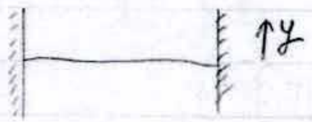
will wilson

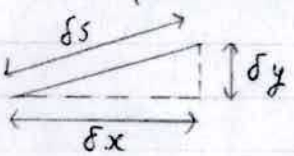
Pset 2 Solutions

Physics 200A Homework 2

(a) 

The string is continuous, so the Lagrangian density, $\mathcal{L}$, is of interest

The potential energy is the work done to stretch the string (from equilibrium)



$$\int dV = \int T\,ds - \int T\,dx$$

tension

$$\Longrightarrow V = \int T\left[1 + \left(\tfrac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx - \int T\,dx$$

$$\Longrightarrow V = \int T\left\{\left[1 + (\partial_x y)^2\right]^{\frac{1}{2}} - 1\right\} dx$$

So, $\mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 - T\left\{\left[1 + (\partial_x y)^2\right]^{\frac{1}{2}} - 1\right\}$ ✓ $\quad L = \int_0^L dx\,\mathcal{L}$

mass per unit length $\quad S = \int dt\,L$

The E-L equations (for $\mathcal{L} = \mathcal{L}(\partial_t y, \partial_x y)$) are given by: → derived by $\delta S = \int dt \int dx \left(\frac{\partial\mathcal{L}}{\partial y}\delta y + \frac{\partial\mathcal{L}}{\partial y'}\delta y'\right) = 0$ ...

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t y)} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial_x y)} = 0 \qquad \boxed{\frac{\partial L}{\partial y} = 0}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t y)} = \mu\,\partial_t y \qquad \frac{\partial\mathcal{L}}{\partial(\partial_x y)} = -T\,\partial_x y\left[1 + (\partial_x y)^2\right]^{-\frac{1}{2}}$$

$$\Longrightarrow \partial_t\,\mu\,\partial_t y = \partial_x\left\{T\,\partial_x y\left[1 + (\partial_x y)^2\right]^{-\frac{1}{2}}\right\}$$

If μ, T constant:

$$\mu \partial_t^2 y = T \partial_x \{ \partial_x y [1 + (\partial_x y)^2]^{-\frac{1}{2}} \}$$

This reduces to a wave equation if $|\partial_x y| << 1$ i.e. for a string whose displacement from equilibrium does not change rapidly along the length of the string

$$\mu \partial_t^2 y = T \partial_x^2 y$$

b) Taking $\mathcal{L} = \frac{1}{2} \mu (\partial_t y)^2 - T[1 + (\partial_x y)^2]^{\frac{1}{2}}$ [The '$-T$' term has no effect on the e.o.m. so has been disregarded]

The momentum density conjugate to y:

$$\pi_y = \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t y)} = \mu \partial_t y$$

$\Rightarrow$ the Hamiltonian density, $\mathcal{H} = \pi \left(\frac{\pi}{\mu}\right) - \frac{1}{2} \mu \left(\frac{\pi}{\mu}\right)^2 + T[1 + (\partial_x y)^2]^{\frac{1}{2}}$

$$\mathcal{H} = \frac{1}{2} \frac{\pi^2}{\mu} + T[1 + (\partial_x y)^2]^{\frac{1}{2}}$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial \pi} = \frac{\pi}{\mu}$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial y} = 0$$

c) $\mathcal{H} = \frac{1}{2}\frac{\pi^2}{\mu} + \underbrace{T\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{1}{2}}}$

$$= T\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \dots\right]$$

$$\mathcal{H} = \frac{1}{2}\frac{\pi^2}{\mu} + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

& also $\mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 - \frac{1}{2}T(\partial_x y)^2$

More generally, taking $\mathcal{H} = \pi\partial_t y - \mathcal{L}$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial t} = \underbrace{\frac{\partial \pi}{\partial t}}_{\substack{= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t y)} \\ = \frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x y)}}}\frac{\partial y}{\partial t} + \cancel{\pi\frac{\partial^2 y}{\partial t^2}} - \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial t} - \cancel{\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_t y)}}_{=\pi}\frac{\partial(\partial_t y)}{\partial t}} - \frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial(\partial_x y)}{\partial t}$$

E-L eqns

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial t} = \cancel{\frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial t}} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial y}{\partial t} - \cancel{\frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial t}} - \frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial(\partial_x y)}{\partial t}$$

$$= -\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial y}{\partial x} - \frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial^2 y}{\partial x \partial t}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\frac{\partial y}{\partial t}\right\}$$ in 1-D. For 3D, $\partial_x \to \nabla$

c.f. the Poynting theorem:

$$-\frac{du}{dt} = \vec{\nabla}\cdot\vec{S} + \vec{J}_f \cdot \vec{E}$$

$\vec{S}$: Poynting vector; $\vec{J}_f$: free current density; $\vec{E}$: electric field; u: energy density

$\mathcal{H} \leftrightarrow u$ energy density

$\frac{\partial \mathcal{L}}{\partial(\partial_x y)} \frac{\partial y}{\partial t} \leftrightarrow S_x$ energy flux density

$$\frac{dH}{dt} = -\int \vec{\nabla}\cdot\vec{S}\, d^3x = -\int \vec{S}\cdot d\vec{S} \quad \text{in general}$$

In this case: $\frac{\partial \mathcal{L}}{\partial(\partial_x y)} = -T\partial_x y$

so $\vec{S} = -T\partial_x y\, \partial_t y\, \hat{e}_1$

unit vector in the $+x$ direction

PSet 2: Problem 2

Tom Zdyrski

Problem 2: Consider a particle with velocity $\vec{v_1}$ moving in a region with constant potential U_1 into a region with constant potential U_2 (c.f. fig. 1).

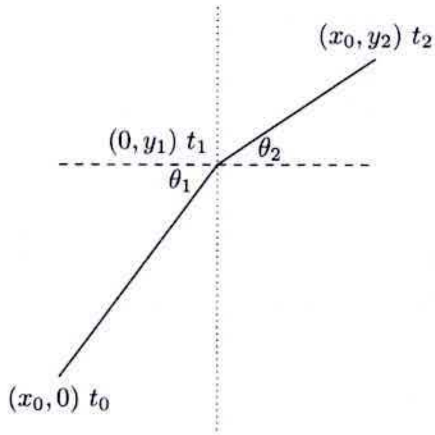


Figure 1: Setup of problem

Part a): *v_1 is the initial velocity. How does the direction change? What is v_2?* For this problem, we fix the end points $(x_0, 0)$ and (x_2, y_2) as well as the end times t_0 and t_2. We have shown in class that free particles move along geodesics (in flat space, straight lines at constant velocity). Since the potential U is constant on either side of the boundary, the force $\vec{F} = \vec{\nabla} U$ is also zero on either side of the boundary. Thus, we immediately know it follows a straight path in either region. Labeling the crossing point as $(0, y_1)$ at time t_1, we get:

$$\begin{aligned}
S = \int_{t_0}^{t_2} L dt &= \int_{t_0}^{t_1} \left(\frac{1}{2}mv^2 - U\right) dt + \int_{t_1}^{t_2} \left(\frac{1}{2}mv^2 - U\right) dt \\
&= \left(\frac{1}{2}mv_1^2 - U_1\right) \int_{t_0}^{t_1} dt + \left(\frac{1}{2}mv_2^2 - U_2\right) \int_{t_1}^{t_2} dt \\
&= \left(\frac{1}{2}mv_1^2 - U_1\right)(t_1 - t_0) + \left(\frac{1}{2}mv_2^2 - U_2\right)(t_2 - t_1) \\
&= \frac{m(x_0^2 + y_1^2)}{2t_1} - U_1 t_1 + \frac{m(x_0^2 + (y_2 - y_1)^2}{2(t_2 - t_1)} - U_2 t_2
\end{aligned}$$

Since the endpoints are fixed, $S = S(y_1, t_1)$. Then, the principle of least action gives:

$$dS = 0\frac{\partial S}{\partial y_1} dy_1 + \frac{\partial S}{\partial t_1} dt_1$$

Since dy_1 and dt_1 are arbitrary, the partials must each be identically zero. Then,

$$\frac{\partial S}{\partial y_1} = m\frac{y_1}{t_1} - m\frac{y_2 - y_1}{t_2 - t_1} \implies mv_1 \sin\theta_1\ mv_2 \sin\theta_2 \implies v_1 \sin\theta_1 = v_2 \sin\theta_2$$

Doing the same for the time-derivative yields,

$$\frac{\partial S}{t_1} = 0 = -\frac{1}{2}m\frac{x_0^2 + y_1^2}{t_2^2} - U_1 + \frac{1}{2}m\frac{x_0^2 + (y_2 - y_1)^2}{(t_2 - t_1)^2} + U_2 \implies \frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2 \implies E_1 = E_2$$

Thus, we have

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 + U_1 - U_2 \implies v_2 = v_1\sqrt{1 + \frac{U_1 - U_2}{\frac{1}{2}mv_1^2}}$$

and

$$\sin\theta_2 = \frac{v_1}{v_2}\sin\theta_1 = \frac{\sin\theta_1}{\sqrt{1 + \frac{U_1 - U_2}{\frac{1}{2}mv_1^2}}}$$

(a) Part b): *Find the ratio of times in the same path for particles with different masses but the same U.* Multiplying the Lagrangian by a constant doesn't change the trajectory of the system. Thus, if $m \to \alpha m$ and $t \to \beta t$, we know the constant potentials $U_{1,2} \to U_{1,2}$ is unchanged. Therefore, we require the kinetic term is also unchanged: $1/2mv^2 \to 1/2mv^2\alpha/\beta^2$. That is, $\beta = \sqrt{\alpha} \implies t \propto \sqrt{m}$ which tells us

$$\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}$$

(b) Part c): *Find the ratio of times in the same path for particles with the same mass but moving [in] different potentials U_1, U_2, where $U_2/U_1 = c$, a constant.* Likewise, if the two setups differ by changing $U \to cU$ (i.e. $U_2 = cU_1$), then we must have $t \to \beta t$ s.t. $1/2mv^2 \to 1/2mv^2/\beta^2 = 1/2mv^2c$. This shows that $\beta = 1/\sqrt{c} \implies t \propto 1/\sqrt{U}$ showing that

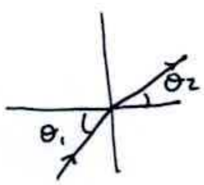
$$\frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$

(c) Part d): *What problem in optics does this problem resemble?* This problem resembles Snell's Law of refraction. Like Snell's law, we see that the particle paths bend when the cross the boundary. However, a key difference exists: In Snell's law, $\sin\theta \propto v$ whereas here $\sin\theta \propto 1/v$. This represents the fact that light waves refract towards areas with slower wave speed whereas particles refract away from areas with slower particle speed.

* Alternative solution to (a):

the potential energy is independent of the coordinates whose axes are parallel to the plane separating the half-spaces

$\Rightarrow$
- $mv_1 \sin\theta_1 = mv_2 \sin\theta_2$
- $\frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2$



$\Rightarrow$ Solution.

Robin Heinonen

3. a $$L = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - V(\vec{r})$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{m\dot{\vec{r}}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

$$\frac{\partial L}{\partial \vec{r}} = -\frac{\partial V(\vec{r})}{\partial \vec{r}} = -\nabla V(\vec{r})$$

The eq. of motion is then

$$\boxed{\frac{d\vec{p}}{dt} = -\nabla V(\vec{r})}$$

since $\boxed{\vec{p} = \gamma m \vec{v}}$ is the correct relativistic momentum, this Lagrangian reproduces relativistic mechanics. Explicitly,

$$\frac{d\vec{p}}{dt} = \frac{m\ddot{\vec{r}}\left(1-\frac{v^2}{c^2}\right)^{1/2} + \frac{m\dot{\vec{r}}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2}}{1-\frac{v^2}{c^2}}$$

$$= \gamma m\ddot{\vec{r}} + \gamma^3 m\dot{\vec{r}}\left(\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2}\right) = -\nabla V(\vec{r})$$

b. As found above,

$$\vec{p} = \gamma m\dot{\vec{r}}$$

so $$H = \vec{p}\cdot\dot{\vec{r}} - L = \gamma m v^2 + \frac{mc^2}{\gamma} + V(\vec{r})$$

$$= \frac{m\left(v^2+c^2\left(1-\frac{v^2}{c^2}\right)\right)}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + V(\vec{r}) = \underline{\gamma mc^2 + V(\vec{r})}$$

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due to printing error

$$\text{Now, } p^2c^2 + m^2c^4 = \frac{m^2c^2v^2}{1 - v^2/c^2} + \frac{m^2c^4\left(1 - \frac{v^2}{c^2}\right)}{(1 - v^2/c^2)}$$

$$= \frac{m^2c^4}{(1 - v^2/c^2)} \quad \text{so indeed}$$

$$H = \gamma mc^2 + V(\vec{r}) = \underline{(p^2c^2 + m^2c^4)^{1/2} + V(\vec{r})}$$

This is a const. of the motion since $\frac{\partial H}{\partial t} = 0$

c. If $V = V(r)$, then

$$\dot{\vec{p}} = -\nabla V = -\frac{\partial}{\partial r} V(r)\,\hat{r}$$

$$\text{and } \frac{d}{dt}(\vec{r} \times \vec{r}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}$$

$$= \gamma m(\dot{\vec{r}} \times \dot{\vec{r}}) + \vec{r} \times (-V'(r)\,\hat{r})$$

$$= 0$$

Hence $\vec{r} \times \vec{p}$ is a const of the motion.

Since angular momentum is conserved, the particle moves in a plane.
Orient $\hat{z}$ normal to this plane, so that $p_\theta = 0$ and $\theta = \frac{\pi}{2}$. Then

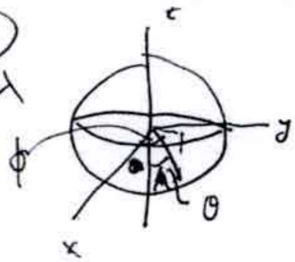
$$p^2 = p_r^2 + \frac{p_\varphi^2}{r^2\sin^2\theta} + \frac{p_\theta^2}{r^2}$$

$$= p_r^2 + \frac{p_\varphi^2}{r^2}$$

$$\Rightarrow H = c\left(p_r^2 + \frac{p_\varphi^2}{r^2} + m^2c^2\right)^{1/2} + V(r)$$

(6.2)

#4



$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\phi}^2) + mgl\cos\theta$$
$$= \tfrac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\phi}^2) + mgl\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \Rightarrow \dot{\theta} = p_\theta / ml^2$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\sin^2\theta\cdot\dot{\phi} \Rightarrow \dot{\phi} = p_\phi / ml^2\sin^2\theta$$

$$H = \sum_\sigma p_\sigma \dot{q}_\sigma - L = \frac{p_\theta^2}{ml^2} + \frac{p_\phi^2}{ml^2\sin^2\theta} - \frac{1}{2}m\left(l^2\cdot\frac{p_\theta^2}{(ml^2)^2} + l^2\sin^2\theta\cdot\frac{p_\phi^2}{(ml^2\sin^2\theta)^2}\right) - mgl\cos\theta$$

$$\mathcal{H} = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} \Rightarrow \frac{p_\phi}{ml^2\sin^2\theta} = \dot{\phi}, \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \Rightarrow \frac{p_\theta}{ml^2} = \dot{\theta} \Rightarrow \dot{p}_\theta = \ddot{\theta}ml^2$$

$$\frac{\partial H}{\partial \phi} = \dot{p}_\phi \Rightarrow 0 = -\dot{p}_\phi \Rightarrow p_\phi = \text{const.}$$

(I'm interpreting ~~expand Hamiltonian~~ as expand Hamiltonian equations of motion...)

$$\frac{\partial H}{\partial\theta} = -\dot{p}_\theta \Rightarrow mgl\sin\theta - \left(\frac{p_\phi^2}{ml^2}\right)\cdot\frac{\cos\theta}{\sin^3\theta} = \dot{p}_\theta$$

$$\Rightarrow mgl\sin\theta - ml^2\dot{\phi}^2\sin\theta\cos\theta = -\ddot{\theta}ml^2$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta - \dot{\phi}^2\sin\theta\cos\theta = 0$$

For uniform circular motion $\ddot{\theta} = \dot{\theta} = 0$.

$$\cos\theta_0 = \frac{g}{l\dot{\phi}^2}$$

Near Unif Circular Motion: Neglect $O((\theta-\theta_0)^2)$

$$\ddot{\theta} + \frac{g}{l}\left(\sin\theta_0 + \cos\theta_0(\theta-\theta_0) - \cancel{\sin\theta_0(\theta-\theta_0)^2}\right) - \dot{\phi}^2\left(\sin\theta_0 + \cos\theta_0(\theta-\theta_0)\right)\left(\cos\theta_0 - \sin\theta_0\cdot(\theta-\theta_0)\right) = 0.$$

$$\ddot{\theta} + \frac{g}{l}\left(\cos\theta_0\cdot(\theta-\theta_0) - \cancel{\sin\theta_0(\theta-\theta_0)^2}\right) - \dot{\phi}^2\left(-\sin^2\theta_0(\theta-\theta_0) + \cos^2\theta_0(\theta-\theta_0) - \cancel{\sin\theta_0\cos\theta_0(\theta-\theta_0)^2}\right)$$

$$\ddot{\theta} + \left[\frac{g}{l}\cos\theta_0 + \dot{\phi}^2\left(1 - 2\cos^2\theta_0\right)\right](\theta-\theta_0)$$

Next Page ↓.

(6.2) continued

This works out better if we use

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \left(\frac{p_\phi}{ml^2}\right)^2 \frac{\cos\theta}{\sin^3\theta} = 0$$

Note: $\dfrac{\cos\theta}{(\sin\theta)^3} \simeq \dfrac{\cos\theta_0 - \sin\theta_0\cdot\epsilon}{(\sin\theta_0 + \epsilon\cdot\cos\theta_0)^3} = \dfrac{\cos\theta_0 - \epsilon\sin\theta_0}{\sin^3\theta_0\left(1+\epsilon\cdot\frac{\cos\theta_0}{\sin\theta_0}\right)^3} \simeq \dfrac{(\cos\theta_0 - \epsilon\cdot\sin\theta_0)\left(1-3\frac{\cos\theta_0}{\sin\theta_0}\epsilon\right)}{\sin^3\theta_0}$

$\theta = \theta_0 + \epsilon$

$$\Rightarrow \ddot{\epsilon} + \frac{g}{l}\left(\sin\theta_0 + \epsilon\cdot\cos\theta_0\right) - \left(\frac{p_\phi}{ml^2}\right)^2\left(\frac{\cos\theta_0}{\sin^3\theta_0} - \frac{3\cos^2\theta_0}{\sin^4\theta_0}\epsilon - \epsilon\cdot\frac{1}{\sin^2\theta_0}\right) = 0.$$

Recall $\cos\theta_0 = \dfrac{g}{l\dot{\phi}_0^2}$, $\dot{\phi}_0 = \dfrac{p_\phi}{ml^2\sin^2\theta_0}$, $\left(\dfrac{p_\phi}{ml^2}\right)^2 = \dot{\phi}_0^2\sin^4\theta_0$

$$\Rightarrow \ddot{\epsilon} + \underbrace{\left(\frac{g}{l}\sin\theta_0 - \dot{\phi}_0^2\sin\theta_0\cos\theta_0\right)}_{=0,\ \text{since } \dot{\phi}_0^2\cos\theta_0 = g/l} + \epsilon\cdot\underbrace{\left(\frac{g}{l}\cos\theta_0 + 3\dot{\phi}_0^2\cos^2\theta_0 + \dot{\phi}_0^2\sin^2\theta_0\right)}_{\epsilon\cdot\dot{\phi}_0^2\left(4\cos^2\theta_0 + 1 - \cos^2\theta_0\right)} = 0$$

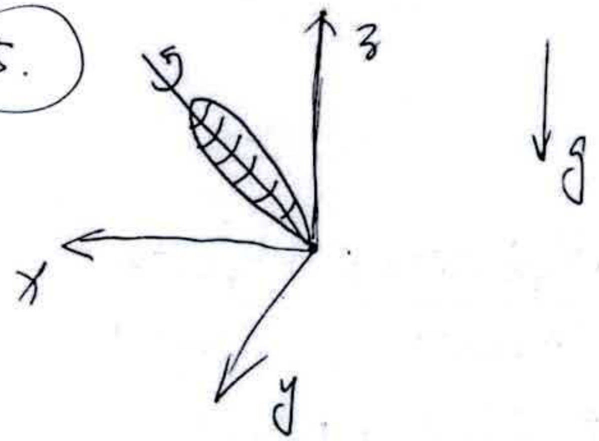
$$\Rightarrow \ddot{\epsilon} = -\epsilon\cdot\dot{\phi}_0^2\left(1 + 3\cos^2\theta_0\right)$$

$$\boxed{\ddot{\epsilon} = -\epsilon\cdot\frac{g}{l\cos\theta_0}\cdot\left(1+3\cos^2\theta_0\right)}$$

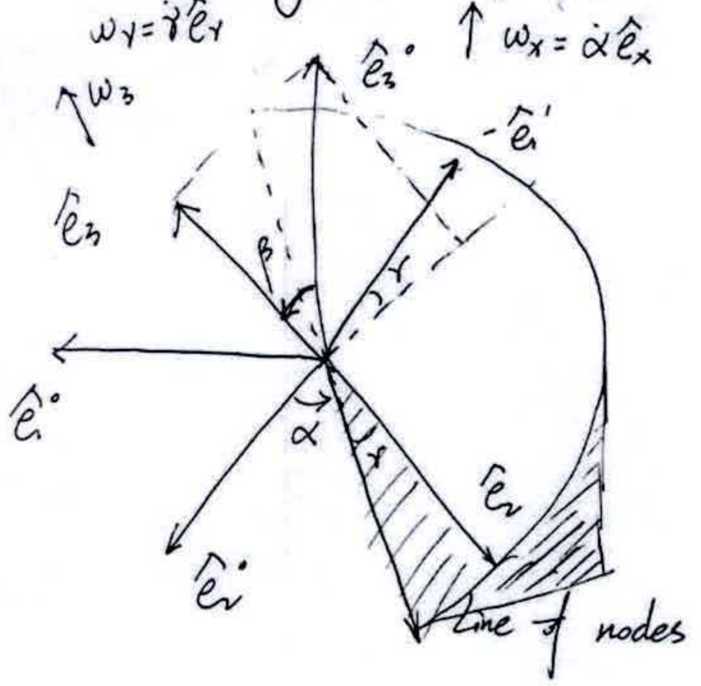
[redacted] Fei A53111966

Problem Set II.

5.

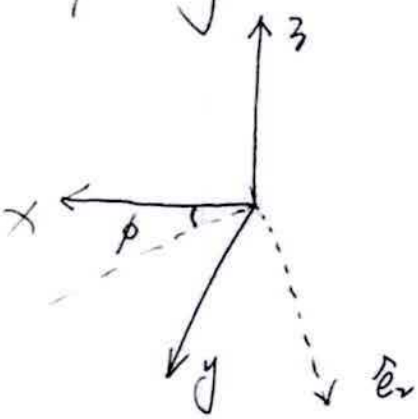


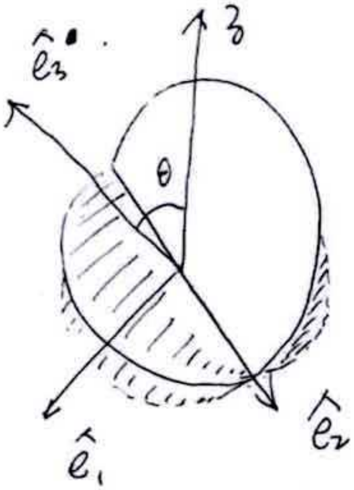
Euler Angles Definition. α. β. γ P155

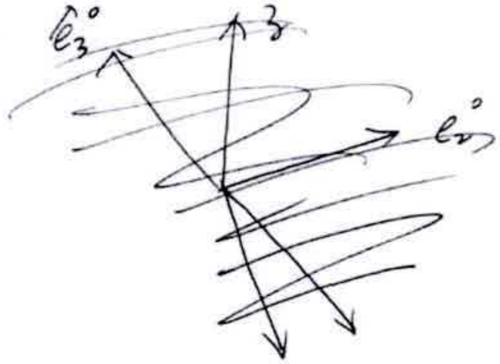


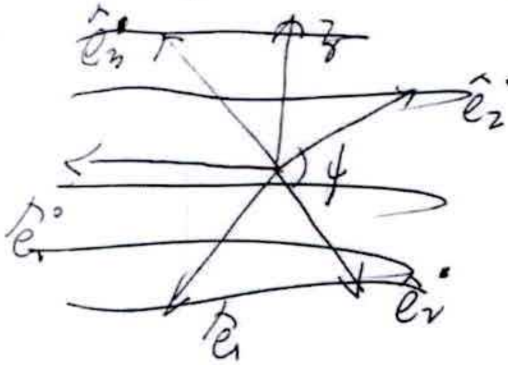
$\omega_\beta = \dot\beta \hat e_\beta$

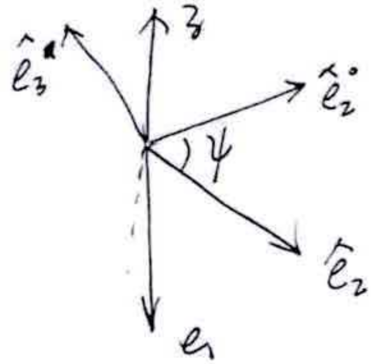
Seperately











spin .

$$\vec{\omega} = \dot{\phi}\,\hat{z} + \dot{\theta}\,\hat{e}_2 + \dot{\psi}\,\hat{e}_3$$

$$\hat{z} = \cos\theta\,\hat{e}_3^\circ - \sin\theta\,\hat{e}_1 \Rightarrow \boxed{\vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1 + \dot{\theta}\,\hat{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3}$$

Angular Momentum $\vec{L}$ $(\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$

$$\vec{L} = -\lambda_1\dot{\phi}\sin\theta\,\hat{e}_1 + \lambda_2\dot{\theta}\,\hat{e}_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$

As a symmetric top. $\lambda_1 = \lambda_2$: $T = \frac{1}{2}\lambda_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 \quad \boxed{\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta}$$

$$U = MgR\cos\theta$$

$$\boxed{P_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \lambda_1\dot{\theta}} \quad \boxed{P_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)} \quad \boxed{P_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + P_\psi\cos\theta}$$

$$\Rightarrow T = \frac{(\lambda_1\dot{\phi}\sin^2\theta)^2}{2\lambda_1\sin^2\theta} + \frac{\lambda_1}{2}\cdot\left(\frac{P_\theta}{\lambda_1}\right)^2 + \frac{\lambda_3}{2}\left(\frac{P_\psi}{\lambda_3}\right)^2 = \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{P_\theta^2}{2\lambda_1} + \frac{P_\psi^2}{2\lambda_3}$$

$$\therefore H = T + MgR\cos\theta = \sum p\dot{q} - \mathcal{L}$$

Show explicity:

$$H = p_\theta \dot{\theta} + p_\psi \dot{\psi} + p_\varphi \dot{\varphi} - \mathcal{L}$$

$$\dot{\theta} = \frac{p_\theta}{\lambda_1} \; ; \; \dot{\psi} = \frac{p_\psi}{\lambda_3} - \dot{\varphi}\cos\theta \; ; \; \dot{\varphi} = \frac{p_\varphi - p_\psi \cos\theta}{\lambda_1 \sin^2\theta}$$

$$\Rightarrow p_\theta \dot{\theta} = \frac{p_\theta^2}{\lambda_1} \; ; \; p_\psi \dot{\psi} + p_\varphi \dot{\varphi} = \frac{p_\psi^2}{\lambda_3} - p_\psi \dot{\varphi}\cos\theta + \frac{p_\varphi^2}{\lambda_1 \sin^2\theta} - \frac{p_\varphi p_\psi \cos\theta}{\lambda_1 \sin^2\theta}$$

$$\underbrace{\begin{aligned} p_\psi &= \lambda_3 \dot{\psi} + \lambda_3 \dot{\varphi}\cos\theta \\ p_\varphi &= \lambda_1 \dot{\varphi} \sin^2\theta + p_\psi \cos\theta \end{aligned}}$$

$$p_\varphi \dot{\varphi} = \frac{p_\varphi}{\lambda_1}\left(p_\varphi - p_\psi \cos\theta\right) \csc\theta^2$$

$$p_\psi \dot{\psi} = \frac{p_\psi^2}{\lambda_3} + \frac{p_\psi}{\lambda_1}\left(p_\psi \cos\theta - p_\varphi\right) \cot(\theta) \csc(\theta)$$

$$\Rightarrow H = \frac{p_\theta^2}{\lambda_1} + \frac{p_\psi^2}{\lambda_3} + \frac{p_\varphi^2}{\lambda_1 \sin^2\theta} - \frac{p_\varphi p_\psi \cos\theta}{\lambda_1 \sin^2\theta} + \frac{p_\psi^2}{\lambda_1}\frac{\cos^2\theta}{\sin^2\theta} - \frac{p_\psi p_\varphi}{\lambda_1}\frac{\cos\theta}{\sin^2\theta} - \mathcal{L}$$

$$\left[H = \frac{p_\theta^2}{\lambda_1} + \frac{(p_\varphi - p_\psi \cos\theta)^2}{\lambda_1 \sin^2\theta} + \frac{p_\psi^2}{\lambda_3} - \mathcal{L}\right]$$

$$\mathcal{L} = \frac{1}{2}\frac{(p_\varphi - p_\psi \cos\theta)^2}{\lambda_1 \sin^2\theta} + \frac{1}{2}\frac{p_\theta^2}{\lambda_1} + \frac{1}{2}\frac{p_\psi^2}{\lambda_3} - MgR\cos\theta$$

$$\Rightarrow \boxed{H = \frac{1}{2}\frac{p_\theta^2}{\lambda_1} + \frac{1}{2}\frac{p_\psi^2}{\lambda_3} + \frac{1}{2}\frac{(p_\varphi - p_\psi \cos\theta)^2}{\lambda_1 \sin^2\theta} + MgR\cos\theta}$$

Hamilton Equations:

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi - P_\psi \cos\theta}{\lambda_1 \sin^2\theta} = 31.3a \text{ in FW}$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{\lambda_1}$$

$$\dot{\psi} = \frac{\partial H}{\partial P_\psi} = -\frac{P_\phi - P_\psi \cos\theta}{\lambda_1 \sin^2\theta}\cos\theta + \frac{P_\psi}{\lambda_3} = 31.3b \text{ in FW}$$

$$\left.\dot{P}_\phi = \dot{P}_\psi = 0 \right\} = 31.2a,b \text{ in FW}$$

$$P_\theta = -\frac{\partial H}{\partial \theta} = MgR\sin\theta + \frac{-2(P_\phi - P_\psi\cos\theta)\overset{P_\psi}{\sin\theta}}{2\lambda_1 \sin^2\theta} + \frac{(P_\phi - P_\psi\cos\theta)^2 \sin\theta\cos\theta}{(\sin^2\theta)^2\, 2\lambda_1}$$

$$= \frac{(P_\psi\cos\theta - P_\phi)\left[\cancel{(P_\psi\cos\theta - P_\phi)}\cos\theta + \sin^2\theta\right]}{\lambda_1 \sin^3\theta} + MgR\sin\theta$$

$$\dot{P_\theta} = -\frac{1}{\lambda_1 \sin^3\theta}\left(-(p_\phi^2 + p_\psi^2)\cos\theta + p_\phi p_\psi(1+\cos^2\theta)\right) + MgR\sin\theta$$

where p_ϕ, p_ψ are constants

so $\lambda_1\ddot{\theta} = \dot{P}_\theta$ reduces to 31.5b in FW

now: $\ddot{\theta}\, d\theta = \frac{d\dot{\theta}}{dt}\, d\theta = \dot{\theta}\, d\dot{\theta}$

$$\Rightarrow \lambda_1 \int \dot{\theta}\, d\dot{\theta} = \int \underbrace{\dot{P}_\theta}_{\text{function of } \theta}\, d\theta$$ } can integrate for solution ✓

#6

a) Lagrangian density

$$\cancel{\mathcal{L}}\ \mathcal{L} = \frac{-\hbar^2}{2m}\left(\frac{d\psi^*}{dx}\right)\left(\frac{d\psi}{dx}\right) - \psi^*(U-E)\psi$$

so equation for ψ^* is given by

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\frac{d\psi^*}{dx}} - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi}$$ which is the time-independent Schrödinger equation

(b) the Lagrangian is invariant under gauge symmetry — not gauge since $\theta \neq \theta(x)$

$$\boxed{\begin{aligned}\psi &\to e^{\frac{i\theta}{\hbar}}\psi \\ \psi^* &\to e^{-\frac{i\theta}{\hbar}}\psi^*\end{aligned}}$$ phase rotation

$$\theta \to \delta\theta \Rightarrow \psi \cancel{\to} \psi + \cancel{\frac{i}{\hbar}\psi(\delta\theta)}\ \psi e^{\frac{i\delta\theta}{\hbar}} \sim \psi + (\delta\theta)Q_\psi$$

$$\psi^* \to \psi^* \cancel{+} + \cancel{e}\psi^* e^{-\frac{i\delta\theta}{\hbar}} \sim \psi^* - \frac{i}{\hbar}\psi^*(\delta\theta)$$

$$= \psi^* + (\delta\theta)Q_{\psi^*}$$

from Noether's theorem

$$\partial^\mu j_\mu \equiv 0$$

where $\boxed{\begin{aligned}Q_\psi &= \frac{i}{\hbar}\psi \\ Q_{\psi^*} &= \frac{-i}{\hbar}\psi\end{aligned}}$

$$\Rightarrow \cancel{j} \quad j_\mu = \frac{\partial L}{\partial(\partial_\mu\phi_a)}\delta\phi_a$$

$$\text{here} \Rightarrow j = -\left(\frac{\partial L}{\partial\frac{d\psi}{dx}}\right)Q_\psi - \left(\frac{\partial L}{\partial\frac{d\psi^*}{dx}}\right)Q_{\psi^*}$$

$$= \frac{\hbar}{2m}\left[\psi^*\frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right]$$

which is the current we want $\cancel{\Rightarrow}$ for time-independent L

$$\Rightarrow \frac{\partial j}{\partial x} = 0$$

for probability $\cancel{p}$ density $P = \psi^*\psi$

$$\frac{\partial P}{\partial t} = \frac{\partial(\psi^*\psi)}{\partial t} = \left(\psi^*\frac{\partial\psi}{\partial t} + \psi^*\frac{\partial\psi}{\partial t}\right)$$

for $\hat{E} = i\hbar\cancel{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}$

$$\Rightarrow \cancel{\partial}\ \frac{d\psi}{dt} = \cancel{\frac{1}{i\hbar}}\ \frac{-i}{\hbar}\left[\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi\right]$$

$$\Rightarrow \frac{d\psi^*}{dt} = \frac{-1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + U\psi^*\right]$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial t}} = \frac{-i}{\hbar}\left[\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2}\psi - U\psi^*\psi - \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\psi^* + U\psi^*\psi\right]$$

$$\boxed{= -\frac{dJ}{dx} = 0}$$

~~or if we know the time dependence Schrödinger equation.~~

$\Rightarrow$ probability is ~~conver~~ conserved

And if we can get time-dependent Schrödinger equation we can use Noether's theorem in 4-vector form

(7) (a) $L = L(q,\dot{q},\ddot{q},t)$

$$S = \int_{t_1}^{t_2} L(q,\dot{q},\ddot{q},t)\,dt \quad , \text{ consider } q = q + \delta q$$

Then $$\delta S = \int_{t_1}^{t_2} L(q,\dot{q},\ddot{q},t)\,dt - \int_{t_1}^{t_2} L(q+\delta q,\dot{q}+\delta\dot{q},\ddot{q}+\delta\ddot{q},t)\,dt$$

$$= \int_{t_1}^{t_2}\Big(L(q,\dot{q},\ddot{q},t) - L(q+\delta q,\dot{q}+\delta\dot{q},\ddot{q}+\delta\ddot{q},t)\Big)\,dt$$

First order Approx in δq

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\delta q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}\right)dt$$

We will use $\delta\dot{q} = \frac{d}{dt}\delta q$, $\delta\ddot{q} = \frac{d^2}{dt^2}\delta q$

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q}\delta q + \underset{①}{\frac{\partial L}{\partial \dot{q}}\frac{d\,\delta q}{dt}} + \underset{②}{\frac{\partial L}{\partial \ddot{q}}\frac{d^2\delta q}{dt^2}}\right)dt$$

①: $\displaystyle\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \dot{q}}\frac{d\delta q}{dt}\right)dt$: let $u = \frac{\partial L}{\partial \dot{q}}$, $dv = \frac{d\delta q}{dt}$

$$= \underbrace{\frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2}}_{0 \text{ b.c. fixed endpoints}} - \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q\right)dt$$

(7) continued #1

$$(2)\quad \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \ddot{q}} \frac{d^2}{dt^2}\delta q\right) dt = \left(\frac{\partial L}{\partial \ddot{q}} \frac{d}{dt}\delta q\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\left(\frac{d}{dt}\delta q\right)\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right)\right] dt$$

$$= \underbrace{\left(\frac{\partial L}{\partial \ddot{q}} \frac{d}{dt}\delta q\right)\Bigg|_{t_1}^{t_2}}_{0 \text{ if we pick } \frac{d\delta q}{dt}(t_1, t_2) = 0} - \underbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \cdot \delta q\right)\Bigg|_{t_1}^{t_2}}_{0 \text{ fixed endpoints}} + \int_{t_1}^{t_2} \left(\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\delta q\right) dt$$

which we will do.

Recap:

$$\delta S = \underbrace{\left(\frac{\partial L}{\partial \dot{q}}\delta q + \frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\delta q\right)}_{\text{All vanish}}\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\right)\delta q \, dt = 0$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0}$$

(7) continued #2

$$0 = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} \Rightarrow \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q}$$

b) $$\frac{dL}{dt} = \dot{q}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\right) + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} + \ddot{q}\frac{\partial L}{\partial \ddot{q}}\right)$$

$$\Rightarrow \frac{d}{dt}\Bigg[\underbrace{\dot{q}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right) + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - L}_{\text{Conserved Quantity (Analog of Energy)}}\Bigg] = 0$$

For $\mathcal{L}(q,\dot{q},t)$ we found $\frac{d}{dt}\Big[\underbrace{\dot{q}\frac{\partial L}{\partial \dot{q}} - L}_{E}\Big] = 0$

Could Express $\frac{d}{dt}\left[\dot{q}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right) + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - L\right] = 0$

or $\frac{d}{dt}\Big[E \underbrace{- \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} + \ddot{q}\frac{\partial L}{\partial \ddot{q}}}_{\text{Corrections to Energy}}\Big] = 0$

⑦ continued #1 (Alternative method (idea) to I.B.P).

$$\delta S = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q + \frac{\partial L}{\partial \ddot{q}} \frac{d^2 \delta q}{dt^2} \right] dt = 0$$

Using i.b.p twice more gets us a term of the form $\left(\frac{\partial L}{\partial \ddot{q}} \frac{d}{dt} \delta q \right) \Big|_{t_1}^{t_2}$, which I see no reason for it to be zero. Instead, we are going to apply a math theorem: Given continuous functions $f_0, f_1, \ldots, f_n$ on (a,b) satisfying

$$\int_a^b \left[f_0(x) h(x) + f_1(x) h'(x) + f_2(x) h''(x) + \cdots + f_n(x) h^{(n)}(x) \right] dx = 0$$ for all smooth functions $h(x)$ on (a,b), then there exist continuously differentiable functions $u_0, u_1, \ldots, u_{n-1}$ on (a,b) such that

$f_0 = u_0'$, $f_1 = u_0 + u_1'$, $\ldots$, $f_{n-1} = u_{n-2} + u_{n-1}'$, $f_n = u_{n-1}$ everywhere in (a,b).

General case ... page on fundamental Lemma of calculus of Variations

For us, $f_0 = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)$, $f_1 = 0$, $f_2 = \frac{\partial L}{\partial \ddot{q}}$, $h(x) = \delta q(t)$.

This theorem tells us:

$f_0 = u_0'$, $f_1 = u_0 + u_1'$, $f_2 = u_1$

Using $f_1 = 0$, $0 = u_0' + u_1''$ $\Rightarrow$ $0 = f_0 + f_2''$ Equation of motion.

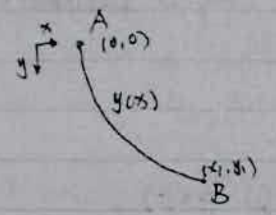
$$\Rightarrow \boxed{0 = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}}$$

Note: this is equivalent to setting $\left(\frac{\partial L}{\partial \ddot{q}} \frac{d}{dt} \delta q \right) \Big|_{t_1}^{t_2} = 0$.

problem of brachistochrone

Yanseng Zhang

8)



Suppose A is the origin (0,0) and B has coordinates (x_1, y_1). $y = y(x)$ is the curve along which we could descend in the least time from A to B.

So $ds = \sqrt{1+(\frac{dy}{dx})^2}\, dx$ along this curve.

And according to energy conservation, we have

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = mgy$$

So $$dt = \sqrt{\frac{1+(\frac{dy}{dx})^2}{2gy}}\, dx$$

We could get the total time taken from A to B.

$$T = \int_0^{x_1} \sqrt{\frac{1+(\frac{dy}{dx})^2}{2gy}}\, dx$$

So we take $\delta T = 0$ for getting least time T.

It's kind of Lagrangian where

$$\boxed{L' = \sqrt{\frac{1+(\frac{dy}{dx})^2}{2gy}}}$$

So we get equation:

$$\boxed{\frac{d}{dx}\left(\frac{\partial L'}{\partial(\frac{dy}{dx})}\right) = \frac{\partial L'}{\partial y}}$$

$L' \Leftrightarrow L$
$x \Leftrightarrow t$
$y \Leftrightarrow x$

However it's hard to solve it. We consider

$$H' = \frac{\partial L'}{\partial(\frac{dy}{dx})} \cdot \frac{dy}{dx} - L' = \frac{(\frac{dy}{dx})^2}{\sqrt{2gy}\sqrt{1+(\frac{dy}{dx})^2}} - \sqrt{\frac{1+(\frac{dy}{dx})^2}{2gy}} = \frac{-1}{\sqrt{2gy}\sqrt{1+(\frac{dy}{dx})^2}}$$

We have $$\frac{dH'}{dx} = \frac{\partial L'}{\partial(\frac{dy}{dx})}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial L'}{\partial(\frac{dy}{dx})}\right) - \frac{\partial L'}{\partial y}\frac{dy}{dx} - \frac{\partial L'}{\partial(\frac{dy}{dx})}\frac{d^2y}{dx^2}$$

$$= 0$$

So $$H' = \frac{-1}{\sqrt{2gy}\sqrt{1+(\frac{dy}{dx})^2}} = \text{const} \triangleq C$$

So $\quad y\left[1+\left(\frac{dy}{dx}\right)^2\right] = \frac{1}{2gC^2} \triangleq 2r$ $\qquad$ Impose variable θ where $x = x(\theta)$

Suppose $\quad \frac{dy}{dx} = \cot\frac{\theta}{2}$

we have $\quad y = 2r\sin^2\frac{\theta}{2} = r(1-\cos\theta)$

Then $\quad dy = r\sin\theta\, d\theta$

we have $\quad \frac{dx}{d\theta} = r\sin\theta \cdot \tan\frac{\theta}{2} = 2r\sin^2\frac{\theta}{2} = r(1-\cos\theta)$

So $\quad x = r(\theta - \sin\theta) + x_0$

Plug $(0,0)$ into $x \cdot y$ we have

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$$

where r could be given by $B(x_1, y_1)$, θ is a real parameter, corresponding to the angle through which the rolling circle has rotated.

Cycloid: A curve traced by a point on the rim of a circular wheel as it rolls along a straight line without slippage.

$$x = r\cos^{-1}\left(1 - \frac{y}{r}\right) - \sqrt{y(2r-y)}$$