$

so $K_{ij;0} = 2\left(\frac{\dot a}{a}\right) K_{ij}$

$$K_{i0;j} = \cancel{K_{i0,j}}^{0} - \cancel{\Gamma^{\lambda}_{ij} K_{\lambda 0}}^{0} - \Gamma^{\lambda}_{0j} K_{\lambda i} = -\left(\frac{\dot a}{a}\right) K_{ij}$$

So $K_{(ij;0)} = (2-1-1)\left(\frac{\dot a}{a}\right) K_{ij} = 0$

Finally

$$K_{ij;l} = K_{ij,l} - \Gamma^{\lambda}_{il} K_{\lambda j} - \Gamma^{\lambda}_{jl} K_{\lambda i}$$

$$K_{ij,l} = a^4 h_{ij,l}$$

Recall $\Gamma^{m}_{il} = \frac{1}{2} g^{mp} (g_{ip,l} + g_{lp,i} - g_{il,p})$

$$= \frac{1}{2} h^{mn} (h_{in,l} + h_{ln,i} - g_{il,n})$$

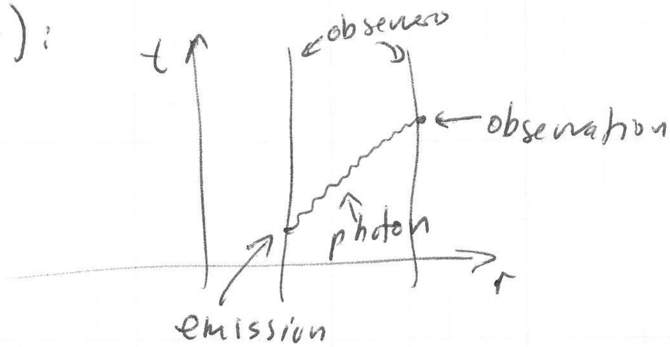
so $\Gamma^{m}_{il} K_{mj} = \frac{1}{2} a^4 h_{mj} \Gamma^{m}_{il} = \frac{1}{2} a^4 (h_{ij,l} + h_{lj,i} - h_{il,j})$

so

$$K_{ij;l} = a^4 \left[h_{ij,l} - \tfrac{1}{2}(h_{ij,l} + h_{lj,i} - h_{il,j}) - \tfrac{1}{2}(h_{ij,l} + h_{il,j} - h_{jl,i}) \right]$$

$= 0$ even before symmetrizing.

Consider two comoving observers (both have $\vec{U}$ as tangent vector):



then, since $K =$ constant

$$\omega_{em} a_{em} = \omega_{obs} a_{obs}$$

or, since $\omega_{em} = \frac{1}{\lambda_{em}}$

$$\boxed{\frac{\lambda_{em}}{a_{em}} = \frac{\lambda_{obs}}{a_{obs}}}$$

That is $\lambda_{obs} = \frac{a_{obs}}{a_{em}} \lambda_{em}$

and since a is increasing $\lambda_{obs} > \lambda_{em} \Rightarrow$ redshift.

Define the redshift as

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{a_{obs}}{a_{em}} - 1$$

or

$$\boxed{\frac{a_{em}}{a_{obs}} = \frac{1}{1+z}}$$

$\Rightarrow$ Measuring z gives the factor by which the universe has grown since emission as $1+z$.

The <u>instantaneous physical distance</u> $d_p(t)$ between isotropic observers is the distance between them on a common $t=$constant surface. Recall

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)\, d\Omega_2^2\right]$$

where $S_{+1} = \sin\chi$ $\quad S_0 = \chi \quad S_{-1} = \sinh\chi$. Then the distance between an isotropic observer at $\chi = 0$ and one at χ is

$$d_p(t) = a(t)\chi$$

Taking $\frac{d}{dt}$, we have $\quad \dot{d}_p = \dot{a}\chi = \dot{a}\left(\frac{d_p}{a}\right) = \left(\frac{\dot{a}}{a}\right) d_p$

So, interpreting $\dot{d}_p = V_p$ as the "velocity of separation" of the isotropic observers (really, the rate at which space is growing between them), we have

$$V_p = H d_p$$

which is Hubble's law. (if we evaluate that today we have

$$V_{p0} = H_0 d_{p0}).$$

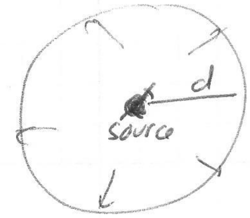
The problem at hand, though, is that H_0, which is of cosmological interest, cannot be directly determined from the above because we have no way of measuring d_{p0} or V_{p0} directly. The problem (beyond other accidental issues, like the fact that Galaxies are not necessarily isotropic observers) is that

(i) we have no ruler to measure d_{p0}, we have to infer it from other observations, like luminosity (see below)

(ii) we cannot observe V_{p0}, the velocity <u>today</u> of an observer far away, because light was emitted in the past. This is a small effect if the time T of light travel is much smaller than H_0^{-1}.

In flat space, the luminosity L (defined as energy/time emitted) of a source, and the flux F (defined as energy/time×area received) are related by

$$L = 4\pi d^2 F$$



So we define a <u>luminosity distance</u>, d_L by

$$d_L^2 = \frac{L}{4\pi F}$$

This is useful if we can identify objects in the sky as "standard candles", ie, objects that have the <u>same</u> intrinsic luminosity. Then measuring the flux at Earth we can directly infer the relative distance, d_L, to Earth.

In a FRW background, ~~pa~~ photons from a source (at $\chi=0$) get redshifted by $(1+z)$. Moreover, since we are looking at energy/time ~~rece~~ emitted vs received, ~~if the~~ energy emitted over an ~~pa~~ time interval δt ~~a~~ is received over a time interval $(1+z)\delta t$. So

$$\frac{F}{L} = \frac{1}{(1+z)^2 A}$$

where A is the area of a sphere centered at $\chi=0$ with comoving radius χ. Now, from ds^2 we have

$$A = 4\pi a_0^2 S_k^2(\chi)$$

So

$$d_L = \sqrt{\frac{L}{4\pi F}} = (1+z)\, a_0 S_k(\chi)$$

(Note: check on δt argument. Emit two photons at $t=0$ and $t=\delta t$. They follow null ~~trajecto~~ geodesics from $\chi=0$ to χ:

$$ds^2 = 0 = -dt^2 + a^2 d\chi^2$$

Or

$$\frac{d\chi}{dt} = a^{-1}$$

$$\Rightarrow \chi = \int_0^t a^{-1}(t')\,dt' = \int_{\delta t}^{t+\delta T} a^{-1}(t')\,dt'$$

and we want δT. But then, from the equality

$$\int_0^{\delta t} a(t')^{-1} dt' = \int_t^{t+\delta T} a(t')^{-1} dt'$$

and if δt is infinitesimal

$$a(0)^{-1}\delta t = a(t)^{-1}\delta T \quad \text{or} \quad \delta T = \left(\frac{a(0)}{a(t)}\right)^{-1}\delta t = \left(\frac{a_{em}}{a_{obs}}\right)^{-1}\delta t \Big).$$

Now, the expression for d_L is not very useful since it depends on χ explicitly, not an observable. However, as in the note above,

$$\chi = \int_0^t a^{-1}(t')\,dt' = \int \frac{dt'}{da}\frac{da}{a} = \int_{a_{em}}^{a_{obs}} \frac{da}{a\dot{a}}$$

Now using $\dfrac{a_{em}}{a_{obs}} = \dfrac{a}{a_0} = \dfrac{1}{1+z}$, where we have $a_{obs} = a_0$ (today)

and $a_{em} = a$, the scale factor at emission corresponding to redshift z, we can change variables from a to z. Using $\dot{a} = Ha$, we have

$$\chi = \int_0^z \left[\frac{a_0}{(1+z')^2}dz'\right]\left[\frac{1}{a^2 H}\right] = \frac{1}{a_0}\int_0^z \frac{dz'}{H(z')}$$

Note added: At this point a solution of Friedmann equations gives $a(t)$, the integral can be done if we insert $t = t(a)$, and then express the result in terms of the redshift. We instead write the integral as an integral over z:

To perform the integral we need a solution to Friedmann equations, which give $H(z)$. Of course,

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

and we know $\rho_i = \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} = \rho_{0i}(1+z)^{3(1+w_i)}$

Moreover, recall that evaluating this today and dividing by H_0^2 we get
$$1 = \sum_i \Omega_{0i}$$

So
$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\sum_i \rho_{0i}(1+z)^{3(1+w_i)} = \sum_i \Omega_{0i}(1+z)^{3(1+w_i)}$$

Let $E(z) = H(z)/H_0$. Then

$$\chi = \frac{1}{a_0 H_0}\int_0^z \frac{dz'}{E(z')} \qquad \text{with } E(z) = \sqrt{\sum_i \Omega_{0i}(1+z)^{3(1+w_i)}}$$

and this can be plugged into $d_L = (1+z)a_0 S_k(\chi)$ to get d_L in terms of z, a_0 and H_0. But the integration has to be done numerically.

Note that now we need a_0 in addition to H_0 and z. But if we know Ω_{0k} we can get a_0 (since $\rho_{0k} = -\frac{3}{8\pi G}\frac{k}{a_0^2}$) except for the case $k=0$. However, for $k=0$ $S_k(\chi) = \chi^2$ and a_0 drops out of d_L. For $k\neq 0$ we can use $\Omega_{0k} = 1-\Omega_0$ to infer Ω_{k0} and use it above. So, ~~since unity~~ recalling that

$$\cancel{a_0}\ \Omega_{0k} = \frac{8\pi G}{3H_0^2}\rho_{0k} = -\frac{k}{H_0^2 a_0^2}$$

then
$$a_0^2 = -\frac{k}{\Omega_{0k}H_0^2} \quad \text{or} \quad a_0 = \frac{1}{H_0\sqrt{|\Omega_{0k}|}} = \frac{1}{H_0\sqrt{|1-\Omega_0|}}$$

(provided $k\neq 0$).

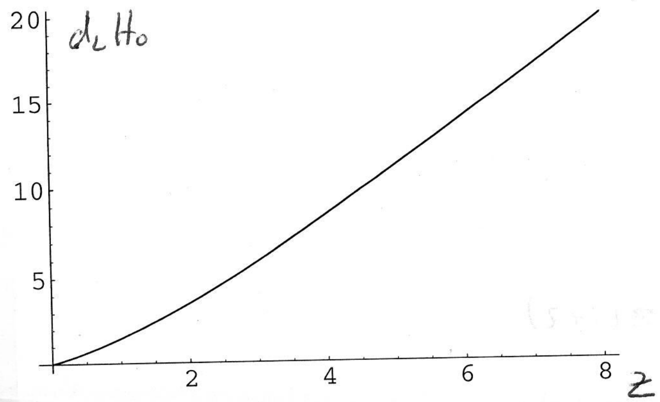
So, finally
$$\boxed{d_L = \frac{(1+z)}{H_0\sqrt{|1-\Omega_0|}}\, S_k\left[\sqrt{|1-\Omega_0|}\int_0^z \frac{dz'}{E(z')}\right]}$$

Exercise: do the integral $\int_0^z \frac{dz'}{E(z')}$ (numerically?) → Ellyptic integral... need numerics to plot anyway.

for the case that we have only Λ and matter (and the three cases $k=0,\pm 1$).

$$E^2(z) = \Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_k(1+z)^2$$

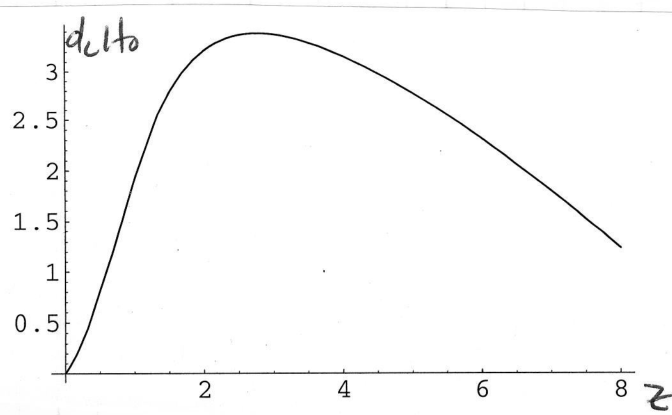
where $\Omega_k = 1-\Omega_m-\Omega_\Lambda$.



$\Omega_m = 0.3$

$\Omega_\Lambda = 0.5$

$\Omega_k = 0.2$ $k=-1$



$\Omega_m = 0.3$

$\Omega_\Lambda = 1.5$

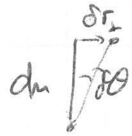
$\Omega_k = -0.8$ $k=+1$

Note the maximum from $S_k[\chi] = \sin\chi$ (eventually has a zero).

There are other measures of distance:

(i) Proper motion distance, d_M.

In flat space



$d_M\ \delta r_1 = d_M\,\delta\theta$ so, divide by δt

So define:

$$d_M = \frac{\dot{r}_1}{\dot{\theta}}$$

(ii) Angular diameter distance, d_A:

In flat space,

, so $d_A = \dfrac{D}{\theta}$.

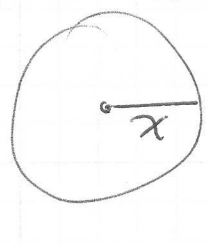
Exercise: Show $d_A = (1+z)^{-2} d_L$ and $d_M = (1+z)^{-1} d_L$.

Ans: For d_A let the observer be at $\chi = 0$ and the light emitted from χ, with θ ranging from 0 to Θ.

Null lines still have $\dot{\chi} = \frac{1}{a}$. But doing this way is problematic since comparing the tangent vectors at the observer (the origin) is bad (coordinate singularity).

Avoiding coordinate singularity is messy.

Easier: (observer at $\chi=0$).



with $\theta = 2\pi$ in $d_A = \frac{D}{\theta}$. But now by geometry, at emission

$$D = 2\pi\, a_{em} S_k(\chi)$$

so
$$d_A = \frac{2\pi a_{em} S_k(\chi)}{2\pi} = a_{em} S_k(\chi) = \frac{1}{1+z} a_0 S_k(\chi) = \frac{d_L}{(1+z)^2}$$

Similarly
$$d_M = \frac{\delta r_1/\delta t_{emiss}}{\delta\theta/\delta t_{obs}} = \frac{\delta r_1}{\delta\theta}\cdot\frac{\delta t_{obs}}{\delta t_{em}}$$

But $\dfrac{\delta r_1}{\delta\theta} = d_A = \dfrac{1}{(1+z)} a_0 S_k(\chi)$ and $\dfrac{\delta t_{obs}}{\delta t_{em}} = (1+z) \Rightarrow d_M = a_0 S_k(\chi) = \dfrac{d_L}{1+z}$

Lookback Time:

If today's time is t_0 and the time when a photon was emitted by a comoving observer (or at an event coinciding with a comoving observer) with coordinate x is t_{em}, then

$$\Delta t = t_0 - t_{em} = \int_{t_{em}}^{t_0} dt = \int_{a_{em}}^{a_0} \frac{da}{\dot{a}} = \int_{a_{em}}^{a_0} \frac{da}{aH}$$

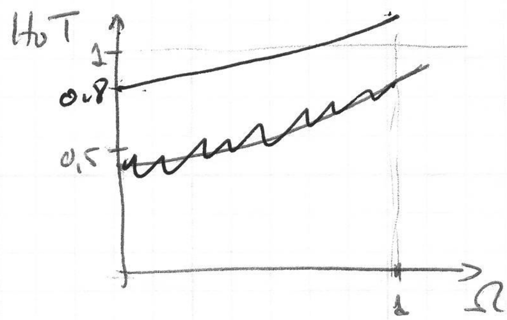
Using $H = H_0 E(z)$ and $a = \frac{a_0}{1+z}$ we have

$$\boxed{\Delta t = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')E(z')}}$$ Lookback time

The integral is dimensionless, the units are set by $H_0^{-1} \sim 10^{10}$ yrs. In particular, as $z \to \infty$ the integral goes to a fixed finite number (that depends on the details of $E(z')$), of order 1. So we are tempted to say

$$T = \text{age of universe} = \frac{1}{H_0} \int_0^\infty \frac{dz'}{(1+z')E(z')} \approx \frac{1}{H_0}$$

In fact, I get (from mathematica) that for $\Omega_m = 0.3$ $\Omega_{radiation} = 0$



So, for fixed Ω_m, T increases with Ω_Λ (albeit slowly).
This is not the whole story because there is also radiation! But adding $\Omega_{rad} = 10^{-3}\Omega_m$ changes the result a negligible amount.