Yanzeng Zheng

1) For an oscillator with slowly varing parameter, action is given by

$$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\oint (2E-\omega^2 q^2)^{1/2} dq$$

$$\text{So } \delta I = \frac{1}{2\pi}\left[\int \delta(2E-\omega^2q^2)^{1/2}dq + \int (2E-\omega^2q^2)^{1/2} d\delta q\right]$$

$$= \frac{1}{2\pi}\left[\int \frac{-\omega^2 q}{(2E-\omega^2q^2)^{1/2}}\cdot\delta q\, dq + \int (2E-\omega^2q^2)^{1/2} d\delta q\right]$$

$$= \frac{1}{2\pi}\left[\int \frac{-\omega^2 q}{(2E-\omega^2q^2)^{1/2}}\delta q\, dq - \int \delta q \cdot d(2E-\omega^2q^2)^{1/2}\right]$$

$$= \frac{1}{2\pi}\left\{\int\left[\frac{-\omega^2 q}{(2E-\omega^2q^2)^{1/2}} + \frac{\omega^2 q}{(2E-\omega^2q^2)^{1/2}}\right]\cdot\delta q\, dq\right\}$$

$$= 0$$ where we have fixed E and λ in one loop

It means that action is an adiabatic invariant for such a system.

On the other hand, $I = \frac{1}{2\pi}\oint p\,dq$

$$= \frac{1}{2\pi}\oint (2E-\omega^2q^2)^{1/2}dq$$

Impose $q = \left(\frac{2E}{\omega^2}\right)^{1/2}\sin\alpha$, then

$$I = \frac{1}{2\pi}\cdot 2E/\omega \int_0^{2\pi}\cos^2\alpha\, d\alpha = \frac{2E}{\omega}\cdot\frac{1}{2\pi}\cdot 4\cdot\frac{\pi}{2}\cdot\frac{1}{2} = E/\omega$$

which means that E/ω is an adiabatic invariant.

From what we have discussed in class, we know that

$$x = \frac{a_0}{\sqrt{\omega}} e^{i\int \omega dt}$$ by WKB methods

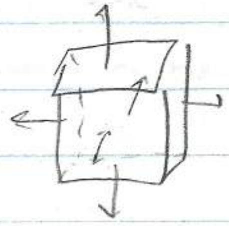
$$\text{So } \overline{\omega x^2} = \frac{\overline{\omega^2 x^2}}{\omega} = E/\omega = a_0^2/2 \quad \text{const}$$

So they are consistant with each other. Actually, they have both used the ~~approximation~~ assumption that $\dot{\lambda}/\lambda \ll \omega$

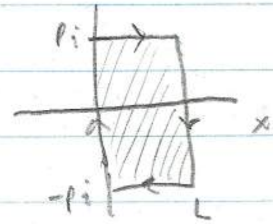
and it's the only approximation they applied. So they should give the same solution.

Phys 200A Pset 4 Robin Heinonen

2. Consider a particle of the gas colliding with a wall.



If the wall is stationary, the particle's momentum in the direction normal to the wall is changed from p_i to $-p_i$. Hence we have the phase diagram



for constant L, E,

and if $\dot{L} \ll \bar{v}$ we have for $i=1,2,3$ the adiabatic invariant

$$I_i = \oint_{E,L} p_i \, dx_i = 2 p_i L = \text{const.}$$

so

$$p_i = \frac{L_0}{L} p_{i,0}$$

It follows that the internal energy of the gas is

$$U = T = N \cdot 3 \cdot \frac{p_i^2}{2m} = \frac{3}{2m} N \frac{L_0^2}{L^2} p_{i,0}^2$$

$$= \frac{L_0^2}{L^2} U_0$$ where U_0 is the energy of the gas in a stationary box of sidelength L.

Thus $U \propto \frac{1}{V^{2/3}}$ so $P = -\frac{\partial U}{\partial V} \propto V^{-5/3}$

i.e. $PV^{5/3} = \text{const.}$

Thermodynamics predicts $PV^{\gamma} = \text{const.}$ for the adiabatic expansion of an ideal gas. Since $\gamma = 5/3$ for a monatomic ideal gas, we are in agreement.

3

Parametric Resonance

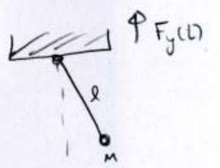
(1) Timescale $\sim 2\omega_0$, ω_0 natural frequency

(2) Approximation(s)
(1) $\omega^2(t) = \omega_0^2(1 + h\cos\omega t)$
slowly varying $\rightarrow$ h small
(2) Close to resonance $\rightarrow$ $\omega - 2\omega_0 = O(\varepsilon)$

(3) Leverage
(1) Near resonance $\rightarrow$ drop terms further away
(2) Floquet Theory $\rightarrow$ Existence of Exponential Sol'ns

(4) Key Features
Given Eq. $\ddot{x} + \omega_0^2(1 + h\cos(\gamma t))x = 0$
$\Rightarrow$ Condition for instability $(\gamma - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$.

(5) Canonical Example: Vertically oscillating pendulum.



$g \rightarrow g - F_y(t)/M$

$0 = \ddot{\theta} + \frac{g}{\ell}\theta \rightarrow \ddot{\theta} + \omega_0^2\Big(1 - \overbrace{\frac{a_0}{\ell\omega_0^2}}^{h}\cos(\alpha t)\Big)\theta = 0$

Instability $(\alpha - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$

$\rightarrow (\alpha - 2\omega_0)^2 < \frac{a_0^2}{4\ell g}$.

Frequency of forcing can couple to natural frequency generating parametric instabilities.

Adiabatic

(1) Timescale: Slowly varying compared to natural frequency.

(2) Requirements / Approximations
(1) Applicable when $H = H(p, q, \lambda(t))$; $\frac{\dot{\lambda}}{\lambda} \ll \Omega$
(2) Periodic Motion for fixed λ.
(3) Approx: $\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} \simeq \overline{\frac{\partial H}{\partial \lambda}} \cdot \frac{d\lambda}{dt}$

Approx Const

$\bar{A} \equiv \frac{1}{\tau}\int_0^{\tau} A(t)\,dt$,

(3) Leverage: Hamiltonian Structure
$\rightarrow \dot{q} = \frac{\partial H}{\partial p} \rightarrow dt = \int dq/(\partial H/\partial p)$
Take $\int dt \rightarrow \oint dq/(\partial H/\partial p)$.

(4) Key Features: $I = \oint_{E,\lambda} \frac{p\,dq}{2\pi}$ Approx const.

$\rightarrow I(E) = \oint \frac{p\,dq}{2\pi} \rightarrow 2\pi\frac{\partial I}{\partial E} = \oint dq/(\partial H/\partial p) = \frac{2\pi}{\omega}$

$\rightarrow \frac{\partial E}{\partial I} = \omega$. I const on $\tau > \omega^{-1}$.

(5) Canonical Example: Pendulum with slowly varying length.

By comparison with H.O. we know $I(E) = E/\omega$

$E = mg\ell\bar{\theta}^2$, $\bar{\theta} \rightarrow \theta_{rms}$, $\omega = \sqrt{g/\ell}$

$\Rightarrow I = \frac{mg\ell\bar{\theta}^2}{(g/\ell)^{1/2}} = \text{const}$

$\Rightarrow \bar{\theta}_{rms} \propto \ell^{-3/4}$

Slowly changing length allows estimate of rms angle as a function of length.

Ponder Motive

(1) Timescale: Rapid oscillations about a mean motion. ($\omega \gg \Omega$).

(2) Approximations / Requirements
(1) Small Amplitude, Rapidly Oscillating periodic forcing function

(3) Leverage
(1) Product of fast oscillators $\rightarrow$ slow beats
(2) Averaging over fast time scales decouples the problem.

(4) Key Feature
Given e.q. of form
$m\ddot{x} = -\frac{dU}{dx} + f$, $\frac{d^2U}{dx^2} \sim \Omega_0^2$

Where $|f|$ small, f has $\omega \gg \Omega_0$.
$f = f_1(x)\cos\omega t + f_2(x)\sin\omega t$

$\Rightarrow U_{eff} = U(x) + \frac{1}{4m\omega^2}(f_1^2 + f_2^2)$.

(5) Canonical Example: Inverted Pendulum.

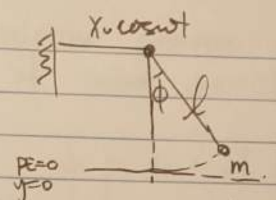
$\ddot{\theta} = -\frac{g\sin\theta}{\ell} + \left(-\frac{y_0\omega^2}{\ell}\right)\cos\omega t \cdot \sin\theta$

$\Rightarrow U_{eff} = -\frac{mg\cos\theta}{\ell} + \frac{1}{4M\omega^2}\left(\frac{Mx_0\omega^2}{\ell}\cos\theta\right)^2$

Rapid forcing function can yield additional stable point.

#4.

(a)



$x = \ell \sin\phi + x_0 \cos\omega t$

$y = \cancel{\ell\cos\phi}\ \ell(1-\cos\phi)$

$\dot{x} = \ell\dot\phi\cos\phi - x_0\omega\sin\omega t$

$\dot{y} = \ell\sin\phi \cdot \dot\phi$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mg\ell(1-\cos\phi)$$

$$= \frac{1}{2}m\left[\ell^2\dot\phi^2 - x_0^2\omega^2\sin^2\omega t - 2\ell x_0\omega\dot\phi\cos\phi\sin\omega t\right] - mg\ell(1-\cos\phi)$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) = \frac{\partial L}{\partial\phi} \Rightarrow$ EOM:

$$\Rightarrow \ddot\phi = \frac{-g}{\ell}\sin\phi + \frac{\omega^2 x_0}{\ell}\cos\omega t\cos\phi$$

let $\phi = \bar\Phi + \tilde\Phi$ $\quad \bar\Phi = \langle\phi\rangle \quad \tilde\Phi \ll \bar\Phi$

$$\Rightarrow \ddot{\bar\Phi} + \ddot{\tilde\Phi} = \frac{-g}{\ell}\sin(\bar\Phi+\tilde\Phi) + \frac{\omega^2 x_0}{\ell}\cos\omega t\cos(\bar\Phi+\tilde\Phi)$$

$$\Rightarrow \begin{cases} \ddot{\tilde\Phi} = -\tilde\Phi\frac{g}{\ell}\cos\bar\Phi + \frac{\omega^2 x_0}{\ell}\cos\omega t\cos\bar\Phi & \text{① fast part} \\ \ddot{\bar\Phi} = \frac{-g}{\ell}\sin\bar\Phi - \frac{\omega^2 x_0}{\ell}\sin\bar\Phi\langle\tilde\Phi\cos\omega t\rangle & \text{②} \end{cases}$$

$$\Rightarrow \ddot{\tilde\Phi} \sim \omega^2\tilde\Phi \gg \frac{g}{\ell}\tilde\Phi$$

$$\Rightarrow \ddot{\tilde\Phi} \approx \frac{\omega^2 x_0}{\ell}\cos\omega t\cos\bar\Phi$$

$$\tilde\Phi = a(\bar\Phi)\cos\omega t \Rightarrow a(\bar\Phi) = \frac{-x_0}{\ell}\cos\bar\Phi$$

$$\langle\tilde\Phi\cos\omega t\rangle = \frac{-x_0}{\ell}\cos\bar\Phi\langle\cos^2\omega t\rangle = \frac{-x_0}{2\ell}\cos\bar\Phi$$

from ② $\Rightarrow$ $\ddot{\bar\Phi} = \frac{-g}{\ell}\sin\bar\Phi + \frac{\omega^2 x_0^2}{4\ell^2}\sin 2\bar\Phi$

$$= -\frac{d}{d\bar\Phi}\left(\frac{-g}{\ell}\cos\bar\Phi + \frac{x_0^2\omega^2}{4\ell^2}\sin^2\bar\Phi\right)$$

$\Rightarrow$ define $U_{eff} = \left(\frac{-g}{\ell}\cos\bar\Phi + \frac{x_0^2\omega^2}{4\ell^2}\sin^2\bar\Phi\right) \qquad \ddot{\bar\Phi} = -\frac{dU_{eff}}{d\bar\Phi}$

Equilibrium position given by $\frac{dU_{eff}}{d\bar{\phi}} = 0$

$$\Rightarrow \frac{g}{\ell}\sin\bar{\phi} - \frac{\omega^2 x_0^2}{4\ell^2}\sin 2\bar{\phi} = 0$$

$$\Rightarrow \bar{\Phi} = \pi \quad \text{or} \quad \bar{\Phi} = \pi$$

$$\text{or} \quad \cos\bar{\phi} = \frac{2g\ell}{\omega^2 x_0^2}$$

stability

$$\frac{d^2U_{eff}}{d\bar{\phi}^2} = +\frac{g}{\ell}\cos\bar{\phi} + \frac{\omega^2 x_0^2}{4\ell^2}\cdot 2\cos 2\bar{\phi}$$

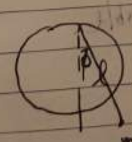
case 1: $\bar{\phi} = \pi \Rightarrow \frac{d^2U_{eff}}{d\bar{\phi}^2} < 0$ unstable

case 2: $\bar{\Phi} = 0$ if stable $\Rightarrow 2g\ell > x_0^2\omega^2$

case 3: $\cos\bar{\phi} = \frac{2g\ell}{\omega^2 x_0^2}$ if stable $\Rightarrow \left(\frac{2g^2}{\omega^2 x_0^2} - \frac{\omega^2 x_0^2}{4\ell^2}\right)(2\cos^2\bar{\phi} - 1) > 0$

$$\Rightarrow \omega^2 x_0^2 > 2g\ell.$$

(b) Similar procedure



$x = r_0\cos\omega t + \ell\sin\phi$

$y = \ell(1-\cos\phi) + r_0\sin\omega t.$

after cancelling and ignore terms has no dependence on ϕ

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell r_0\omega^2\sin(\phi-\omega t) + mg\ell\cos\phi$$

$$\Rightarrow \text{EOM} \quad \ddot{\phi} = \frac{-g}{\ell}\sin\phi + \frac{\omega^2 r_0}{\ell}\cos\phi\cos\omega t + \frac{\omega^2 r_0}{\ell}\sin\phi\sin\omega t$$

$$\phi = \bar{\Phi} + \tilde{\Phi} \qquad \bar{\Phi} = \langle\phi\rangle \gg \tilde{\Phi}$$

$$\Rightarrow \ddot{\bar\phi} + \ddot{\tilde\phi} = \frac{-g}{\ell}\sin\bar\phi - \frac{g}{\ell}\tilde\phi\cos\bar\phi + \frac{\omega^2 r_0}{\ell}\cos\omega t\cos\bar\phi - \tilde\phi\frac{\omega^2 r_0}{\ell}\sin\bar\phi\cos\omega t$$

$$+ \frac{\omega^2 r_0}{\ell}\sin\bar\phi\sin\omega t + \frac{\omega^2 r_0}{\ell}\cos\bar\phi\sin\omega t\,\tilde\phi.$$

because $\omega^2 \gg \frac{g}{\ell}$

$$\Rightarrow \begin{cases} \ddot{\tilde\phi} = \underbrace{\frac{-g}{\ell}\tilde\phi\cos\bar\phi}_{\text{ignore}} + \frac{\omega^2 r_0}{\ell}\cos\omega t\cos\bar\phi + \frac{\omega^2 r_0}{\ell}\sin\bar\phi\sin\omega t \\ \ddot{\bar\phi} = \frac{-g}{\ell}\sin\bar\phi - \frac{\omega^2 r_0}{\ell}\sin\bar\phi\langle\tilde\phi\cos\omega t\rangle + \frac{\omega^2 r_0}{\ell}\cos\bar\phi\langle\tilde\phi\sin\omega t\rangle \end{cases}$$

suppose $\tilde\phi = a(\bar\phi)\cos\omega t + b(\bar\phi)\sin\omega t$

$$\Rightarrow \omega^2[-a(\bar\phi)\cos\omega t - b(\bar\phi)\sin\omega t] = \frac{\omega^2 r_0}{\ell}\cos\omega t\cos\bar\phi + \frac{\omega^2 r_0}{\ell}\sin\omega t\sin\bar\phi$$

$$a(\bar\phi) = \frac{-r_0}{\ell}\cos\bar\phi \qquad b(\bar\phi) = \frac{-r_0}{\ell}\sin\bar\phi$$

$$\langle\tilde\phi\cos\omega t\rangle = \frac{-r_0}{2\ell}\cos\bar\phi \qquad \langle\tilde\phi\sin\omega t\rangle = \frac{-r_0}{2\ell}\sin\bar\phi$$

$$\Rightarrow \ddot{\bar\phi} = \frac{-g}{\ell}\sin\bar\phi + \frac{\omega^2 r_0^2}{4\ell^2}\sin 2\bar\phi - \frac{\omega^2 r_0^2}{4\ell^2}\sin 2\bar\phi = \frac{-g}{\ell}\sin\bar\phi$$

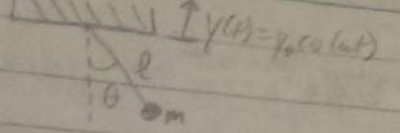
define $U_{eff} = \frac{-g}{\ell}\cos\bar\phi$

$$\frac{dU_{eff}}{d\bar\phi} = 0 \Rightarrow \bar\phi = 0 \text{ or } \pi$$

$$\frac{d^2U_{eff}}{d\bar\phi} = \frac{g}{\ell}\cos\bar\phi \begin{cases} > 0 & \text{when } \bar\phi = 0 \quad \text{stable} \\ < 0 & \text{when } \bar\phi = \pi \quad \text{unstable.} \end{cases}$$

Tom Zdyrski

5)



The pendulum has oscillating support $y(t)=y_0\cos(\omega t)$ with $\omega = 2\omega_0+\varepsilon$, $\omega_0=\sqrt{\frac{g}{\ell}}$ and $\varepsilon \ll \omega_0$

So, the coordinates of the mass are

$x' = \ell\sin\theta$, $y' = y - \ell\cos\theta = y_0\cos\omega t - \ell\cos\theta$

$\Rightarrow \dot{x}' = \ell\dot\theta\cos\theta$, $\dot{y}' = -y_0\omega\sin\omega t + \ell\dot\theta\sin\theta$

$\Rightarrow T = \frac{1}{2}m(\dot{x}'^2+\dot{y}'^2) = \frac{1}{2}m(\ell^2\dot\theta^2\cos^2\theta + \ell^2\dot\theta^2\sin^2\theta + y_0^2\omega^2\sin^2\omega t$
$- 2\ell y_0\omega\dot\theta\sin\omega t\sin\theta)$

$= \frac{1}{2}m(\ell^2\dot\theta^2 + y_0^2\omega^2\sin^2\omega t) - m\ell y_0\omega\dot\theta\sin\omega t\sin\theta$

$V = mgy' = mgy_0\cos\omega t - mg\ell\cos\theta$

$\Rightarrow L = T - V = \frac{1}{2}m(\ell^2\dot\theta^2 + y_0^2\omega^2\sin^2\omega t) - m\ell y_0\omega\dot\theta\sin\omega t\sin\theta - mgy_0\cos\omega t + mg\ell\cos\theta$

From the Euler-Lagrange Equations:

$\frac{\partial L}{\partial\theta} = -m\ell y_0\omega\dot\theta\sin\omega t\cos\theta - mg\ell\sin\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)$

$= \frac{d}{dt}(m\ell^2\dot\theta - m\ell y_0\omega\sin\omega t\sin\theta) = m\ell^2\ddot\theta - m\ell y_0\omega\dot\theta\sin\omega t\cos\theta - m\ell y_0\omega^2\cos\omega t\sin\theta$

$\Rightarrow m\ell^2\ddot\theta = -mg\ell\sin\theta + m\ell y_0\omega^2\cos\omega t\sin\theta$

$\Rightarrow \ddot\theta = -\frac{g}{\ell}\sin\theta + \frac{y_0}{\ell}\omega^2\cos\omega t\sin\theta$

For $\theta \ll 1$ (which only holds for a subset of the solution time since θ will grow exponentially)

$\ddot\theta \approx \frac{-g}{\ell}\theta + \frac{y_0}{\ell}\omega^2(\cos\omega t)\theta$

$\Rightarrow 0 = \ddot\theta + \omega_0^2\left(1 - \frac{y_0}{\ell}\left(\frac{\omega}{\omega_0}\right)^2\cos\omega t\right)\theta = 0$ where $\omega_0^2 = \frac{g}{\ell}$

Defining $h = \frac{-y_0}{\ell}\left(\frac{\omega}{\omega_0}\right)^2$ and $\varepsilon = \omega - 2\omega_0$, we have $\varepsilon \ll \omega_0$

$0 = \ddot\theta + \omega_0^2(1 + h\cos(2\omega_0 + \varepsilon))\theta = 0$ Mathieu's Eq.

Ansatz: $\theta = A(t)e^{i(\omega_0+\varepsilon/2)t}$, with $\frac{\dot A}{A} \ll \omega$ since

$\varepsilon \ll \omega_0$ (so the coefficients are slowly varying)

$$\Rightarrow 0 = \left[\ddot{\tilde{A}} + 2i(\omega_0 + \tfrac{\varepsilon}{2})\dot{\tilde{A}} - (\omega_0 + \tfrac{\varepsilon}{2})^2 \tilde{A}\right] e^{i(\omega_0 + \varepsilon/2)t} + \omega_0^2 (1 + h\cos(2\omega_0 + \varepsilon)t)\tilde{A} e^{i(\omega_0 + \varepsilon/2)t}$$

$$\Rightarrow 0 = [\ddot{\tilde{A}} + 2i(\omega_0 + \tfrac{\varepsilon}{2})\dot{\tilde{A}} - \omega_0^2 \tilde{A} - \omega_0 \varepsilon \tilde{A} - \tfrac{\varepsilon^2}{4}\tilde{A} + \omega_0^2 \tilde{A} + h\cos(2\omega_0 + \varepsilon)t\, \tilde{A}]$$

Since $\varepsilon_0 \ll \omega_0$, drop ε^2 term. Further, $\tilde{A}(t)$ varies slowly, so $\ddot{\tilde{A}} \ll \dot{\tilde{A}}\omega_0$, so drop $\ddot{\tilde{A}}$ term

$$\Rightarrow 0 = [2i(\omega_0 + \tfrac{\varepsilon}{2})\dot{\tilde{A}} - \omega_0 \varepsilon \tilde{A} + \tilde{A}\omega_0^2 h \cos(2\omega_0 + \varepsilon)t] e^{i(\omega_0 + \varepsilon/2)t}$$

Note, for the resonant term, $\cos(2\omega_0 + \varepsilon)t$, we have

$$\cos(2\omega_0 + \varepsilon)\hat{A} e^{i(\omega_0 + \varepsilon/2)t} = |\hat{A}| \cos(2\omega_0 + \varepsilon) e^{i(\omega_0 + \varepsilon/2)t + i\phi} \qquad \phi = \tan^{-1}\left(\frac{\mathrm{Im}\,\hat{A}}{\mathrm{Re}\,\hat{A}}\right)$$

$$= |\hat{A}| \cos(2\omega_0 + \varepsilon)\cos[(\omega_0 + \tfrac{\varepsilon}{2})t + \phi] + i|\hat{A}|\cos(2\omega_0 + \varepsilon)\sin[(\omega_0 + \tfrac{\varepsilon}{2})t + \phi]$$

$$= \tfrac{1}{2}|\hat{A}| \cos[3(\omega_0 + \tfrac{\varepsilon}{2})t + \phi] + \tfrac{1}{2}|\hat{A}|\cos[(\omega_0 + \tfrac{\varepsilon}{2})t - \phi]$$
$$+ i\tfrac{1}{2}|\hat{A}| \sin[3(\omega_0 + \tfrac{\varepsilon}{2})t + \phi] - \tfrac{1}{2}|\hat{A}| \sin[(\omega_0 + \tfrac{\varepsilon}{2})t - \phi]$$

Ignoring the higher frequency/off resonant terms, this gives

$$= \tfrac{1}{2}\tilde{A}^* e^{-i(\omega_0 + \varepsilon/2)t}$$

So $$0 = [2i(\omega_0 + \tfrac{\varepsilon}{2})\dot{\hat{A}} - \omega_0 \varepsilon \hat{A}] e^{i(\omega_0 + \varepsilon/2)t} + \tfrac{1}{2}\hat{A}^* h\omega_0^2 e^{-i(\omega_0 + \varepsilon/2)t}$$

Dropping the $\varepsilon/2$ term (since $\varepsilon \ll \omega_0$) gives

$$0 = (\dot{\tilde{A}} 2i\omega_0 - \omega_0 \varepsilon \tilde{A}) e^{i(\omega_0 + \varepsilon/2)t} + \tfrac{1}{2} h\omega_0^2 \tilde{A}^* e^{-i(\omega_0 + \varepsilon/2)t}$$

Taking the imaginary part gives

$$0 = 2|\dot{\hat{A}}|\omega_0 \cos[(\omega_0 + \tfrac{\varepsilon}{2})t + \phi] - \omega_0 \varepsilon |\hat{A}| \sin[(\omega_0 + \tfrac{\varepsilon}{2})t + \phi] - \tfrac{1}{2}h\omega_0^2 |\hat{A}| \sin\ldots$$

$$\Rightarrow 0 = \sin[(\omega_0 + \tfrac{\varepsilon}{2})t][-2\omega_0 |\dot{\hat{A}}|\sin\phi - \omega_0 \varepsilon |\hat{A}|\cos\phi - \tfrac{1}{2}h\omega_0^2 |\hat{A}|\cos\phi]$$
$$+ \cos[(\omega_0 + \tfrac{\varepsilon}{2})t][2\omega_0 |\dot{\hat{A}}|\cos\phi - \omega_0 \varepsilon |\hat{A}|\sin\phi + \tfrac{1}{2}h\omega_0^2 |\hat{A}|\sin\phi]$$

defining $b(t) = \mathrm{Re}\{\hat{A}(t)\} = |\hat{A}|\cos\phi$, $a(t) = \mathrm{Im}\{\hat{A}(t)\} = |\hat{A}|\sin\phi$ and equating cos/sin coefficients to zero gives

$$0 = 2\omega_0 \dot{a} + \omega_0 \varepsilon b + \tfrac{1}{2}h\omega_0^2 b$$
$$0 = 2\omega_0 \dot{b} - \omega_0 \varepsilon a + \tfrac{1}{2}h\omega_0^2 a$$

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$\Rightarrow \quad 0 = \dot{a} + \frac{\varepsilon}{2} b + \frac{\omega_0 h}{4} b$

$\quad\; 0 = \dot{b} - \frac{\varepsilon}{2} a + \frac{\omega_0 h}{4} a$

Choosing $\quad a = a_0 e^{st} \qquad b = b_0 e^{st}$ for ansatz gives

$\quad 0 = s a_0 + \left(\frac{\varepsilon}{2} + \frac{\omega_0 h}{4}\right) b_0$

$\quad 0 = \left(\frac{\varepsilon}{2} + \frac{\omega_0 h}{4}\right) a_0 + s b_0$

$\Rightarrow \quad s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\varepsilon^2}{4} = \frac{1}{4}\left(\frac{\omega_0^2 h^2}{4} - \varepsilon^2\right)$

$\Rightarrow \quad s = \pm \frac{1}{4} \sqrt{(\omega_0 h)^2 - (2\varepsilon)^2}$

So, if $s^2 > 0 \Rightarrow \left|\frac{\omega_0 h}{2\varepsilon}\right| > 1$ then the growth

rate is $s = \frac{\sqrt{(\omega_0 h)^2 - (2\varepsilon)^2}}{4}$, or plugging in

$h = -\frac{\gamma_0}{\ell}\left(\frac{\omega}{\omega_0}\right)^2$ we get the instability criterion

$\boxed{\left|\frac{\gamma_0 \omega^2}{2\ell\varepsilon\omega_0}\right| > 1}$ and growth rate $\boxed{s = \frac{\sqrt{(\gamma_0 \omega^2)^2 - (2\varepsilon\ell\omega_0)^2}}{4\ell\omega_0}}$

Jialing Fei A53111966

6. Mathieu's equation

$$\ddot{\phi} + \gamma\dot{\phi} + \omega_0^2\phi(1 + h\cos 2\omega t) = 0$$

$\omega = \omega_0 + \frac{\epsilon}{2}$ half the forcing frequency results in the parametric resonance

$$h = \frac{4y_0}{l}$$

assume solution in the form $\phi = a\cos\omega t + b\sin\omega t$

$$\dot{\phi} = \omega(-a\sin\omega t + b\cos\omega t)$$

$$\ddot{\phi} = -\omega^2(a\cos\omega t + b\sin\omega t)$$

$\Rightarrow$

$$-a\omega^2\cos\omega t - b\omega^2\sin\omega t - \omega\gamma a\sin\omega t + \omega\gamma b\cos\omega t$$

$$+ a\omega_0^2\cos\omega t + b\omega_0^2\sin\omega t + a\omega_0^2 h\cos 2\omega t\cos\omega t + b\omega_0^2 h\cos 2\omega t\sin\omega t = 0$$

$$a\omega_0^2 h\cos 2\omega t\cos\omega t = \frac{a\omega_0^2 h}{2}(\cos\omega t + \cos 3\omega t)$$

$$b\omega_0^2 h\cos 2\omega t\sin\omega t = \frac{b\omega_0^2 h}{2}(\sin 3\omega t - \sin\omega t)$$

on resonance; off-resonance ✗ throw away ω_3, it doesn't contribute to the instability

$$\omega^2 = \left(\omega_0 + \frac{\epsilon}{2}\right)^2 = \omega_0^2 + \omega_0\epsilon + \frac{\epsilon^2}{4}$$

$\frac{\epsilon^2}{4}$ — omit

$$\left(-a\omega_0\epsilon + \gamma b\omega_0 + \frac{1}{2}a\omega_0^2 h\right)\cos\omega t - \left(b\omega_0\epsilon + \gamma a\omega_0 + \frac{1}{2}b\omega_0^2 h\right)\sin\omega t = 0$$

$$\begin{vmatrix} -\epsilon_0 + \frac{1}{2}\omega_0 h & \gamma \\ \gamma & \epsilon_0 + \frac{1}{2}\omega_0 h \end{vmatrix} = 0$$

$$\epsilon_0^2 = \left(\frac{\omega_0 y_0}{l}\right)^2 - \gamma^2$$

for any $\epsilon < \sqrt{\left(\frac{\omega_0 y_0}{l}\right)^2 - \gamma^2}$ will cause instability

letting $\epsilon \to 0$. $y_{0,min} = \frac{\gamma l}{\omega_0}$

$y_0 > \frac{\gamma l}{\omega_0}$ or the damping term prevents the parametric instability

7.

a) We know that

$$\begin{cases} \dot{q} = \partial H/\partial p = \frac{\partial H_0}{\partial p} = p/m \\ \dot{p} = -\partial H/\partial q = -\partial H_0/\partial q - \frac{\partial V(q)}{\partial q}\frac{d^2A}{dt^2} = -\frac{\partial V_0}{\partial q} - \frac{\partial V(q)}{\partial q}\frac{d^2A}{dt^2} \end{cases}$$

where we assume that $H_0 = p^2/2m + V_0(q)$

So $\ddot{q} = \dot{p}/m = -\frac{1}{m}\frac{\partial V_0}{\partial q} - \frac{1}{m}\frac{\partial V(q)}{\partial q}\frac{d^2A}{dt^2}$ (*)

Because $A(t)$ is periodic with period τ

So $\frac{d^2A}{dt^2}$ should be periodic with same period.

Then solution to Eq (*) should be a mean motion plus a fast quiver

Suppose $q = Q + \varepsilon$ where $Q = \langle q \rangle$ ($\langle\ \rangle$ means a short time average)

Then we have

$$\ddot{Q} + \ddot{\varepsilon} = \underset{①}{-\frac{1}{m}\frac{\partial V_0}{\partial Q}} \underset{②}{-\frac{\varepsilon}{m}\frac{\partial^2 V_0}{\partial Q^2}} \underset{③}{-\frac{1}{m}\frac{\partial V}{\partial Q}\frac{d^2A}{dt^2}} \underset{④}{-\frac{\varepsilon}{m}\frac{\partial^2 V}{\partial Q^2}\frac{d^2A}{dt^2}}$$

Because $\varepsilon \sim \cos(\frac{2\pi}{\tau}t + \varphi_0)$ is fast time scale term.

We know that ② and ③ are fast time scale terms, either $(A \sim \cos(\frac{2\pi}{\tau}t))$

and ① and ④ are slow time scale terms.

So we have

$$\begin{cases} \ddot{Q} = -\frac{1}{m}\frac{\partial V_0}{\partial Q} - \frac{1}{m}\frac{\partial^2 V}{\partial Q^2}\langle \varepsilon \frac{d^2A}{dt^2} \rangle \\ \ddot{\varepsilon} = -\frac{\varepsilon}{m}\frac{\partial^2 V_0}{\partial Q^2} - \frac{1}{m}\frac{\partial V}{\partial Q}\frac{d^2A}{dt^2} \end{cases}$$

Because $\ddot{\varepsilon} \sim (\frac{2\pi}{\tau})^2 \varepsilon \gg (\frac{2\pi}{T})^2 \varepsilon \sim \frac{\varepsilon}{m}\frac{\partial^2 V_0}{\partial Q^2}$

So we get

$$\varepsilon = \frac{1}{m}\left(\frac{T}{2\pi}\right)^2 \frac{\partial V}{\partial Q}\frac{d^2A}{dt^2}$$

So $$\ddot{Q} = -\frac{1}{m}\frac{\partial V_0}{\partial Q} - \frac{1}{m^2}\left(\frac{T}{2\pi}\right)^2\left(\frac{\partial V}{\partial Q}\right)\frac{\partial^2 V}{\partial Q^2}\left\langle\left(\frac{d^2A}{dt^2}\right)^2\right\rangle \qquad (*)$$

$$= -\frac{1}{m}\frac{\partial V_0}{\partial Q} - \frac{1}{m^2}\cdot\left(\frac{\partial V}{\partial Q}\right)\frac{\partial^2 V}{\partial Q^2}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle \qquad \text{where } \frac{d^2A}{dt^2} \sim \frac{2\pi}{T}\frac{dA}{dt}$$

Eq (*) is the mean field equation for this system.

b.) $$K(p,q) = H_0(p,q) + \frac{1}{\cancel{4}m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\left(\frac{\partial V(q)}{\partial q}\right)^2$$

So $$\dot{p} = -\frac{\partial K}{\partial q} = -\partial H_0/\partial q - \frac{1}{\cancel{4}m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle 2\frac{\partial V(q)}{\partial q}\frac{\partial^2 V(q)}{\partial q^2}$$

$$= -\frac{\partial V_0}{\partial q} - \frac{1}{m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\frac{\partial V}{\partial q}\frac{\partial^2 V}{\partial q^2}$$

$$\dot{q} = \partial K/\partial p = \partial H_0/\partial p = p/m$$

So $$\ddot{q} = -\frac{1}{m}\frac{\partial V_0}{\partial q} - \frac{1}{m^2}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\frac{\partial V}{\partial q}\cdot\frac{\partial^2 V}{\partial q^2} \qquad (**)$$

We get same equation with Eq (*)

It denotes that effective Hamitonian

$$K(p,q) = H_0(p,q) + \frac{1}{2m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\left(\frac{\partial V}{\partial q}\right)^2$$

Note: When $m\ddot{x} = -\frac{dU}{dx} + f$ we know that

$$U_{eff} = U_0 + \frac{1}{4m\omega^2}\langle f_1^2 + f_2^2\rangle \qquad \text{where } f = f_1\cos\omega t + f_2\sin\omega t$$

In this case, $f = \frac{\partial V}{\partial q}\frac{d^2A}{dt^2}$ $\qquad A \sim a_0\cos(\omega t)$

So $$U_{eff} = U_0 + \frac{\omega^2}{4m\omega^2}\left(\frac{\partial V}{\partial q}\right)^2\cdot a_0^2$$

~~because~~ And $$\frac{1}{2m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\left(\frac{\partial V}{\partial q}\right)^2 = \frac{\omega^2}{2m}\langle a_0^2\cos^2\omega t\rangle\left(\frac{\partial V}{\partial q}\right)^2 = \frac{a_0^2}{4m}\left(\frac{\partial V}{\partial q}\right)^2$$

So the factor should be $\frac{1}{2}$ rather than $\frac{1}{4}$, given that $\langle\frac{d^2A}{dt^2}\rangle$ providing another $\frac{1}{2}$

(8)

$$L = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2) \qquad \text{(Polar coords)}.$$

$$\frac{\partial L}{\partial \theta} = 0 \rightarrow \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} = \text{const.} = p$$

$$H = \frac{1}{2}m\left(\dot{R}^2 + \frac{p^2}{R^2}\right)$$

(a) $I = \oint_{E,R} \frac{p\,dq}{2\pi}$; At fixed R, $\dot{R} = 0$.
At fixed E, $H = E$.
$\rightarrow$ $p = R\sqrt{2mE}$

$$I = \oint_{E,\lambda} R\sqrt{2mE}\,\frac{d\theta}{2\pi} = R\sqrt{2mE} \overset{\text{(approx)}}{=} \text{const}.$$

$$E = \frac{I^2}{2mR^2} \qquad R \rightarrow \alpha R \Rightarrow E \rightarrow \frac{I^2}{2mR^2\alpha^2} = \frac{E}{\alpha^2}.$$

(b) Show action is exact invariant, should this be the case, it should hold for the particular case $R(t) = (1+\varepsilon t)R_0$, however

$$S = \int L\,dt = \frac{1}{2}m\int\left[(\varepsilon R_0)^2 + R^2\left(\frac{p}{mR^2}\right)^2\right]dt = \frac{1}{2}m\int\left[\varepsilon^2R_0^2 + \frac{p^2}{m^2R(t)^2}\,dt\right]$$

$$= \frac{1}{2}m\left[\varepsilon^2R_0^2t - \frac{p^2}{\varepsilon m^2R_0R(t)}\right]$$

Has explicit time dependence !

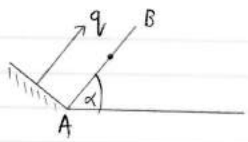
— $R(t) = (1+\varepsilon t)R_0$. From above, $mR^2\dot{\theta} = p \rightarrow \theta = \int \frac{p}{mR^2}dt$

$$\Rightarrow \theta = \int\frac{p}{m(1+\varepsilon t)^2R_0^2}dt = -\frac{p}{\varepsilon mR_0^2}\frac{1}{(1+\varepsilon t)} = \boxed{\frac{-p}{\varepsilon mR_0R(t)} = \theta(t)}$$

$$E = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2) = \frac{1}{2}m\left(\overbrace{(\varepsilon\cancel{(1+\varepsilon t)})^2}^{O(\varepsilon^2)} + \cancel{(1+\varepsilon t)^2}\left(\frac{p}{\cancel{m}\cancel{(1+\varepsilon t)^2}R_0^2}\right)\right)$$

$$R_0 \rightarrow \alpha R_0 \Rightarrow E \rightarrow \frac{p}{2\alpha^2R_0^2} = \frac{E}{\alpha^2}.$$

9)



Potential $V(q_1) = mg\, q \sin(\alpha)$

The action variable

$$I(E) = \frac{1}{\pi} \int_0^{q_1} \left[2m(E - mg\, q \sin(\alpha)) \right]^{\frac{1}{2}} dq$$

q_1 is the amplitude

$$= \frac{2\sqrt{2}\left(g\, m q_1 \sqrt{m(E - g\, m q_1 \sin(\alpha))} + E \frac{1}{\sin(\alpha)} \left(\sqrt{Em} - \sqrt{m(E - g\, m q_1 \sin(\alpha))} \right) \right)}{3 g\, m \pi}$$

$$E = mg\, q_1 \sin(\alpha)$$

$$\Longrightarrow I(E) = \frac{2\sqrt{2}}{3g\sqrt{m}\sin(\alpha)\pi} E^{\frac{3}{2}}$$

$$\Longrightarrow H = \left(\frac{3g\sqrt{m}\sin(\alpha)\, I\pi}{2\sqrt{2}} \right)^{\frac{2}{3}}$$

$$q_1 = \text{amplitude} = \frac{E(t)}{mg\sin(\alpha)} = \frac{g\, I^2 (3\pi)^{\frac{2}{3}} \sin(\alpha)}{2(g\, I\sqrt{m}\sin(\alpha))^{4/3}}$$

#10. If action variable is defined to be equal to the area under phase curve, that is

$$I = \oint p \, dq = \int_{\varepsilon \ell} dq \, dp$$

then the Hamilton's principal function, action S., increases

$$\Delta S = \oint p \, dq = I \quad \text{during each period.}$$

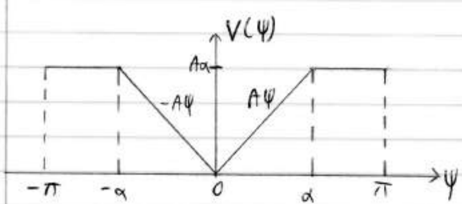
$\Rightarrow$ the conjugate variable of action variable is angle-action ~~#~~ variable and generated by

$$Q_\alpha = \frac{\partial S}{\partial P_\alpha}$$

$$\Rightarrow \theta = \frac{\partial}{\partial I} S \Rightarrow \Delta\theta = \frac{\partial \Delta S}{\partial I} = 1.$$

That is angle-action increases by unity during each period under this definition.

11)



$$H = \frac{p^2}{2m} + V(\psi)$$

$$I = \frac{1}{2\pi} \oint p\,dq = \frac{1}{2\pi} \oint \frac{\partial S}{\partial \psi}\, d\psi = \frac{\sqrt{2m}}{2\pi} \oint \sqrt{E - V(\psi)}\, d\psi$$

$$= \frac{\sqrt{2m}}{\pi} \left\{ \int_0^{\alpha} \sqrt{E - A\psi}\, d\psi + \sqrt{E - A\alpha} \int_{\alpha}^{\pi} d\psi \right\}$$

$$= \frac{\sqrt{2m}}{\pi} \left\{ \frac{-2}{3A} (E - A\alpha)^{\frac{3}{2}} + \frac{2}{3A} E^{\frac{3}{2}} + (\pi - \alpha)(E - A\alpha)^{\frac{1}{2}} \right\}$$

$$I = \frac{\sqrt{2m}}{\pi} \left\{ \frac{2}{3A} \left(E^{\frac{3}{2}} - (E - A\alpha)^{\frac{3}{2}} \right) + (\pi - \alpha)(E - A\alpha)^{\frac{1}{2}} \right\}$$

$$\dot{\theta} = \omega = \frac{\partial H}{\partial I} = \frac{1}{\frac{\partial I}{\partial E}} = \frac{\sqrt{2}\, A \pi \sqrt{E - A\alpha}}{\sqrt{m}\left(A(\pi + \alpha) - 2E + 2\sqrt{E}\sqrt{E - A\alpha}\right)}$$

$$\theta = \omega t + \theta_0$$