Department of Physics, UCSD
Physics 225B, General Relativity
Winter 2014
Homework 2, solutions

1. (i) Let's compute the Lie derivative at a point p for a metric satisfying $\phi_t^* g = \Omega_t^2 g$:

$$\begin{aligned}\mathcal{L}_K g|_p &= \lim_{t\to 0}\frac{1}{t}(\phi_t^* g|_p - g|_p)\\ &= \lim_{t\to 0}\frac{1}{t}(\Omega_t^2 g|_p - g|_p)\\ &= \lim_{t\to 0}\frac{1}{t}(\Omega_t^2 - 1)g|_p\\ &= \frac{d\Omega_t^2}{dt}\bigg|_{t=0} g|_p\end{aligned}$$

More explicitly, we know from class the left hand side has components $2K_{(\mu;\nu)}$, while the right hand side is $\partial_t\Omega^2|_{t=0}g_{\mu\nu} = 2\omega g_{\mu\nu}$. Hence $K_{(\mu;\nu)} = \omega g_{\mu\nu}$.

(ii) Contracting indices, $g^{\mu\nu}K_{(\mu;\nu)} = \omega g^{\mu\nu}g_{\mu\nu} = n\omega$. Hence $\omega = \frac{1}{n}g^{\mu\nu}K_{(\mu;\nu)} = \frac{1}{n}g^{\mu\nu}K_{\mu;\nu}$. Using this in the conformal Killing condition, $K_{(\mu;\nu)} = \omega g_{\mu\nu}$, we have

$$K_{(\mu;\nu)} = \frac{1}{n}K^{\lambda}{}_{;\lambda}g_{\mu\nu} \tag{0.1}$$

or

$$K_{\mu;\nu} + K_{\nu;\mu} - \frac{2}{n}K^{\lambda}{}_{;\lambda}g_{\mu\nu} = 0.$$

(iii) Using $g_{\mu\nu} = \eta_{\mu\nu}$ above we have $\partial_\nu K_\mu + \partial_\mu K_\nu - \frac{2}{n}\partial\cdot K = 0$.

(iv) Taking ∇^ν of (0.1) we have $\nabla^2 K_\mu + \nabla^\nu\nabla_\mu K_\nu - \frac{2}{n}\nabla_\mu\nabla_\nu K_\nu = 0$. Now note that the last two terms cancel if $n = 2$ *and* the covariant derivatives commute, that is, flat spacetime. Specializing to this, we have $\partial^2 K_\mu = 0$ each component of K_μ satisfied the wave equation in $1+1$ dimensions. This is solved by arbitrary left and right moving waves: $K_\mu(t,x) = L_\mu(x+t) + R_\mu(x-t)$. These are all the solutions, and there are infinitely many of them: take any complete set of functions of the real line, $f_i(x)$, with i a positive integer, say. Then you can expand $L_\mu(x+t) = \sum_{i=1}^{\infty}\ell_{i\mu}f_i(x+t)$, where $\ell_{i\mu}$ are expansion coefficients. And similarly for R_μ.

Another way to see this is instructive. Introduce light cone coordinates, $u = x+t$ and $v = x-t$. Then $\partial^2 K_\mu = 0$ is $\partial_u\partial_v K_\mu = 0$, which is solved by K_μ being independent of either u or v.

(v) We already saw in (iv) the result of acting with ∇^μ, which after going to flat space gives $\partial^2 K_\mu + (1-\frac{2}{n})\partial^\nu\partial\cdot K = 0$. Taking the divergence of this we get $\partial^2(\partial\cdot K) = 0$. So

$\partial \cdot K(x)$ is linear in x and $K_\mu(x)$ at most quadratic. We are looking for a solution of the conformal killing equation in (iii), and we make an ansatz

$$K_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + \tfrac{1}{2}c_{\mu\nu\lambda}x^\nu x^\lambda.$$

Notice that by construction $c_{\mu\nu\lambda} = c_{\mu\lambda\nu}$. Plugging into the equation and equating to zero separately the different powers of x we find

$$b_{\mu\nu} + b_{\mu\nu} = \tfrac{2}{n}b^\lambda{}_\lambda\eta_{\mu\nu} \tag{0.2}$$

$$c_{\mu\nu\lambda} + c_{\nu\mu\lambda} = \tfrac{2}{n}c^\rho{}_{\rho\lambda}\eta_{\mu\nu} \tag{0.3}$$

The first of these can be solved by separating the matrix $b_{\mu\nu}$ into symmetric and antisymmetri parts, $b_{\mu\nu} = \mathcal{A}_{\mu\nu} + \mathcal{S}_{\mu\nu}$ with $\mathcal{A}_{\mu\nu} = -\mathcal{A}_{\nu\mu}$ and $\mathcal{S}_{\mu\nu} = \mathcal{S}_{\nu\mu}$. Then Eq. (0.2) gives no constraint on $\mathcal{A}_{\mu\nu}$ and gives that $\mathcal{S}_{\mu\nu}$ is proportional to the metric, $\mathcal{S}_{\mu\nu} = s\eta_{\mu\nu}$. In order to solve (0.3) we use a trick you may have seen before (in connection with solving for the connection, $\Gamma_{\mu\nu\lambda}$): re-write the equation twice, with the indices cyclically permuted,

$$\begin{aligned}
c_{\mu\nu\lambda} + c_{\nu\mu\lambda} &= \tfrac{2}{n}c^\rho{}_{\rho\lambda}\eta_{\mu\nu}\\
c_{\nu\lambda\mu} + c_{\lambda\nu\mu} &= \tfrac{2}{n}c^\rho{}_{\rho\mu}\eta_{\nu\lambda}\\
c_{\lambda\mu\nu} + c_{\mu\lambda\nu} &= \tfrac{2}{n}c^\rho{}_{\rho\nu}\eta_{\lambda\mu}
\end{aligned}$$

and subtract the middle one from the sum of the outer ones:

$$c_{\mu\nu\lambda} = \tfrac{1}{n}(c^\rho{}_{\rho\lambda}\eta_{\mu\nu} + c^\rho{}_{\rho\nu}\eta_{\lambda\mu} - c^\rho{}_{\rho\mu}\eta_{\nu\lambda}) \equiv c_\lambda\eta_{\mu\nu} + c_\nu\eta_{\lambda\mu} - c_\mu\eta_{\nu\lambda}$$

Combining these results we have

$$K_\mu = a_\mu + \mathcal{A}_{\mu\nu}x^\nu + sx_\mu + 2c\cdot xx_\mu - c_\mu x^2$$

We recognize a_μ as generating translations and $\mathcal{A}_{\mu\nu}$ generating Lorentz transformations. Together they form the Poincare group. These are isometries (not merely up to conformal transformations) and as we know there are $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ of them. It is easy to recognize s as generating dilatations, $x^\mu \to e^s x^\mu$. The last one is harder to understand. $\delta x^\mu = 2c\cdot xx_\mu - c_\mu x^2$ is an infinitesimal version of a conformal transformation. The finite form of the transformation is most easily described as an inversion ($x^\mu \to -x^\mu/x^2$) followed by a translation ($x^\mu \to x^\mu + a^\mu$) followed by another inversion.

(vi) Compute, compute, compute... Start from $\delta A_\mu = (\mathcal{L}_K A)_\mu = K^\lambda\partial_\lambda A_\mu + \partial_\mu K^\lambda A_\lambda$ and compute the variation of the Lagrangian density, $-\frac{1}{4}\delta(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} = F^{\mu\nu}\delta\partial_\nu A_\mu$, or

$$\begin{aligned}
\delta(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}) &= F^{\mu\nu}\partial_\nu(K^\lambda\partial_\lambda A_\mu + \partial_\mu K^\lambda A_\lambda) \\
&= F^{\mu\nu}(\partial_\nu K^\lambda\partial_\lambda A_\mu + K^\lambda\partial_\nu\partial_\lambda A_\mu + \partial_\nu\partial_\mu K^\lambda A_\lambda + \partial_\mu K^\lambda\partial_\nu A_\lambda) \\
&= F^{\mu\nu}(\partial_\nu K^\lambda\partial_\lambda A_\mu + \partial_\mu K^\lambda\partial_\nu A_\lambda + \tfrac{1}{2}K^\lambda\partial_\lambda F_{\nu\mu}) \\
&= \partial_\nu K^\lambda F^{\mu\nu}(\partial_\lambda A_\mu - \partial_\mu A_\lambda) + \tfrac{1}{4}\partial_\lambda(K^\lambda F^{\mu\nu}F_{\nu\mu}) - \tfrac{1}{4}\partial_\lambda K^\lambda F^{\mu\nu}F_{\nu\mu} \\
&= \tfrac{1}{2}F^{\mu\nu}F^\lambda{}_\mu(\partial_\nu K_\lambda + \partial_\lambda K_\nu - \tfrac{1}{2}\eta_{\lambda\nu}\partial\cdot K) + \tfrac{1}{4}\partial_\lambda(K^\lambda F^{\mu\nu}F_{\nu\mu})
\end{aligned}$$

(Steps: line 2 to 3, moved the fourth term in line 2 to second in line 3, and used antisymmetry in $\mu \leftrightarrow \nu$ toset to zero the antisymmetric combination of $\partial_\mu\partial_\nu$ and replace $\frac{1}{2}F_{\nu\mu}$ for $\partial_\nu A_\mu$; line 3 to 4, combined the first two terms, integrated by parts the third). Now, the first term has a factor that vanishes if k is a conformal Killing vector in $n = 4$ and the last term is a total derivate which integrates in the action integral, by Stoke's theorem, to a surface integral at infinity, which vanishes.

2. (i) We use polar coordinates $X^1 = r\cos\theta$ and $X^2 = r\sin\theta$. Eliminate X^0: use $(X^0)^2 = 1 + (X^1)^2 + (X^2)^2 = 1 + r^2$ so that $X^0dX^0 = -X^1dX^1 - X^2dX^2 = -rdr$ and for the metric we need

$$(dX^0)^2 = (\frac{rdr}{X^0})^2 = \frac{r^2}{1+r^2}dr^2 \tag{0.4}$$

so that the metric is

$$ds^2 = -\frac{r^2}{1+r^2}dr^2 + dr^2 + r^2d\theta^2 = \frac{dr^2}{1+r^2} + r^2d\theta^2.$$

Since r and θ are polar coordinates for $X^{1,2}$, which are unrestricted, we have $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$.

(ii) It is easiest to first find an embedding coordinates for the embedding $-(X^0)^2 + (X^1)^2 + (X^2)^2 = -1$ that gives the usual Poincare half-plane metric. To this end let

$$X^0 + X^1 = u$$
$$X^2 = xu$$

This is motivated by the a similar choice made in class for deSitter space, that had $\hat{t} = \ln(w + u)$, $\hat{x} = x/(w + u)$; see lecture notes. A third relation, needed to fix all three coordinates X follows from $-(X^0)^2 + (X^1)^2 + (X^2)^2 = -1$ using $-(X^0)^2 + (X^1)^2 = (-X^0 +$

$X^1)(X^0 + X^1) = (-X^0 + X^1)u$. So we have

$$\begin{aligned} X^0 &= \frac{1}{2}\left(u + \frac{1 + x^2u^2}{u}\right) \\ X^1 &= \frac{1}{2}\left(u - \frac{1 + x^2u^2}{u}\right) \\ X^2 &= xu \end{aligned}$$

Computing the pull back of this map (the embedding) we get

$$ds^2 = \frac{du^2 + u^4 dx^2}{u^2}.$$

Not quite what we wanted, but close. Form the second term it is apparent we want $y = 1/u$, or to be explicit,

$$\begin{aligned} X^0 &= \frac{1}{2}\left(\frac{1}{y} + y + \frac{x^2}{y}\right) \\ X^1 &= \frac{1}{2}\left(\frac{1}{y} - y - \frac{x^2}{y}\right) \\ X^2 &= x/y \end{aligned} \tag{0.5}$$

and the pull back is the desired metric.

We can now exhibit the relation between our (x, y) and (r, θ) coordinates:

$$\begin{aligned} y^{-1} = u = X^0 + X^1 &= \pm\sqrt{1 + r^2} + r\cos\theta \\ x/y = X^2 &= r\sin\theta \end{aligned} \tag{0.6}$$

Of course, to get x explicitly you can divide the second by the first. There are tow signs in the square root, taking the upper sign gives $y > 0$ while the lower gives $y < 0$.

(iii) The question, by design, is a bit ambiguous. What I hoped you would explore is the question of whether one of the coordinate system covers more of the manifold than the other. From the embedding in terms of (x, y) it is apparent that one could take $y < 0$ just as well as $y > 0$, but something really bad happens at $y = 0$. If you plot the surface $(X^0)^2 = 1 + (X^1)^2 + (X^2)^2$ you realize immediately that it consists of two disconnected pieces, one for $X^0 \geqslant 1$ and the other for $X^0 \leqslant 1$. Our manifold, $\mathbb{H}^2$, corresponds to one of the disconnected components, $X^0 \geqslant 1$, and correspondingly, $X^0 + X^1 > 0$. In the (x, y) coordinate system this means $y > 0$, while in the (r, θ) system that means we have taken the positive square root in defining X^0, see, eg, Eq. (0.6).

(iv) We now have $(X^0)^2 = -1 + (X^1)^2 + (X^2)^2 = -1 + r^2$ so that we still have $X^0 dX^0 = -X^1 dX^1 - X^2 dX^2 = -r dr$ but Eq. (0.4) is now

$$(dX^0)^2 = (\frac{r dr}{X^0})^2 = -\frac{r^2}{1 - r^2} dr^2.$$

So now the metric is

$$ds^2 = -\frac{-r^2}{1-r^2}dr^2 + dr^2 + r^2 d\theta^2 = \frac{dr^2}{1-r^2} + r^2 d\theta^2.$$

Following the hint, $\int \frac{dr}{\sqrt{1-r^2}} = \arcsin r$. That is, $ds^2 = d\Omega_2^2$, the metric on S^2.

3. (i) This is much like what we did in class for maximally symmetric spaces, and we went through the logic then. Choose:

$$\begin{aligned}
X^0 &= \cosh\chi \\
X^1 &= \sinh\chi\cos\theta_1 \\
&\vdots \\
X^n &= \cosh\chi\sin\theta_1\cdots\sin\theta_{n-1}\sin\theta_n
\end{aligned}$$

To be clear, the spacelike vector $(X^1,\ldots,X^n)$ of magnitude $\sinh\chi$ is written in spherical coordinates in terms of $n-1$ angles that parametrize points on the unit n-dimensional sphere.

Now compute the pull-back. Recall, in general, if the map between manifolds (or rather, between the corresponding coordinate patches) is $y^a = y^a(x^\mu)$ then the pull back of g $(\phi^* g)_{\mu\nu}(x) = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} g_{ab}(y)$. Note that this corresponds formally to replace $\frac{\partial y^a}{\partial x^\mu}\mathrm{d}x^\mu$ for $\mathrm{d}y^a$ in $ds^2 = g_{ab}\mathrm{d}y^a\mathrm{d}y^b$. So we proceed that way:

$$ds^2 = -(\sinh\chi d\chi + 0 + \cdots + 0)^2 + (\cosh\chi\cos\theta_1 d\chi - \sinh\chi\sin\theta_1 d\theta_1 + 0 + \cdots + 0)^2 + \cdots$$

By construction the second through last term correspond to spherical coordinates with radius $\sinh\chi$ se we know they add to $(d\sin\chi)^2 + \sin\chi^2 d\Omega_{n-1}^2 = \cosh^2 d\chi^2 + \sin\chi^2 d\Omega_{n-1}^2$. Combining with the first term we have

$$ds^2 = -\sinh\chi^2 d\chi^2 + \cosh^2 d\chi^2 + \sin\chi^2 d\Omega_{n-1}^2 = d\chi^2 + \sin\chi^2 d\Omega_{n-1}^2.$$

(ii) Getting the metric into the analog of the form in Eq. (3) of the assignment is straightforward: simply let $r = \sinh\chi$, so that $dr = \cosh\chi d\chi$, or $d\chi = dr/\cosh\chi = dr/\sqrt{1+r^2}$.

To get the metric as in Eq. (2) of the assignment we repeat the procedure in the $n=2$ case, generalizing in an obvious way:

$$\begin{aligned}
X^0 &= \frac{1}{2}\left(u + \frac{1+(\sum_i (x^i)^2)u^2}{u}\right) \\
X^1 &= \frac{1}{2}\left(u - \frac{1+(\sum_i (x^i)^2)u^2}{u}\right) \\
X^{1+i} &= x^i u
\end{aligned}$$

where $i = 1, \ldots, n-1$, and $u = 1/y$. Then

$$ds^2 = \frac{1}{y^2}(dy^2 + \sum_i (dx^i)^2).$$

4. A geodesic is an extremum of the path length, $\int ds$. To incorporate a constraint into it we can use the method of Lagrange multipliers, thus:

$$\delta S = 0, \qquad \text{where} \qquad S = \int (ds + d\tau \lambda f(X)) \, .$$

Here $\lambda(\tau)$ is the lagrange multiplier, a function of the affine parameter τ, ds is the square root of ds^2 given by

$$\begin{aligned} ds^2 &= -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^4)^2 \qquad && \text{for deSitter, } dS \\ ds^2 &= -(dX^0)^2 - (dX^1)^2 + \cdots + (dX^4)^2 \qquad && \text{for anti-deSitter, } AdS \end{aligned}$$

and $f(X)$ stands form the constraint that defines the embedded submanifold,

$$\begin{aligned} f(X) &= -(X^0)^2 + (X^1)^2 + \cdots + (X^4)^2 - \alpha^2 \qquad && \text{for deSitter, } dS \\ f(X) &= -(X^0)^2 - (X^1)^2 + \cdots + (X^4)^2 + \alpha^2 \qquad && \text{for anti-deSitter, } AdS \end{aligned}$$

Let's go through this explicitly for the dS case. Writing $ds^2 = \eta_{MN} dX^M dX^N$ and $f(X) = \eta_{MN} X^M X^N - \alpha^2$, we have

$$\delta S = \int d\tau \left(\frac{1}{e(\tau)} \eta_{MN} \frac{dX^M}{d\tau} \frac{d\delta X^N}{d\tau} + \delta\lambda f(X) + \lambda \eta_{MN} X^M \delta X^N \right) = 0.$$

Using the fact that we have chosen τ to be an affine parameter, we have, after integration by parts, the conditions

$$\begin{aligned} -\frac{d^2 X^N}{d\tau^2} + \lambda X^N &= 0 \\ \eta_{MN} X^M X^N - \alpha^2 &= 0 \end{aligned}$$

The first of these can be rewritten,

$$\frac{1}{X^0} \frac{d^2 X^0}{d\tau^2} = \frac{1}{X^1} \frac{d^2 X^1}{d\tau^2} = \cdots = \frac{1}{X^4} \frac{d^2 X^4}{d\tau^2} = \lambda \tag{0.7}$$

while taking two derivatives on the second we have

$$\eta_{MN} X^M \frac{d^2 X^N}{d\tau^2} + \eta_{MN} \frac{dX^M}{d\tau} \frac{dX^N}{d\tau} = 0.$$

Note that the second term in this expression is set to a constant by our choice of affine parameter, and the first term can be simplified, using (0.7), thus

$$\lambda \eta_{MN} X^M X^N = -\eta_{MN} \frac{dX^M}{d\tau}\frac{dX^N}{d\tau} = \text{constant}$$

Since the factor multiplying λ is α^2, a constant, we learn that $\lambda =$constant. We can then solve the Eqs. (0.7) trivially,

$$X^N(\tau) = a^N e^{\sqrt{\lambda}\tau} + b^N e^{-\sqrt{\lambda}\tau}$$

where a and b are arbitrary constants. Of course, if $\lambda < 0$ we have cosine and sine solutions.