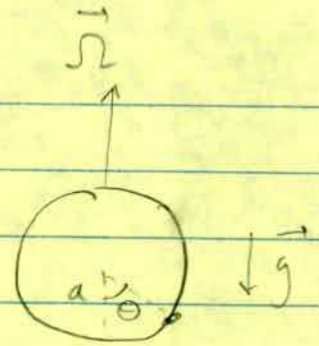
Robin Heinonen

3.1

a.



$$\begin{cases} x = a\cos\Omega t \sin\theta \\ y = a\sin\Omega t \sin\theta \\ z = -a\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = -a\Omega \sin\Omega t \sin\theta + a\dot{\theta}\cos\Omega t \cos\theta \\ \dot{y} = a\Omega\cos\Omega t \sin\theta + a\dot{\theta}\sin\Omega t \cos\theta \\ \dot{z} = a\dot{\theta}\sin\theta \end{cases}$$

$$\Rightarrow T = \tfrac{1}{2}m\left(a^2\Omega^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\theta}^2\sin^2\theta\right)$$

$$= \tfrac{1}{2}ma^2\left(\Omega^2\sin^2\theta + \dot{\theta}^2\right)$$

$$\Rightarrow \underline{L = \tfrac{1}{2}ma^2\left(\Omega^2\sin^2\theta + \dot{\theta}^2\right) + mga\cos\theta}$$

b. The eq. of motion is

$$ma^2\ddot{\theta} = ma^2\Omega^2\sin\theta\cos\theta - mga\sin\theta$$

$$\Rightarrow \ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta$$

At equilibrium, this is zero, so

$$0 = \Omega^2\sin\theta_0\cos\theta_0 - \frac{g}{a}\sin\theta_0 = \sin\theta_0\left(\Omega^2\cos\theta_0 - \frac{g}{a}\right)$$

The nontrivial solution is

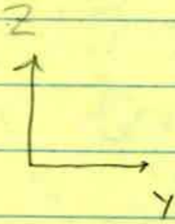
$\boxed{\cos\theta_0 = \dfrac{g}{a\Omega^2}}$, provided $a\Omega^2 > g$.

31 b (cont'd)

In the co-rotating frame, we have

$\ddot{z} = -g$

$\ddot{y} = \Omega^2 y$



The tangential components are

$$a_{t,z} = -g \sin\theta$$
$$a_{t,y} = \Omega^2 y \cos\theta = \Omega^2 (a \sin\theta) \cos\theta$$

so we must have $\Omega^2 a \sin\theta \cos\theta = g \sin\theta$
and we arrive at the same result.

c. Write $\theta = \theta_0 + \eta(t)$

Then $\sin\theta \simeq \sin\theta_0 + \eta \cos\theta_0$
$\cos\theta \simeq \cos\theta_0 - \eta \sin\theta_0$

To first order in η, then,

$$\ddot{\eta} = \Omega^2 \left(\sin\theta_0 \cos\theta_0 + \eta \cos^2\theta_0 - \eta \sin^2\theta_0\right)$$
$$- \frac{g}{a}\left(\sin\theta_0 + \eta \cos\theta_0\right)$$

(using $\cos\theta_0 = \frac{g}{a\Omega^2}$)

$$= \frac{g}{a}\sin\theta_0 - \frac{g}{a}\sin\theta_0 + \eta\left(\frac{g^2}{a^2\Omega^2} - \frac{g^2}{a^2\Omega^2} - \Omega^2 \sin^2\theta_0\right)$$

$$\doteq -\Omega^2 \eta \sin^2\theta_0$$

This is the equation of an oscillator with frequency $\boxed{\omega^2 = \Omega^2 \sin^2\theta_0}$

3.1c (cont'd) It is stable since

$\Omega^2 \sin^2\theta_0 > 0$ always, so the acceleration m experiences is restoring.

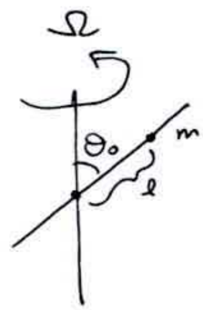
d. If $a\Omega^2 < g$ then there is no equilibrium point for $0 < \theta < \pi$.

Instead, now $\theta = 0$ is a stable equilibrium, since for $\theta \ll 1$

$$\ddot{\theta} \simeq \left(\Omega^2 - \frac{g}{a}\right)\theta$$

and the coefficient in parentheses is less than 0.

FW 3.2 - Emily Nardoni



$$\boxed{\mathcal{L} = \frac{1}{2} m \dot{\ell}^2 + \frac{1}{2} m \Omega^2 \ell^2 \sin^2\theta_0 - mg\ell \cos\theta_0}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\ell}} - \frac{\partial \mathcal{L}}{\partial \ell} = 0 \Rightarrow m\ddot{\ell} = m\Omega^2 \sin^2\theta_0\, \ell - mg\cos\theta_0$$

Equilibrium: $\ddot{\ell} = 0 \quad \Rightarrow \Omega^2 \sin^2\theta_0\, \ell_0 = g\cos\theta_0$

$$\boxed{\ell_0 = \frac{g\cos\theta_0}{\Omega^2 \sin^2\theta_0}}$$

Stability against small displacements:

$$\ell = \ell_0 + \delta\ell$$

$$\left.\begin{aligned} m\,\delta\ddot{\ell} &= m\Omega^2 \sin^2\theta_0 (\ell_0 + \delta\ell) - mg\cos\theta_0 \\ &= m\Omega^2 \sin^2\theta_0\, \delta\ell \end{aligned}\right\} \quad \delta\ddot{\ell} = \underbrace{+\Omega^2 \sin^2\theta_0}_{\text{positive}}\, \delta\ell \Rightarrow \boxed{\text{unstable}}$$

Balance of forces in non-inertial reference frame:

- $-mg$
- centrifugal $m\Omega^2 \ell \sin\theta_0$ out from rotational orbit
- reaction force of wire $m\Omega^2 \sin\theta_0 \cos\theta_0\, \ell + mg\sin\theta_0$ $\perp$ to wire

via Lagrange multipliers, let $f(\theta) = \theta_0$ - constraints,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

$$\Rightarrow m\ell^2\ddot{\theta} = m\Omega^2 \ell^2 \cos\theta \sin\theta + mg\ell \sin\theta + \lambda$$

$\ddot{\theta} = 0,\ \theta = \theta_0$ equilibrium:

$\Rightarrow$ normal force $\underline{N = \frac{\lambda}{\ell} = -m\Omega^2 \ell \cos\theta_0 \sin\theta_0 - mg\sin\theta_0}$

PSet 1: Problem 3: FW 3.3

3.3 A simple pendulum of mass m_2 and length l is constrained to move in a single plane. The point of support is attached to a mass m_1 which can move on a horizontal line in the same plane. Find the Lagrangian of the system in terms of suitable generalized coordinates. Derive the equations of motion.

Answer: First, define the position of the pivot as x and the angle the pivot makes with the vertical as θ. Then, defining $(x_1, 0)$ and (x_2, y_2) as the coordinates of the pivot and pendulum, respectively, we have:

$$\begin{aligned} x_1 &= x & &\implies \dot{x_1} = \dot{x} \\ x_2 &= x + l\sin\theta & &\implies \dot{x_2} = \dot{x} + xl\dot{\theta}\cos\theta \\ y_2 &= -l\cos\theta & &\implies \dot{y_2} = l\dot{\theta}\sin\theta \end{aligned}$$

Then, the kinetic energy T is given by

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta) \\ &= \frac{1}{2}\dot{x}^2(m_1+m_2) + \frac{1}{2}m_2l^2\dot{\theta}^2 + m_2\dot{x}\dot{\theta}l\cos\theta \end{aligned}$$

Furthermore, the potential energy V is given by $V = mgy_2 = -mgl\cos\theta$. Combining these gives the expression for the potential energy:

$$\boxed{L = T - V = \frac{1}{2}\dot{x}^2(m_1+m_2) + \frac{1}{2}m_2l^2\dot{\theta}^2 + m_2\dot{x}\dot{\theta}l\cos\theta + mgl\cos\theta}$$

Now, to get the equations of motion, we apply the Euler-Lagrange equations. Considering first the x-component gives

$$\frac{\partial L}{\partial x} = 0 \implies \frac{\partial L}{\partial \dot{x}} = \dot{x}(m_1+m_2) + m_2\dot{\theta}l\cos\theta = P \text{ constant}$$

Thus, we have obtained a conserved momentum P.

Next, considering the θ-component, we get

$$\begin{aligned} \frac{\partial L}{\partial \theta} = -m_2\dot{x}l\dot{\theta}\sin\theta - m_2gl\sin\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m_2l^2\dot{\theta} + m_2\dot{x}l\cos\theta\right) \\ = m_2l^2\ddot{\theta} + m_2\ddot{x}l\cos\theta - m_2\dot{x}l\dot{\theta}\sin\theta \end{aligned}$$

However, we know that

$$\dot{x} = \frac{P - \dot{\theta}m_2l\cos\theta}{m_1+m_2} \implies \ddot{x} = \frac{\dot{\theta}^2m_2l\sin\theta - \ddot{\theta}m_2l\cos\theta}{m_1+m_2}$$

Substituting this into the expression for θ gives

$$\begin{aligned} 0 &= \ddot{\theta}m_2l^2\left(1 - \frac{m_2\cos^2\theta}{m_1+m_2}\right) + \ddot{theta}\frac{m_2^2l^2\cos\theta\sin\theta}{m_1+m_2} + m_2gl\sin\theta \\ &= \ddot{\theta}m_2l^2\left(\frac{m_1+m_2\sin^2\theta}{m_1+m_2}\right) + \ddot{theta}\left(\frac{m_2^2l^2\cos\theta\sin\theta}{m_1+m_2}\right) + m_2gl\sin\theta \end{aligned}$$

Next, we will assume $\theta << 1$ and only keep terms to first order. Note: $\dot{\theta}^2$ is second order small and hence will be dropped. Approximating $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, we have

$$0 = \frac{\ddot{\theta} l^2 m_1 m_2}{m_1 + m_2} + \theta m_2 g l \implies 0 = \ddot{\theta} + \theta \frac{g(m_1 + m_2)}{l m_1}$$

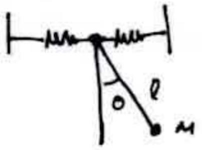
Therefore, θ oscillates with a frequency given by

$$\boxed{\omega = \sqrt{\frac{g}{l}\frac{(m_1 + m_2)}{m_2}}}$$

(a) L & EOM, (b) Small Angle → Result

Student sol

3.7



$$T = \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

$x_m = x + l\sin\theta \Rightarrow \dot{x}_m = \dot{x} + l\dot{\theta}\cos\theta$

$y_m = -l\cos\theta \Rightarrow \dot{y}_m = -l\dot{\theta}\sin\theta$

$$V = -mgl\cos\theta + \frac{1}{2}kx^2$$

$$\boxed{L = \frac{1}{2}m\left[\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right] + mgl\cos\theta - \frac{1}{2}kx^2}$$

$$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{2}m\left[2l\dot{x}\cos\theta + 2l^2\dot{\theta}\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + l^2\ddot{\theta}m$$

$$\frac{\partial L}{\partial\theta} = -lm\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$\Rightarrow \ddot{x}\cos\theta - \cancel{\dot{x}\dot{\theta}\sin\theta} + l\ddot{\theta} = -\cancel{\dot{x}\dot{\theta}\sin\theta} - g\sin\theta$$

(1) $\ddot{x}\cos\theta + l\ddot{\theta} = -g\sin\theta$

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x} + ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial x} = -kx$$

(2) $\Rightarrow \ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta = \frac{-k}{m}x$

Small Angle Approx.

(1) $\ddot{x} + l\ddot{\theta} = -g\theta$

(2) $\ddot{x} + l\ddot{\theta} - \overbrace{l\dot{\theta}^2\theta}^{\text{Neglect}} = \frac{-k}{m}x$

$\Big\} \rightarrow$ $g\theta = \frac{k}{m}x$, $\ddot{x} = \frac{mg}{k}\ddot{\theta}$

$$\Rightarrow \left(l + \frac{mg}{k}\right)\ddot{\theta} = -g\theta$$

$$\boxed{\ddot{\theta} = \frac{-g}{(l + mg/k)}\theta}$$

Pendulum Eq. $l' = l + mg/k$

Yanzeng Zhang

3.8

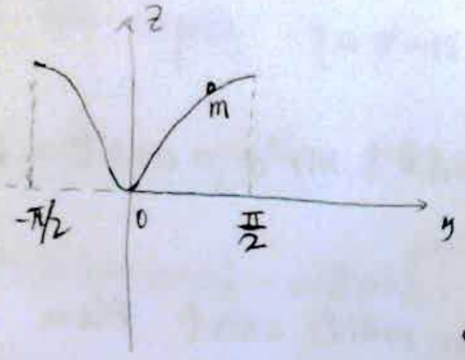


fig (8)

Considering the mass m, its location is

$$\vec{r} = (r, \Phi, z)$$

So $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\Phi}^2 + \dot{z}^2)$

$U = mgz \qquad$ where $z = \alpha \sin(r/R)$

That is $\quad T = \frac{m}{2}\left\{\dot{r}^2 + r^2\dot{\Phi}^2 + \left[\frac{\alpha}{R}\dot{r}\cos(\frac{r}{R})\right]^2\right\}$

We get
$$\mathcal{L} = T - U$$
$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{\alpha^2}{R^2}\dot{r}^2\cos^2\frac{r}{R}\right) - mg\alpha\sin\frac{r}{R}$$

equations of motion:

For r:
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} + m\frac{\alpha^2}{R^2}\dot{r}\cos^2\frac{r}{R}$$
$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\Phi}^2 - m\frac{\alpha^2}{R^2}\dot{r}^2\cdot\frac{1}{R}\sin\frac{r}{R}\cos\frac{r}{R} - mg\alpha\cdot\frac{1}{R}\cos\frac{r}{R}$$

so $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}$ gives

$$\left(1 + \frac{\alpha^2}{R^2}\cos^2\frac{r}{R}\right)\ddot{r} - 2\frac{\alpha^2}{R^3}\cos\frac{r}{R}\sin\frac{r}{R}\dot{r}^2 = r\dot{\Phi}^2 - \frac{\alpha^2}{R^3}\sin\frac{r}{R}\cos\frac{r}{R}\dot{r}^2 - g\frac{\alpha}{R}\cos\frac{r}{R}$$

that is
$$\boxed{\left(1 + \frac{\alpha^2}{R^2}\cos^2\frac{r}{R}\right)\ddot{r} = r\dot{\Phi}^2 + \frac{\alpha^2}{R^3}\sin\frac{r}{R}\cos\frac{r}{R}\dot{r}^2 - g\frac{\alpha}{R}\cos\frac{r}{R}}$$

For Φ

$$\frac{\partial L}{\partial \dot\Phi} = mr^2\dot\Phi \qquad \frac{\partial L}{\partial \Phi} = 0$$

So $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\Phi}\right) - \frac{\partial L}{\partial \Phi} = 0$ gives

$\frac{d}{dt}(mr^2\dot\Phi) = 0$ which means $mr^2\dot\Phi = \text{const} = 2L$

and L is the angular momentum

(b) If stationary horizontal circular orbits exist then

$z = \alpha \sin\frac{r}{R} = \text{const}$ which means $r = \text{const}$ and $\ddot r = \dot r = 0$

So the motion equation for r gives

$$r\dot\Phi^2 - \frac{g\alpha}{R}\cos\frac{r}{R} = 0 \qquad \text{since } mr^2\dot\Phi = 2L$$

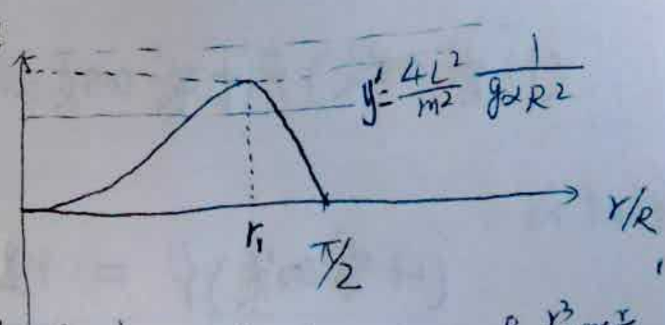
we got $\left(\frac{2L}{m}\right)^2\frac{1}{r^3} - \frac{g\alpha}{R}\cos\frac{r}{R} = 0$

Then $\boxed{r^3\cos\frac{r}{R} = \frac{4L^2}{m^2}\frac{R}{g\alpha}}$ (*) this is the equation that r meets for stationary horizontal circular orbits

If we draw the curve of $y = \frac{r^3}{R^3}\cos\frac{r}{R}$ and $\frac{4L^2}{m^2}\frac{1}{g\alpha R^2} = y'$

we got fig(9)

So if $y' = \frac{4L^2}{m^2}\frac{1}{g\alpha R^2}$ is less than the maximum of $\frac{r^3}{R^3}\cos\frac{r}{R}$, then there are two rs that meet the equation (*)



there is only one root r if $y' = \frac{4L^2}{m^2}\frac{1}{g\alpha R^2}$ equals the maximum of $\frac{r^3}{R^3}\cos\frac{r}{R}$.

Certainly, if $\frac{4L^2}{m^2}\frac{1}{g\alpha R^2} > \frac{r^3}{R^3}\cos\frac{r}{R}$, there is no such r_0.

(c) We impose $\eta = r - r_0$ then $\dot\eta = \dot r$ $\ddot\eta = \ddot r$ (Assume r_0s exist)

So motion equation gives

$\sin\frac{2(r_0+\eta)}{R}$

$$\left[1+\frac{\alpha^2}{R^2}\left(\cos\frac{r_0}{R}\cos\frac{\eta}{R}-\sin\frac{r_0}{R}\sin\frac{\eta}{R}\right)^2\right]\ddot\eta = \frac{1}{(r_0+\eta)^3}\frac{4L^2}{m^2} + \frac{\alpha^2}{2R^3}\left(\sin\frac{2r_0}{R}\cos\frac{2\eta}{R}+\sin\frac{2\eta}{R}\cos\frac{2r_0}{R}\right)\dot\eta^2$$

$$-\frac{g\alpha}{R}\left(\cos\frac{r_0}{R}\cos\frac{\eta}{R}-\sin\frac{r_0}{R}\sin\frac{\eta}{R}\right)$$

then if η is a small variable and we only take the first-order (that η, $\dot\eta$, $\ddot\eta$)

we get approximation of above eq.

$$\left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r_0}{R}\right)\ddot\eta = \frac{4L^2}{m^2 r_0^3}\left(1-3\frac{\eta}{r_0}\right) - \frac{g\alpha}{R}\left(\cos\frac{r_0}{R} - \sin\frac{r_0}{R}\cdot\frac{\eta}{R}\right)$$

Because of $\frac{4L^2}{m^2 r_0^3} = \frac{g\alpha}{R}\cos\frac{r_0}{R}$, so we get

$$\left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r_0}{R}\right)\ddot\eta = \left(\frac{g\alpha}{R^2}\sin\frac{r_0}{R} - \frac{12L^2}{m^2 r_0^4}\right)\eta$$

Just like the analysis of 3.1 and 3.2, we know that r_0 is a stable orbit only if $\frac{g\alpha}{R^2}\sin\frac{r_0}{R} - \frac{12L^2}{m^2 r_0^4} < 0$

that is $\frac{g\alpha}{R^2}\sin\frac{r_0}{R} - \frac{3g\alpha}{r_0 R}\cos\frac{r_0}{R} < 0$

$$\boxed{\frac{r_0}{R}\tan\frac{r_0}{R} < 3}$$

Actually for $y=\frac{r^3}{R^3}\cos\frac{r}{R}$ we have

$$\frac{dy}{dr}=\frac{3r^2}{R^3}\cos\frac{r}{R}-\frac{r^3}{R^4}\sin\frac{r}{R}$$

$$=\frac{r^2}{R^3}\left(3-\frac{r}{R}\tan\frac{r}{R}\right)\cos\frac{r}{R}$$

So $\frac{r_0}{R}\tan\frac{r_0}{R}<3$ means that

$\frac{dy}{dr}>0$ that is the smaller r is stable

So the only stable orbit is given by smaller r_0 $\left(\frac{4L^2}{m^2}\frac{1}{g\alpha R^2}<\left(\frac{r^3}{R^3}\cos\frac{r}{R}\right)_{max}\right)$

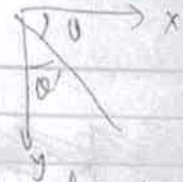
(d) From above analysts we have

$$\left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r_0}{R}\right)\ddot{\eta}=-\frac{g\alpha}{R}\left(\frac{3}{r_0}\cos\frac{r_0}{R}-\frac{1}{R}\sin\frac{r_0}{R}\right)\eta$$

So the frequency is

$$\omega=\sqrt{\frac{g\alpha}{R}\left(\frac{3}{r_0}\cos\frac{r_0}{R}-\frac{1}{R}\sin\frac{r_0}{R}\right)\Big/\left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r_0}{R}\right)}$$

3.13 (a)



$\theta' = \frac{\pi}{2} - \theta$

$\dot\theta' = -\dot\theta$

$x = l\sin\theta' = l\cos\theta$

$y = l\cos\theta' = l\sin\theta$

$T = \frac{m}{2}(\dot x^2 + \dot y^2) + mgy$

(b) $y = \sqrt{l^2 - x^2}$ $\quad \dot y = \frac{-x\dot x}{\sqrt{l^2-x^2}}$

$\Rightarrow L = \frac{m}{2}\left(\dot x^2 + \frac{x^2\dot x^2}{l^2-x^2}\right) + mg\sqrt{l^2-x^2}$

$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right)$

simplifies to ✓ $\rightarrow \frac{\ddot x l^2}{l^2-x^2} + \frac{\dot x^2 x}{(l^2-x^2)^2} + \frac{gx}{\sqrt{l^2-x^2}} = 0$

$$\Rightarrow \boxed{\frac{mx\dot x^2}{l^2-x^2} + \frac{mx^3\dot x^2}{(l^2-x^2)^2} - \frac{mgx}{\sqrt{l^2-x^2}} = \frac{ml^2\ddot x}{l^2-x^2} + \frac{2ml^2\dot x^2 x}{(l^2-x^2)^2}}$$

$\dot x = l\cos\theta'\dot\theta'$ $\quad \ddot x = -l\sin\theta'\dot\theta' + l\cos\theta'\ddot\theta'$

$$\Rightarrow \frac{ml^2}{l^2\cos^2\theta}\left(-l\sin\theta'\dot\theta'^2 + l\cos\theta'\ddot\theta'\right) + \frac{2ml^2\, l^2\cos^2\theta'\,\dot\theta'^2\, l\sin\theta}{l^4\cos^4\theta'}$$

$$= \frac{ml\sin\theta\, l^2\cos^2\theta'\,\dot\theta'^2}{l^2\cos^2\theta'} + \frac{ml^3\sin^3\theta'\cos^2\theta'\,\dot\theta'^2}{l^4\cos^4\theta'} - \frac{mgl\sin\theta'}{l\cos\theta'}$$

$\Rightarrow \frac{ml\ddot\theta' + mgl\sin\theta'}{\cos\theta'} = 0$

$\Rightarrow l\ddot\theta' + g\sin\theta' = 0$ ---- (16.5) In 16.5 example θ is ~~what we pick θ'~~ θ' we pick here

$\frac{r_{12}}{12} < \quad 3g2 \quad r_{12}$

(c) 18.10 $\Rightarrow \int_{t_1}^{t_2} \left[\sum \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] dt = 0$

in terms of δx δy

$$\Rightarrow \int_{t_1}^{t_2} \delta x \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) + \delta y \left(\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \right) \Big] dt = 0$$

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + mgy$$

$$\Rightarrow \int_{t_1}^{t_2} \delta x (-m\ddot{x}) + \delta y (mg - m\ddot{y}) \Big] = 0$$

$y = \sqrt{l^2 - x^2}$ $\quad \delta y = \frac{-x\,\delta x}{\sqrt{l^2-x^2}}$ $\quad \dot{y} = \frac{-x\dot{x}}{\sqrt{l^2-x^2}}$ $\quad \ddot{y} = \frac{-x\ddot{x}}{\sqrt{l^2-x^2}} - \frac{\dot{x}^2}{\sqrt{l^2-x^2}} - \frac{x^2\dot{x}^2}{(l^2-x^2)^{3/2}}$

$$\Rightarrow \left(-m - \frac{mx^2}{l^2-x^2} \right) \ddot{x} - \frac{mgx}{\sqrt{l^2-x^2}} - \frac{mx\dot{x}^2}{l^2-x^2} - \frac{mx^3\dot{x}^2}{(l^2-x^2)^2} - \frac{ml^2}{l^2-x^2}\ddot{x} - \frac{mgx}{\sqrt{l^2-x^2}}$$

$$+ \frac{mx\dot{x}^2}{l^2-x^2} + \frac{mx^3\dot{x}^2}{(l^2-x^2)^2} - \frac{2m\dot{x}^2 x\,(l^2-x^2+x^2)}{(l^2-x^2)^2} = 0$$

so $\frac{ml^2}{l^2-x^2}\ddot{x} + \frac{2ml^2\dot{x}^2 x}{(l^2-x^2)^2} = \frac{mx\dot{x}}{l^2-x^2} + \frac{mx^3\dot{x}^2}{(l^2-x^2)^2} - \frac{mgx}{\sqrt{l^2-x^2}}$

it is the same equation in (b)

(d) $\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + mgy$ $\qquad f = x^2 + y^2 - l^2 = 0$

$$\begin{cases} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial f}{\partial x} = -m\ddot{x} + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda \frac{\partial f}{\partial y} = mg - m\ddot{y} + 2\lambda y = 0 \end{cases}$$

for 19.13 in polar coordinates

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + const$$

constraint $f = r - l = 0$

$$\begin{cases} \dfrac{\partial L}{\partial r} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} + \lambda \dfrac{\partial f}{\partial r} = 0 \\[2ex] \dfrac{\partial L}{\partial \theta} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} + \lambda \dfrac{\partial f}{\partial \theta} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} ml\dot{\theta}^2 + mg\cos\theta + \lambda = 0 & (1) \\ -mgl\sin\theta - ml^2\ddot{\theta} = 0 & (2) \end{cases}$$

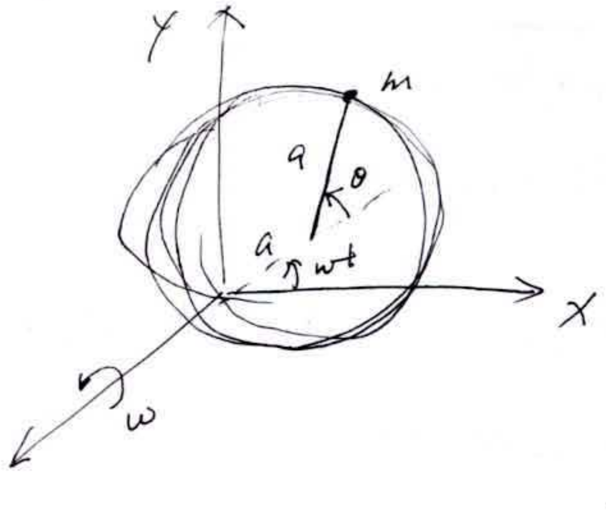
(2) $\Rightarrow$ equation of θ

and then back to (1) can get λ

$\Rightarrow$ it is easier than using x, y coordinates

Jialing

3.15



~~No~~ Ignore friction & Gravity.

$$x = a\cos\omega t + a\cos(\omega t+\theta)$$

$$y = a\sin\omega t + a\sin(\omega t+\theta)$$

$$\therefore \dot{x} = -a\omega\sin\omega t - a(\omega+\dot\theta)\sin(\omega t+\theta)$$

$$\dot{y} = a\omega\cos\omega t + a(\omega+\dot\theta)\cos(\omega t+\theta)$$

$$\dot{x}^2 = a^2\omega^2\sin^2\omega t + 2a^2\omega(\omega+\dot\theta)\sin\omega t\sin(\omega t+\theta) + a^2(\omega+\dot\theta)^2\sin^2(\omega t+\theta)$$

$$\cancel{\dot{y}^2 = a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\dot\theta)}$$

$$\dot{y}^2 = a^2\omega^2\cos^2\omega t + a^2(\omega+\dot\theta)^2\cos^2(\omega t+\theta) + 2a^2\omega(\omega+\dot\theta)\cos\omega t\cos(\omega t+\theta)$$

$$\dot{x}^2+\dot{y}^2 = a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\dot\theta)[\cos\omega t\cos(\omega t+\theta) + \sin\omega t\sin(\omega t+\theta)]$$

$$= a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\dot\theta)\cos\theta$$

$$T = \frac{1}{2}m\left(a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\theta)\cos\theta\right)$$

$$= \frac{ma^2}{2}\left(\omega^2 + (\omega+\dot\theta)^2 + 2\omega(\omega+\dot\theta)\cos\theta\right)$$

$$L = T.$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left[\frac{ma^2}{2}(2(\omega+\dot\theta) + 2\omega\cos\theta)\right] + \frac{ma^2}{2}\cdot 2\omega(\omega+\dot\theta)\sin\theta$$

$$= \frac{ma^2}{2}(2\ddot\theta - 2\omega\sin\theta\,\dot\theta) + ma^2\omega(\omega+\dot\theta)\sin\theta = 0.$$

$$\Rightarrow \ddot\theta = -\omega^2\sin\theta$$

Jiang

a). $x = a\cos\omega t + \Omega\cos(\omega t+\theta)$

$y = a\sin\omega t + \Omega\sin(\omega t+\theta)$

$\dot{x} = -\omega a\sin\omega t + \dot{\Omega}\cos(\omega t+\theta) - \Omega(\omega+\dot{\theta})\sin(\omega t+\theta)$

$\dot{y} = \omega a\cos\omega t + \dot{\Omega}\sin(\omega t+\theta) + \Omega(\omega+\dot{\theta})\cos(\omega t+\theta)$

$\therefore\ \dot{x}^2+\dot{y}^2 = \dot{\Omega}^2\cos^2(\omega t+\theta) - 2\dot{\Omega}\cos(\omega t+\theta)\left[\omega a\sin\omega t + \Omega(\omega+\dot{\theta})\sin(\omega t+\theta)\right] + (\omega a\sin\omega t + \Omega(\omega+\dot{\theta})\ldots)^2$

$+\ \dot{\Omega}^2\sin^2(\omega t+\theta) + 2\dot{\Omega}\sin(\omega t+\theta)\left[\omega a\cos\omega t + \Omega(\omega+\dot{\theta})\cos(\omega t+\theta)\right] + (\omega a\cos\omega t + \Omega(\omega+\dot{\theta})\cos(\omega t+\theta))^2$

$L = T$ (ignore gravity) only kinetic.

$= \frac{1}{2}m\left(a^2\omega^2 + \Omega^2(\omega+\dot{\theta})^2 + 2a\Omega\omega(\omega+\dot{\theta})\cos\theta + \dot{\Omega}^2 + 2\omega a\dot{\Omega}\sin\theta\right)$

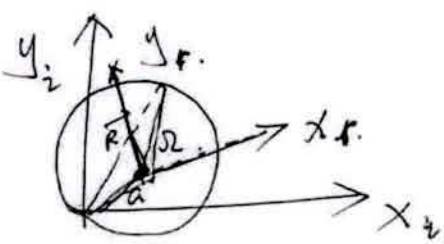
generalized force. $Q = \frac{d}{dt}\frac{\partial L}{\partial\dot{\Omega}} - \frac{\partial L}{\partial\Omega}$

$= \frac{d}{dt}(m\dot{\Omega} + m\omega a\sin\theta) - \left[m\Omega(\omega+\dot{\theta})^2 + ma\omega(\omega+\dot{\theta})\cos\theta\right]$

$= m\ddot{\Omega} + m\omega a\dot{\theta}\cos\theta - m\Omega(\omega+\dot{\theta})^2 - ma\omega(\omega+\dot{\theta})\cos\theta$

$= \underline{-ma(\omega+\dot{\theta})^2 - ma\omega^2\cos\theta}$ $\quad(\ddot{\Omega}=0)$ and $\Omega = a$.

b). $\left(\frac{d\vec{\Omega}}{dt}\right)_i = \left(\frac{d\vec{\Omega}}{dt}\right)_r + \vec{\omega}\times\vec{\Omega}$.



$\vec{R} = \vec{a} + \vec{\Omega}$

$\left(\frac{d\vec{R}}{dt}\right)_i = \left(\frac{d\vec{a}}{dt}\right)_i + \left(\frac{d\vec{\Omega}}{dt}\right)_i = \omega a\hat{\theta}_\omega + \Omega\dot{\theta}\hat{\theta} + \omega\Omega\hat{\theta}$

$$\therefore \left(\frac{d^2\vec{r}}{dt^2}\right)_i = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= -r\dot{\theta}^2\hat{r} - 2\omega r\dot{\theta}\hat{r} - \omega^2 r\hat{r}$$

$$= -a\dot{\theta}^2\hat{r} - 2a\omega a\dot{\theta}\hat{r} - \omega^2 a\hat{r}$$

$$\therefore F = m\left(\frac{d^2\vec{r}}{dt^2}\right) = m\left[a\omega^2\cos\theta\,\hat{r} + a\omega^2\sin\theta\,\hat{\theta} - \underline{\hat{r}(a\dot{\theta}^2 - 2\omega a\dot{\theta} - \omega^2 a)}\right]$$

Component of normal force due to acceleration of the centre of the hoop.

not hat of the normal force.

Centrifugal force.

离心力.