PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW SOLUTIONS #2 : STOCHASTIC PROCESSES

(1) Show that for time scales sufficiently greater than γ^{-1} that the solution $x(t)$ to the Langevin equation $\ddot{x} + \gamma\dot{x} = \eta(t)$ describes a Markov process. You will have to construct the matrix M defined in Eqn. 2.60 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle\eta(s)\rangle = 0$ and $\langle\eta(s)\,\eta(s')\rangle = \Gamma\,\delta(s-s')$.

Solution:

The probability distribution is

$$P(x_1, t_1\,;\,\ldots\,;\,x_N, t_N) = \text{det}^{-1/2}(2\pi M)\,\exp\left\{-\tfrac{1}{2}\sum_{j,j'=1}^{N} M^{-1}_{jj'}\,x_j\,x_{j'}\right\},$$

where

$$M(t,t') = \int_0^t\! ds \int_0^{t'}\! ds'\; G(s-s')\,K(t-s)\,K(t'-s')\,,$$

and $K(s) = (1-e^{-\gamma s})/\gamma$. Thus,

$$\begin{aligned} M(t,t') &= \frac{\Gamma}{\gamma^2}\int_0^{t_{\text{min}}}\! ds\;(1-e^{-\gamma(t-s)})(1-e^{-\gamma(t'-s)}) \\ &= \frac{\Gamma}{\gamma^2}\left\{t_{\text{min}} - \frac{1}{\gamma} + \frac{1}{\gamma}\left(e^{-\gamma t} + e^{-\gamma t'}\right) - \frac{1}{2\gamma}\left(e^{-\gamma|t-t'|} + e^{-\gamma(t+t')}\right)\right\}. \end{aligned}$$

In the limit where t and t' are both large compared to γ^{-1}, we have $M(t,t') = 2D\,\text{min}(t,t')$, where the diffusions constant is $D = \Gamma/2\gamma^2$. Thus,

$$M = 2D\begin{pmatrix} t_1 & t_1 & t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & t_3 & t_3 & \cdots & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_4 & \cdots & t_4 \\ t_1 & t_2 & t_3 & t_4 & t_5 & \cdots & t_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \end{pmatrix}.$$

To find the determinant of M, subtract row #1 from rows #2 through #N, then subtract row #2′ from the rows #3′ through #N', *etc.* The result is

$$\widetilde{M} = 2D\begin{pmatrix} t_1 & t_1 & t_1 & t_1 & t_1 & \cdots & t_1 \\ 0 & t_2-t_1 & t_2-t_1 & t_2-t_1 & t_2-t_1 & \cdots & t_2-t_1 \\ 0 & 0 & t_3-t_2 & t_3-t_2 & t_3-t_2 & \cdots & t_3-t_2 \\ 0 & 0 & 0 & t_4-t_3 & t_4-t_3 & \cdots & t_4-t_3 \\ 0 & 0 & 0 & 0 & t_5-t_4 & \cdots & t_5-t_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & t_N-t_{N-1} \end{pmatrix}.$$

Since $\widetilde{M}$ is obtained from M by consecutive row additions, we have

$$\det M = \det \widetilde{M} = t_1(t_2-t_1)(t_3-t_2)\cdots(t_N-t_{N-1})\,(2D)^N \quad .$$

The inverse is

$$M^{-1} = \frac{1}{2D}\begin{pmatrix} \frac{t_2}{t_1}\frac{1}{t_2-t_1} & -\frac{1}{t_2-t_1} & 0 & \cdots & & & & \\ -\frac{1}{t_2-t_1} & \frac{t_3-t_1}{(t_2-t_1)(t_3-t_2)} & -\frac{1}{t_3-t_2} & 0 & \cdots & & & \\ \cdots & 0 & -\frac{1}{t_n-t_{n-1}} & \frac{t_{n+1}-t_{n-1}}{(t_n-t_{n-1})(t_{n+1}-t_n)} & -\frac{1}{t_{n+1}-t_n} & 0 & \cdots & \\ & & & \ddots & & & & \\ & & & & \cdots & 0 & -\frac{1}{t_N-t_{N-1}} & \frac{1}{t_N-t_{N-1}} \end{pmatrix} .$$

This yields the general result

$$\sum_{j,j'=1}^{N} M^{-1}_{j,j'}(t_1,\ldots,t_N)\,x_j\,x_{j'} = \sum_{j=1}^{N}\left(\frac{1}{t_{j-1,j}}+\frac{1}{t_{j,j+1}}\right)x_j^2 - \frac{2}{t_{j,j+1}}\,x_j\,x_{j+1} \quad ,$$

where $t_{kl} \equiv t_l - t_k$ and $t_0 \equiv 0$ and $t_{N+1} \equiv \infty$. Now consider the conditional probability density

$$\begin{aligned} P(x_1,t_1\,|\,x_2,t_2\,;\,\ldots\,;\,x_N,t_N) &= \frac{P(x_1,t_1\,;\,\ldots\,;\,x_N,t_N)}{P(x_2,t_2\,;\,\ldots\,;\,x_N,t_N)} \\ &= \frac{\det^{1/2} 2\pi M(t_2,\ldots,t_N)}{\det^{1/2} 2\pi M(t_1,\ldots,t_N)}\,\frac{\exp\Big\{-\frac{1}{2}\sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1,\ldots,t_N)\,x_j\,x_{j'}\Big\}}{\exp\Big\{-\frac{1}{2}\sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2,\ldots,t_N)\,x_k\,x_{k'}\Big\}} \end{aligned}$$

We have

$$\begin{aligned} \sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1,\ldots,t_N)\,x_j\,x_{j'} &= \left(\frac{1}{t_{0,1}}+\frac{1}{t_{1,2}}\right)x_1^2 - \frac{2}{t_{1,2}}\,x_1\,x_2 + \left(\frac{1}{t_{1,2}}+\frac{1}{t_{2,3}}\right)x_2^2 + \ldots \\ \sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2,\ldots,t_N)\,x_k\,x_{k'} &= \left(\frac{1}{t_{0,2}}+\frac{1}{t_{2,3}}\right)x_2^2 + \ldots \end{aligned}$$

Subtracting, and evaluating the ratio to get the conditional probability density, we find

$$P(x_1,t_1\,|\,x_2,t_2\,;\,\ldots\,;\,x_N,t_N) = \sqrt{\frac{t_{0,1}+t_{1,2}}{4\pi D\,t_{0,1}\,t_{1,2}}}\,\exp\left\{-\frac{1}{2}\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix} t_{0,1}^{-1}+t_{1,2}^{-1} & -t_{1,2}^{-1} \\ -t_{1,2}^{-1} & t_{1,2}^{-1}-t_{0,2}^{-1}\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}\right\},$$

which depends only on $\{x_1,t_1,x_2,t_2\}$, *i.e.* on the current and most recent data, and not on any data before the time t_2.

(2) Provide the missing steps in the solution of the Ornstein-Uhlenbeck process described in §2.4.3 of the lecture notes. Show that applying the method of characteristics to Eqn. 2.78 leads to the solution in Eqn. 2.79.

Solution:

We solve

$$\frac{\partial \hat{P}}{\partial t} + \beta k \, \frac{\partial \hat{P}}{\partial k} = -Dk^2 \hat{P} \tag{1}$$

using the method of characteristics, writing $t = t_\zeta(s)$ and $k = k_\zeta(s)$, where s parameterizes the curve $\big(t_\zeta(s), k_\zeta(s)\big)$, and ζ parameterizes the initial conditions, which are $t(s=0) = 0$ and $k(s=0) = \zeta$. The above PDE in two variables is then equivalent to the coupled system

$$\frac{dt}{ds} = 1 \qquad , \qquad \frac{dk}{ds} = \beta k \qquad , \qquad \frac{d\hat{P}}{ds} = -Dk^2 \hat{P} \, .$$

Solving, we have

$$t_\zeta = s \qquad , \qquad k_\zeta = \zeta \, e^{\beta s} \qquad , \qquad \frac{d\hat{P}}{ds} = -D \, \zeta^2 \, e^{2\beta s} \, \hat{P} \, ,$$

and therefore

$$\hat{P}(s,\zeta) = f(\zeta) \, \exp\left\{ -\frac{D\zeta^2}{2\beta} \big(e^{2\beta s} - 1\big) \right\} .$$

We now identify $f(\zeta) = \hat{P}(k \, e^{-\beta t}, t = 0)$, hence

$$\hat{P}(k,t) = \exp\left\{ -\frac{D}{2\beta} \big(1 - e^{-2\beta t}\big) k^2 \right\} \hat{P}(k,0) \, .$$

(3) Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is $\frac{1}{2}p$ and the probability to take a step of length 2 in either direction is $\frac{1}{2}(1-p)$. Define the generating function

$$\hat{P}(k,t) = \sum_{n=-\infty}^{\infty} P_n(t) \, e^{-ikn} \, ,$$

where $P_n(t)$ is the probability to be at position n at time t. Solve for $\hat{P}(k,t)$ and provide an expression for $P_n(t)$. Evaluate $\sum_n n^2 \, P_n(t)$.

Solution:

We have the master equation

$$\frac{dP_n}{dt} = \tfrac{1}{2}(1-p) \, P_{n+2} + \tfrac{1}{2}p \, P_{n+1} + \tfrac{1}{2}p \, P_{n-1} + \tfrac{1}{2}(1-p) \, P_{n-2} - P_n \, .$$

Upon Fourier transforming,

$$\frac{d\hat{P}(k,t)}{dt} = \Big[(1-p)\cos(2k) + p\cos(k) - 1\Big]\,\hat{P}(k,t) \ ,$$

with the solution

$$\hat{P}(k,t) = e^{-\lambda(k)\,t}\,\hat{P}(k,0) \ ,$$

where

$$\lambda(k) = 1 - p\cos(k) - (1-p)\cos(2k) \ .$$

One then has

$$P_n(t) = \int\limits_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{ikn}\,\hat{P}(k,t) \ .$$

The average of n^2 is given by

$$\big\langle n^2 \big\rangle_t = -\frac{\partial^2 \hat{P}(k,t)}{\partial k^2}\bigg|_{k=0} = \Big[\lambda''(0)\,t - \lambda'(0)^2\,t^2\Big] = (4-3p)\,t \ .$$

Note that $\hat{P}(0,t) = 1$ for all t by normalization.

(4) Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in §2.6.1 of the lecture notes, and produce a figure similar to Fig. 2.3.

Solution:

Most computing languages come with a random number generating function which produces uniform deviates on the interval $x \in [0,1]$. Suppose we have a prescribed function $y(x)$. If x is distributed uniformly on $[0,1]$, how is y distributed? Clearly

$$\big|p(y)\,dy\big| = \big|p(x)\,dx\big| \qquad \Rightarrow \qquad p(y) = \bigg|\frac{dx}{dy}\bigg|\,p(x) \quad ,$$

where for the uniform distribution on the unit interval we have $p(x) = \Theta(x)\,\Theta(1-x)$. For example, if $y = -\ln x$, then $y \in [0,\infty]$ and $p(y) = e^{-y}$ which is to say y is exponentially distributed. Now suppose we want to specify $p(y)$. We have

$$\frac{dx}{dy} = p(y) \qquad \Rightarrow \qquad x = F(y) = \int\limits_{y_0}^{y} d\tilde{y}\; p(\tilde{y}) \quad ,$$

where y_0 is the minimum value that y takes. Therefore, $y = F^{-1}(x)$, where F^{-1} is the inverse function.

To generate normal (Gaussian) deviates with a distribution $p(y) = (4\pi D\varepsilon)^{-1/2}\exp(-y^2/4D\varepsilon)$, we have

$$F(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int\limits_{-\infty}^{y} d\tilde{y}\; e^{-\tilde{y}^2/4D\varepsilon} = \tfrac{1}{2} + \tfrac{1}{2}\,\text{erf}\bigg(\frac{y}{\sqrt{4D\varepsilon}}\bigg) \quad .$$

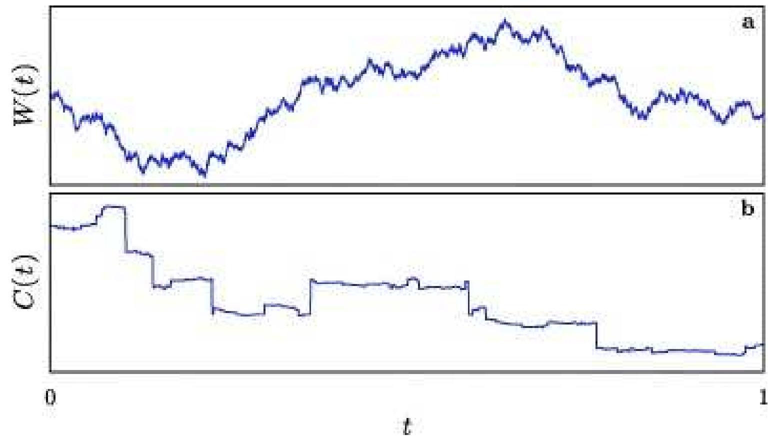

Figure 1: (a) Wiener process sample path $W(t)$. (b) Cauchy process sample path $C(t)$. From K. Jacobs and D. A. Steck, *New J. Phys.* **13**, 013016 (2011).

We now have to invert the error function, which is slightly unpleasant.

A slicker approach is to use the *Box-Muller* method, which used a two-dimensional version of the above transformation,

$$p(y_1, y_2) = p(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| \quad .$$

This has an obvious generalization to higher dimensions. The transformation factor is the Jacobian determinant. Now let x_1 and x_2 each be uniformly distributed on $[0, 1]$, and let

$$x_1 = \exp\left(-\frac{y_1^2 + y_2^2}{4D\varepsilon}\right) \qquad\qquad y_1 = \sqrt{-4D\varepsilon \ln x_1}\,\cos(2\pi x_2)$$

$$x_2 = \frac{1}{2\pi}\tan^{-1}(y_2/y_1) \qquad\qquad y_2 = \sqrt{-4D\varepsilon \ln x_1}\,\sin(2\pi x_2)$$

Then

$$\frac{\partial x_1}{\partial y_1} = -\frac{y_1\, x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_1} = -\frac{1}{2\pi}\,\frac{y_2}{y_1^2 + y_2^2}$$

$$\frac{\partial x_1}{\partial y_2} = -\frac{y_2\, x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_2} = \frac{1}{2\pi}\,\frac{y_1}{y_1^2 + y_2^2}$$

and therefore the Jacobian determinant is

$$J = \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| = \frac{1}{4\pi D\varepsilon}\, e^{-(y_1^2 + y_2^2)/4D\varepsilon} = \frac{e^{-y_1^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \cdot \frac{e^{-y_2^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \quad ,$$

which says that y_1 and y_2 are each independently distributed according to the normal distribution $p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)$. Nifty!

For the Cauchy distribution, with

$$p(y) = \frac{1}{\pi}\,\frac{\varepsilon}{y^2+\varepsilon^2} \quad ,$$

we have

$$F(y) = \frac{1}{\pi}\int\limits_{-\infty}^{y}\!d\tilde{y}\,\frac{\varepsilon}{\tilde{y}^2+\varepsilon^2} = \tfrac{1}{2} + \tfrac{1}{\pi}\tan^{-1}(y/\varepsilon) \quad ,$$

and therefore

$$y = F^{-1}(x) = \varepsilon \tan\big(\pi x - \tfrac{\pi}{2}\big) \quad .$$

(5) Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don't obey Ohm's law in the limit where the 'inelastic mean free path' is greater than the sample dimensions, which you may assume here. Rather, let $\mathcal{R}(L) = e^2 R(L)/h$ be the dimensionless resistance of a quantum wire of length L, in units of $h/e^2 = 25.813\,\text{k}\Omega$. The dimensionless resistance of a quantum wire of length $L + \delta L$ is then given by

$$\begin{aligned}\mathcal{R}(L+\delta L) &= \mathcal{R}(L) + \mathcal{R}(\delta L) + 2\,\mathcal{R}(L)\,\mathcal{R}(\delta L)\\ &\qquad\qquad + 2\cos\alpha\,\sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]}\ ,\end{aligned}$$

where α is a *random phase* uniformly distributed over the interval $[0, 2\pi)$. Here,

$$\mathcal{R}(\delta L) = \frac{\delta L}{2\ell}\ ,$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \lesssim \ell$, where ℓ is the 'elastic mean free path'.

(a) Show that the distribution function $P(\mathcal{R}, L)$ for resistances of a quantum wire obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell}\,\frac{\partial}{\partial \mathcal{R}}\bigg\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\bigg\}\ .$$

(b) Show that this equation may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\,e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\,\frac{1}{\mathcal{R}}\,e^{-(\ln\mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle\mathcal{R}\rangle$ in the former case, and $\langle\ln\mathcal{R}\rangle$ in the latter case.

Solution:

(a) From the composition rule for series quantum resistances, we derive the phase averages

$$\begin{aligned}
\langle \delta\mathcal{R} \rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} \\
\big\langle (\delta\mathcal{R})^2 \big\rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)^2 \left(\frac{\delta L}{2\ell}\right)^2 + 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell}\left(1 + \frac{\delta L}{2\ell}\right) \\
&= 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} + \mathcal{O}\big((\delta L)^2\big)\ ,
\end{aligned}$$

whence we obtain the drift and diffusion terms

$$F_1(\mathcal{R}) = \frac{2\mathcal{R} + 1}{2\ell} \qquad , \qquad F_2(\mathcal{R}) = \frac{2\mathcal{R}(1 + \mathcal{R})}{2\ell}\ .$$

Note that $2F_1(\mathcal{R}) = dF_2/d\mathcal{R}$, which allows us to write the Fokker-Planck equation as

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}}\left\{\frac{\mathcal{R}\,(1 + \mathcal{R})}{2\ell}\,\frac{\partial P}{\partial \mathcal{R}}\right\}.$$

(b) Defining the dimensionless length $z = L/2\ell$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}}\left\{\mathcal{R}\,(1 + \mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\right\}.$$

In the limit $\mathcal{R} \ll 1$, this reduces to

$$\frac{\partial P}{\partial z} = \mathcal{R}\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}}\ ,$$

which is satisfied by $P(\mathcal{R}, z) = z^{-1}\,\exp(-\mathcal{R}/z)$. For this distribution one has $\langle \mathcal{R} \rangle = z$.

In the opposite limit, $\mathcal{R} \gg 1$, we have

$$\begin{aligned}
\frac{\partial P}{\partial z} &= \mathcal{R}^2\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + 2\,\mathcal{R}\,\frac{\partial P}{\partial \mathcal{R}} \\
&= \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu}\ ,
\end{aligned}$$

where $\nu \equiv \ln \mathcal{R}$. This is solved by the log-normal distribution,

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\,e^{-(\nu + z)^2/4z}\ .$$

Note that

$$P(\mathcal{R}, z)\,d\mathcal{R} = (4\pi z)^{-1/2}\,\exp\left\{-\,\frac{(\ln \mathcal{R} - z)^2}{4z}\right\} d\ln \mathcal{R}\ .$$

One then obtains $\langle \ln \mathcal{R} \rangle = z$.