Lecture Notes on partial differential equations

These four lectures follow a basic introduction to Laplace and Fourier transforms.

Emphasis is laid on the notion of initial and boundary problems which provides a wide receptacle to many engineering disciplines.

Many exercises are framed into a particular discipline, in order to show to the student that the methods exposed go over the basic academic manipulations. The point is not to 'ethnicize' the mathematical tools, but rather to show that the same tools are in fact used by several disciplines, although with different jargons. This fact in itself highlights the powerfulness of the tools.

Moreover, the various branches of physics provide rich issues that are a feast to applied mathematicians. It would not be wise not to take advantage of their variety.

Therefore, examples borrow from the various fields that are taught in this unit, namely strength of materials, elasticity, structural dynamics, wave propagation, heat diffusion, compressible and incompressible fluid mechanics, free surface flow, flow in porous media, particle flow, transport engineering, and electrical engineering.

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Chapter I

Solving IBVPs with the Laplace transform

I.1 General perspective

Partial differential equations (PDEs) of mathematical physics [1] are classified in three types, as indicated in the Table below:

Table delineating the three types of PDEs

type	governing equation	unknown u	prototype	appropriate transform
(E) elliptic	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	u(x,y)	potential	Fourier
(P) parabolic	$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$	u(x,t)	heat diffusion	Laplace, Fourier
(H) hyperbolic	$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$	u(x,t)	wave propagation	Laplace, Fourier

We will come back later to a more systematic treatment and classification of PDEs.

The one dimensional examples exposed below intend to display some basic features and differences between parabolic and hyperbolic partial differential equations. The constitutive assumptions that lead to the partial differential field equations whose mathematical nature is of prime concern are briefly introduced.

Problems of mathematical physics, continuum mechanics, fluid mechanics, strength of materials, hydrology, thermal diffusion $\cdots$, can be cast as IBVPs:

IBVPs: **Initial and Boundary Value Problems**

We will take care to systematically define problems in that broad framework. To introduce the ideas, we delineate three types of relations, as follows.

I.1.1 (FE) Field Equations

The field equations that govern the problems are partial differential equations in space and time. As indicated just above, they can be classified as elliptic, parabolic and hyperbolic. This

[1] Posted November 22, 2008; Updated, March 26, 2009

chapter will consider prototypes of the two later types, and emphasize their physical meaning, interpretation and fundamental differences.

We can not stress enough that

(P) for a parabolic equation, the information diffuses at infinite speed, and progressively,

while

(H) for a hyperbolic equation, the information propagates at finite speed and discontinuously.

These field equations should be satisfied *within the body*, say Ω. They are not required to hold on the boundary $\partial\Omega$. Ω is viewed as an *open* set of points in space, in the topological sense.

In this chapter, the solutions $u(x,t)$ are obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{I.1.1}$$

and we shamelessly admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{I.1.2}$$

In contrast, we will see that the use of the Fourier transform adopts the dual rule, in as far as the Fourier variable is the space variable x,

$$u(x,t) \to U(\alpha,t) = \mathcal{F}\{u(x,t)\}(\alpha)\,. \tag{I.1.3}$$

A qualitative motivation consists in establishing a correspondence between the time variable, a positive quantity, and the definition of the (one-sided) Laplace transform which involves an integration over a positive variable.

In contrast, the Fourier transform involves the integration over the whole real set, which is interpreted as a spatial axis.

Attention should be paid to the interpretation of the physical phenomena considered. For example, hyperbolic problems in continuum mechanics involves the speed of propagation of elastic waves. This wave speed should not be confused with the velocity of particles. To understand the difference, consider a wave moving over a fluid surface. When we follow the phenomenon, we attach our eyes to the top of the wave. At two subsequent times, the particles at the top of the waves are not the same, and therefore the wave speed and particle velocity are two distinct functions of space and time. In fact, in the examples to be considered, the wave speeds are constant in space and time.

I.1.2 (IC) Initial Conditions

Initial Conditions come into picture when the physical time is involved, namely in (P) and (H) equations.

Roughly, the number of initial conditions depends on the order of the partial derivative(s) wrt time in the field equation (FE). For example, one initial condition is required for the heat diffusion problem, and two for the wave propagation problem.

I.1.3 (BC) Boundary Conditions

Similarly, the number of boundary conditions depends on the order of the partial derivative(s) wrt space in the field equation. For example, for the bending of a beam, four conditions are required since the field equation involves a fourth order derivative in space.

This rule applies when the body is finite. The treatment of semi-infinite or infinite bodies is both easier and more delicate, and often requires some knowledge-based decisions, which still are quite easy to enter. Typically, since the boundary is rejected to infinity, the boundary condition is replaced by either an asymptotic requirement, e.g. the solution should remain finite at infinity $\cdots$, or by a *radiation condition*. For the standard problems considered here, the source of disturbance is located at finite distance, and the radiation condition ensures that the information diffuses/propagates toward infinity. In contrast, for a finite body, waves would be reflected by the boundary, at least partially, back to the body. Alternatively, one could also consider diffraction problems, where the source of the disturbance emanates from infinity.

Boundary conditions are phrased in terms of the main unknown, or its time or space derivative(s).

For example, for an elastic problem, the displacement may be prescribed at a boundary point: its gradient (strain) should be seen as participating to the response of the structure. The velocity may be substituted for the displacement. Conversely, instead of prescribing the displacement, one may prescribe its gradient: then the displacement participates to the response of the structure.

Attention should be paid not to prescribe incompatible boundary conditions: e.g. for a thermal diffusion problem, either the temperature or the heat flux (temperature gradient) may be prescribed, but *not both* simultaneously.

The situation is more complex for higher order problems, e.g. for the bending of beams. Of course, any boundary condition is definitely motivated by the underlying physics.

It is time now to turn to examples.

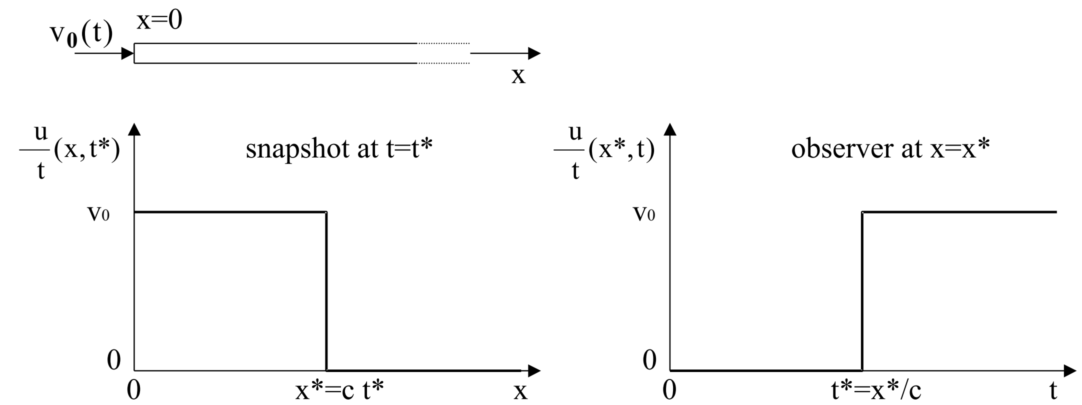

Figure I.1 A semi-infinite elastic bar is subject to a velocity discontinuity. Spatial and time profiles of the particle velocity. The discontinuity propagates along the bar at the wave speed $c = \sqrt{E/\rho}$ where $E > 0$ is Young's modulus and $\rho > 0$ the mass density.

I.2 An example of hyperbolic PDE: propagation of a shock wave

A semi-infinite elastic plane, or half-space, is subject to a load normal to its boundary. The motion is one dimensional and could also be viewed as due to an axial load on the end of an elastic bar with vanishing Poisson's ratio. The bar is initially at rest, and the loading takes the form of an arbitrary velocity discontinuity $v = v_0(t)$.

The issue is to derive the axial displacement $u(x,t)$ and axial velocity $v(x,t)$ of the points of the bar while the mechanical information propagates along the bar with a wave speed,

$$c = \sqrt{\frac{E}{\rho}}, \tag{I.2.1}$$

where $E > 0$ and $\rho > 0$ are respectively the Young's modulus and mass density of the material.

The governing equations of dynamic linear elasticity are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x > 0;\\
&\text{(IC) initial conditions} \quad u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 0\,;\\
&\text{(BC) boundary condition} \quad \frac{\partial u}{\partial t}(0,t) = v_0(t); \quad \text{radiation condition (RC)}\,.
\end{aligned} \tag{I.2.2}$$

The field equation is obtained by combining

$$\begin{aligned}
&\text{momentum balance} \quad \frac{\partial \sigma}{\partial x} - \rho\,\frac{\partial^2 u}{\partial x^2} = 0;\\
&\text{elasticity} \quad \sigma = E\,\frac{\partial u}{\partial x}\,,
\end{aligned} \tag{I.2.3}$$

where σ is the axial stress.

The first initial condition (IC) means that the displacement is measured from time $t = 0$, or said otherwise, that the configuration (geometry) at time $t = 0$ is used as a reference. The second (IC) simply means that the bar is initially at rest.

The radiation condition (RC) is intended to imply that the mechanical information propagates in a single direction, and that the bar is either of infinite length, or, at least, that the signal has not the time to reach the right boundary of the bar in the time window of interest. Indeed, any function of the form $f_1(x - c\,t) + f_2(x + c\,t)$ satisfies the field equation. A function of $x - c\,t$ represents a signal that propagates toward increasing x. To see this, let us keep the eyes on some given value of f_1, corresponding to $x - c\,t$ equal to some constant. Then clearly, sine the wave speed c is a positive quantity, the point we follow moves in time toward increasing x.

The solution is obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{I.2.4}$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{I.2.5}$$

Therefore,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\,U(x,p) - p\,\underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0,\ \text{(IC)}}\Big) = 0\\
&\Rightarrow \quad U(x,p) = a(p)\,\exp(-\frac{p}{c}x) + \underbrace{b(p)\,\exp(+\frac{p}{c}x)}_{b(p)=0,\ \text{(RC)}}\,.
\end{aligned} \tag{I.2.6}$$

The term $\exp(-p\,x/c)$ gives rise to a wave which propagates toward increasing value of x, and the term $\exp(p\,x/c)$ gives rise to a wave which propagates toward decreasing value of x. Where do these assertions come from? There is no direct answer, simply they can be checked on the result to be obtained. Thus the radiation condition implies to set $b(p)$ equal to 0.

In turn, taking the Laplace transform of the (BC),

$$\begin{aligned} &(\text{BC})\ p\,U(0,p) - \underbrace{u(0,0)}_{=0,\ (\text{IC})} = \mathcal{L}\{v_0(t)\}(p) \\ \Rightarrow \quad & U(x,p) = \frac{1}{p}\exp(-\frac{p}{c}x)\,\mathcal{L}\{v_0(t\}(p)\,. \end{aligned} \tag{I.2.7}$$

The inverse Laplace transform is a convolution integral,

$$u(x,t) = \int_0^t \mathcal{H}(t-\frac{x}{c}-\tau)\,v_0(\tau)\,d\tau\,, \tag{I.2.8}$$

which, for a shock $v_0(t) = V_0\,\mathcal{H}(t)$, simplifies to

$$u(x,t) = V_0\,(t-\frac{x}{c})\,\mathcal{H}(t-\frac{x}{c}), \quad \frac{\partial u}{\partial t}(x,t) = V_0\,\mathcal{H}(t-\frac{x}{c})\,. \tag{I.2.9}$$

The analysis of the velocity of the particles reveals two main characteristics of a partial differential equation of the hyperbolic type in non dissipative materials, Fig. I.1:

(H1). **the mechanical information propagates at finite speed**, namely c, which is therefore termed elastic wave speed;

(H2). **the wave front carrying the mechanical information propagates** undistorted and **with a constant amplitude**.

Besides, this example highlights the respective meanings of elastic wave speed c and velocity of the particles $\partial u/\partial t(x,t)$.

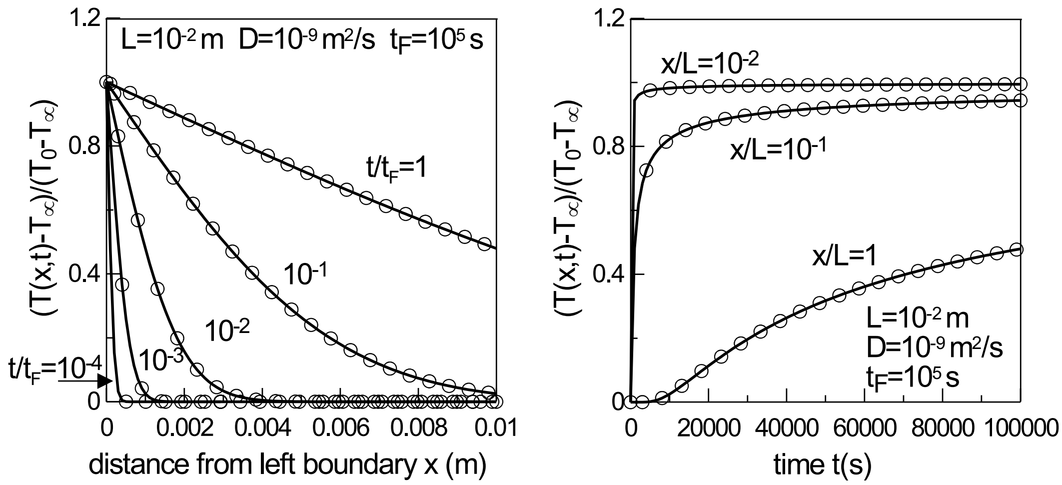

Figure I.2 Heat diffusion on a semi-infinite bar subject to a heat shock at its left boundary $x = 0$. The characteristic time t_F that can be used to describe the information that reaches the position $x = L$ is equal to L^2/D.

I.3 A parabolic PDE: diffusion of a heat shock

As a second example of PDE, let us consider the diffusion of a heat shock in a semi-infinite body.

A semi-infinite plane, or half-space, or bar, is subject to a given temperature $T = T_0(t)$ at its left boundary $x = 0$. The thermal diffusion is one dimensional. The initial temperature along the bar is uniform, $T(x,t) = T_\infty$. It is more convenient to work with the field $\theta(x,t) = T(x,t) - T_\infty$ than with the temperature itself.

For a rigid material, in absence of heat source, the energy equation links the divergence of the heat flux $\mathbf{Q}$ [unit: kg/s^3], and the time rate of the temperature field $\theta(x,t)$, namely

$$\text{energy equation} \quad \operatorname{div}\mathbf{Q} + C\,\frac{\partial\theta}{\partial t} = 0\,, \tag{I.3.1}$$

or in cartesian axes and using the convention of summation over repeated indices (the index i varies from 1 to n in a space of dimension n),

$$\frac{\partial Q_i}{\partial x_i} + C\,\frac{\partial\theta}{\partial t} = 0\,. \tag{I.3.2}$$

Here $C >$ [unit: kg/m/s^2/°K] is the heat capacity per unit volume. The heat flux is related to the temperature gradient by Fourier law

$$\text{Fourier law} \quad \mathbf{Q} = -k_T\,\nabla\theta\,, \tag{I.3.3}$$

or componentwise,

$$Q_i = -k_T\,\frac{\partial\theta}{\partial x_i}\,, \tag{I.3.4}$$

where $k_T > 0$ [unit: kg×m/s^3/°K] is the thermal conductivity.

Inserting Fourier's law in the energy equation shows that a single material coefficient D [unit: m^2/s],

$$D = \frac{k_T}{C} > 0, \tag{I.3.5}$$

that we shall term diffusivity, appears in the field equation.

The initial and boundary value problem (IBVP) is thus governed by the following set of equations:

$$\begin{aligned}
&\text{field equation (FE)} \quad D\,\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} = 0, \quad t>0,\ x>0;\\
&\text{initial condition (IC)} \quad \theta(x,0) = 0\,;\\
&\text{boundary condition (BC)} \quad \theta(0,t) = \theta_0(t); \quad \text{radiation condition (RC)}\,.
\end{aligned} \tag{I.3.6}$$

The radiation condition is intended to imply that the thermal information diffuses in a single direction, namely toward increasing x, and that the bar is either of infinite length.

The solution is obtained through the Laplace transform

$$\theta(x,t) \to \Theta(x,p) = \mathcal{L}\{\theta(x,t)\}(p)\,. \tag{I.3.7}$$

Therefore,

$$
\begin{aligned}
&\text{(FE)}\ D\,\frac{\partial^2\Theta(x,p)}{\partial x^2} - \Big(p\,\Theta(x,p) - \underbrace{\theta(x,0)}_{=0,\ \text{(IC)}}\Big) = 0,\\
&\quad\Rightarrow\quad \Theta(x,p) = a(p)\,\exp(-\sqrt{\frac{p}{D}}\,x) + \underbrace{b(p)\,\exp(\sqrt{\frac{p}{D}}\,x)}_{b(p)=0,\ \text{(RC)}}\\
&\text{(BC)}\ \Theta(0,p) = \mathcal{L}\{\theta_0(t\}(p)\\
&\quad\Rightarrow\quad \Theta(x,p) = \mathcal{L}\{\theta_0(t\}(p)\,\exp(-\sqrt{\frac{p}{D}}\,x)\,.
\end{aligned}
\tag{I.3.8}
$$

The multiform complex function $\sqrt{p}$ has been made uniform by introducing a branch cut along the negative axis $\Re p \le 0$, and by defining $p = |p|\,\exp(i\,\theta)$ with $\theta \in]-\pi,\pi]$, and $\sqrt{p} = \sqrt{|p|}\,\exp(i\,\theta/2)$ so that $\Re\sqrt{p} \ge 0$. Then the term $\exp(\sqrt{p/D}\,x)$ gives rise to heat propagation toward decreasing x, in contradiction with the radiation condition, and this justifies why $b(p)$ has been forced to vanish.

The inverse transform is a convolution product,

$$
\theta(x,t) = \int_0^t \theta_0(t-u)\ \mathcal{L}^{-1}\{\exp\Big(-x\sqrt{\frac{p}{D}}\Big)\}(u)\,du\,. \tag{I.3.9}
$$

Using the relation, established later in Exercise I.6,

$$
\mathcal{L}^{-1}\{\exp\Big(-x\sqrt{\frac{p}{D}}\Big)\}(u) = \frac{x}{2\sqrt{\pi\,D\,u^3}}\,\exp\Big(-\frac{x^2}{4\,Du}\Big), \tag{I.3.10}
$$

then

$$
\theta(x,t) = \frac{x}{2\sqrt{D\pi}}\int_0^t \theta_0(t-u)\ \frac{1}{u^{3/2}}\,\exp\Big(-\frac{x^2}{4\,Du}\Big)\,du\,. \tag{I.3.11}
$$

With the further change of variable

$$
u \to v \quad \text{with} \quad v^2 = \frac{x^2}{4D\,u}, \tag{I.3.12}
$$

the inverse Laplace transform takes the form,

$$
\theta(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{D\,t}}^{\infty} \theta_0(t-\frac{x^2}{4\,D\,v^2})\,\exp(-v^2)\,dv\,, \tag{I.3.13}
$$

which, for a shock $\theta_0(t) = (T_0 - T_\infty)\,\mathcal{H}(t)$, simplifies to

$$
T(x,t) - T_\infty = (T_0 - T_\infty)\ \text{erfc}\Big(\frac{x}{2\sqrt{D\,t}}\Big)\,. \tag{I.3.14}
$$

The plot of the evolution of the spatial profiles of the temperature reveals two main characteristics of a partial differential equation of the parabolic type, Fig. I.2:

(P1). **the thermal information diffuses at infinite speed**: indeed, the fact that the boundary $x = 0$ has been heated is known instantaneously at any point of the bar;

(P2). however, **the amplitude of the thermal shock applied at the boundary requires time to fully develop** in the bar. In fact, the temperature needs an infinite time to equilibrate along the bar.

Note however, that a modification of the Fourier's law, or of the energy equation, which goes by the name of Cataneo, provides finite propagation speeds. The point is not considered further here.

In contrast to wave propagation, where time and space are involved in linear expressions $x \pm c\,t$, here the space variable is associated with the square root of the time variable. Therefore diffusion over a distance $2\,L$ requires a time interval four times larger than the time interval of diffusion over a length L.

The error and complementary error functions

Use has been made of the error function

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-v^2}\,dv\,, \tag{I.3.15}$$

and of the complementary error function

$$\text{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-v^2}\,dv\,. \tag{I.3.16}$$

Note the relations

$$\text{erf}(y) + \text{erfc}(y) = 1, \quad \text{erf}(-y) = -\text{erf}(y), \quad \text{erfc}(-y) = 2 - \text{erfc}(y)\,, \tag{I.3.17}$$

and the particular values $\text{erfc}(0) = 1$, $\text{erfc}(\infty) = 0$.

I.4 An equation displaying advection-diffusion

The diffusion of a species dissolved in a fluid at rest obeys the same field equation as thermal diffusion. Diffusion takes place so as to homogenize the concentration $c = c(x,t)$ of a solute in space. Usually however, the fluid itself moves due to different physical phenomena: for example, seepage of the fluid through a porous medium is triggered by a gradient of fluid pressure and governed by Darcy law. Let us assume the velocity v of the fluid, referred to as *advective velocity*, to be a given constant.

Thus, we assume the fluid to move with velocity v and the solute with velocity v_s. In order to highlight that diffusion is relative to the fluid, two fluxes are introduced, a diffusive flux $\mathbf{J}^d$ and an absolute flux $\mathbf{J}$,

$$\underbrace{\mathbf{J} = c\,v_s}_{\text{absolute flux}}\,, \quad \underbrace{\mathbf{J}^d = c\,(v_s - v)}_{\text{diffusive flux}}\,, \tag{I.4.1}$$

whence,

$$\mathbf{J} = \mathbf{J}^d + c\,v\,. \tag{I.4.2}$$

The diffusion phenomenon is governed by Fick's law that relates the *diffusive* flux $\mathbf{J}^d = c\,(v_s - v)$ to the gradient of concentration via a coefficient of molecular diffusion D [unit: m^2/s],

$$\text{Fick's law} \quad \mathbf{J}^d = -D\,\nabla c\,, \tag{I.4.3}$$

and by the mass balance, which, in terms of concentration and *absolute* flux $\mathbf{J} = c\,v_s$, writes,

$$\text{balance of mass} \quad \frac{\partial c}{\partial t} + \text{div}\,\mathbf{J} = 0\,. \tag{I.4.4}$$

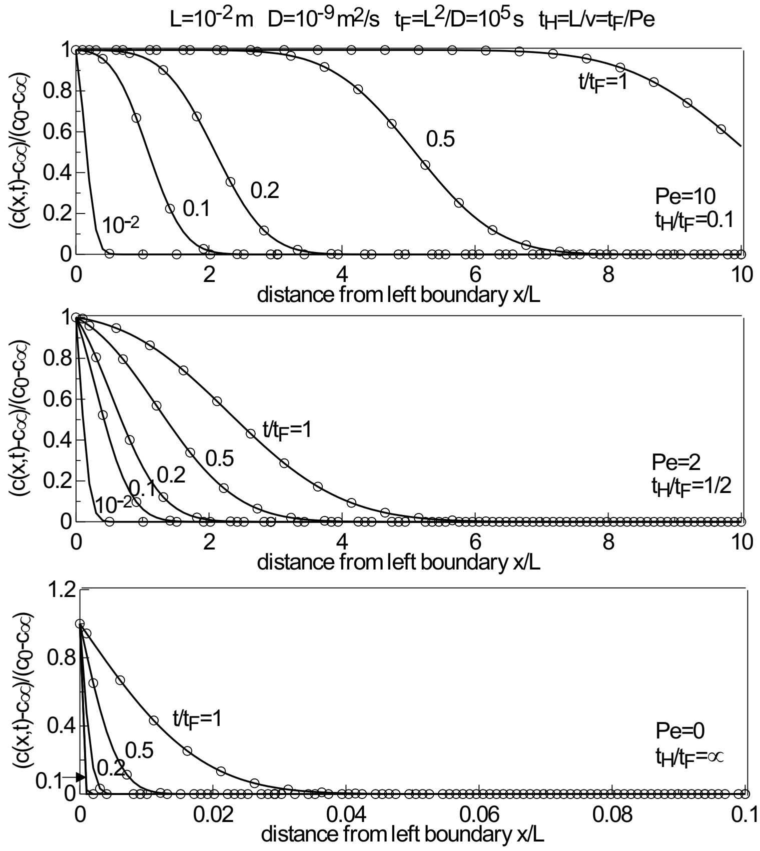

Figure I.3 Advection-diffusion along a semi-infinite bar of a species whose concentration is subject at time $t = 0$ to a sudden increase at the left boundary $x = 0$. The fluid is animated with a velocity v such that the Péclet number Pe is equal to 10, 2 and 0 (pure diffusion) respectively. Focus is on the events that take place at point $x = L$. The time which characterizes the propagation phenomenon is $t_H = L/v$ while the characteristic time of diffusion is $t_F = L^2/D$, with D the diffusion coefficient. Therefore $\text{Pe} = L\,v/D = t_H/t_F$.

The equations of diffusion analyzed in Sect. I.3 modify to

$$\begin{aligned}
&\text{(FE) field equation} \quad \overbrace{D\,\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial t}}^{\substack{\text{diffusive}\\ \text{terms}}} = \overbrace{v\,\frac{\partial c}{\partial x}}^{\substack{\text{advective}\\ \text{term}}}, \quad t > 0,\ x > 0;\\
&\text{(IC) initial condition} \quad c(x,0) = c_i\,;\\
&\text{(BC) boundary condition} \quad c(0,t) = c_0(t); \quad \text{and (RC) a radiation condition}\,.
\end{aligned} \tag{I.4.5}$$

A concentration discontinuity is imposed at $x = 0$, namely $c(0,t) = c_0\,\mathcal{H}(t)$.

The solution $c(x,t)$ is obtained through the Laplace transform $c(x,t) \to C(x,p)$. First, the field equation becomes an ordinary differential equation wrt space where the Laplace variable

p is viewed as a parameter,

$$\text{(FE)}\quad D\,\frac{d^2\tilde{C}}{dx^2} - v\,\frac{d\tilde{C}}{dx} - p\,\tilde{C} = 0, \quad \tilde{C}(x,p) \equiv C(x,p) - \frac{c_i}{p}\,. \tag{I.4.6}$$

The solution,

$$\tilde{C}(x,p) = a(p)\,\exp\Big(\frac{v\,x}{2D} - \sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\,x\Big) + b(p)\,\exp\Big(\frac{v\,x}{2D} + \sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\,x\Big)\,, \tag{I.4.7}$$

involves two unknowns $a(p)$ and $b(p)$. The unknown $b(p)$ is set to 0, because it multiplies a function that would give rise to diffusion from right to left: the radiation condition (RC) intends to prevent this phenomenon. The second unknown $a(p)$ results from the (BC):

$$\text{(RC)}\quad b(p) = 0, \quad \text{(BC)}\quad a(p) = \frac{c_0 - c_i}{p}\,. \tag{I.4.8}$$

The resulting complete solution in the Laplace domain

$$C(x,p) = \frac{c_i}{p} + \frac{c_0 - c_i}{p}\,\exp\Big(-x\,\sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\Big)\,\exp\Big(\frac{v\,x}{2D}\Big)\,, \tag{I.4.9}$$

may be slightly transformed to

$$C(x,P) = \frac{c_i}{P-\alpha} + \frac{c_0 - c_i}{P-\alpha}\,\exp\Big(-x\,\sqrt{\frac{P}{D}}\Big)\,\exp\Big(\frac{v\,x}{2D}\Big), \quad P \equiv p + \alpha, \quad \alpha \equiv \frac{v^2}{4\,D}\,. \tag{I.4.10}$$

The inverse transform of the second term is established in Exercise I.7,

$$\begin{aligned}&\mathcal{L}^{-1}\{\frac{\exp(-x\,\sqrt{P/D})}{P-\alpha}\}(t)\\ &= \frac{\exp(\alpha\,t)}{2}\Big(\exp\,(x\,\sqrt{\frac{\alpha}{D}})\,\text{erfc}(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha\,t}) + \exp\,(-x\,\sqrt{\frac{\alpha}{D}})\,\text{erfc}(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha\,t})\Big)\,.\end{aligned} \tag{I.4.11}$$

Whence, in view of the rule,

$$\mathcal{L}\{c(x,t)\}(P) = \mathcal{L}\{e^{-\alpha\,t}\,c(x,t)\}(p = P - \alpha)\,, \tag{I.4.12}$$

the solution can finally be cast in the format, which holds for positive or negative velocity v,

$$c(x,t) = c_i + \frac{1}{2}\,(c_0 - c_i)\,\Big(\,\text{erfc}\Big(\frac{x - v\,t}{2\sqrt{D\,t}}\Big) + \exp(\frac{v\,x}{D})\,\text{erfc}\Big(\frac{x + v\,t}{2\sqrt{D\,t}}\Big)\,\Big)\,. \tag{I.4.13}$$

Note that any arbitrary function $f = f(x - v\,t)$ leaves unchanged the first order part of the field equation (I.4.5). The solution can thus be seen as displaying a front propagating toward $x = \infty$, which is smoothed out by the diffusion phenomenon. The dimensionless Péclet number Pe quantifies the relative weight of advection and diffusion,

$$\text{Pe} = \frac{L\,v}{D}\begin{cases} \ll 1 & \text{diffusion dominated flow} \\ \gg 1 & \text{advection dominated flow}\end{cases} \tag{I.4.14}$$

The length L is a characteristic length of the problem, e.g. mean grain size in granular media, or length of the column for breakthrough tests in a column of finite length.

For transport of species, the Péclet number represents the ratio of the number, or mass, of particles transported by advection and diffusion. For heat transport, it gives an indication of the ratio of the heat transported by advection and by conduction.

Exercise I.1: **Playing with long darts**.

A dart, moving at uniform horizontal velocity $-v_0$, is headed toward a vertical wall, located at the position $x = 0$. Its head hits the wall at time $t = 0$, and thereafter remains glued to the wall. Three snapshots have been taken, at different times, and displayed on Fig. I.5.

The dart is long enough, so that during the time interval of interest, no wave reaches its right end. In other words, for the present purpose, the dart can be considered as semi-infinite.

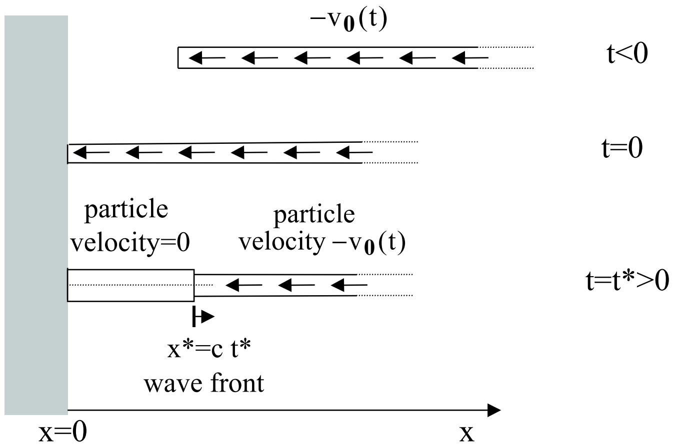

Figure I.4 An elastic dart moving at speed $-v_0$ hits a rigid target at time $t = 0$. The shock then propagates along the dart at the speed of elastic longitudinal waves. The part of the dart behind the wave front is set to rest and it undergoes axial compression: although the analysis here is one-dimensional, we have visualized this aspect by a (mechanically realistic) lateral expansion.

The dart is assumed to be made in a linear elastic material, in which the longitudinal waves propagate at speed c. The equations of dynamic linear elasticity governing the axial displacement $u(x,t)$ of points of the dart are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x > 0;\\
&\text{(IC) initial conditions} \quad u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = -v_0\,; \qquad (1)\\
&\text{(BC) boundary condition} \quad u(0,t) = 0; \quad \text{radiation condition (RC)} \ \frac{\partial u}{\partial x}(\infty,t) = 0\,.
\end{aligned}$$

Find the displacement $u(x,t)$ and give a vivid interpretation of the event.

Solution:

The solution is obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{2}$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{3}$$

Therefore,

$$
\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\,U(x,p) - p\,\underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=-v_0,\ \text{(IC)}}\Big) = 0 \\
&\frac{d^2}{dx^2}\Big(U(x,p) + \frac{v_0}{p^2}\Big) = \frac{p^2}{c^2}\Big(U(x,p) + \frac{v_0}{p^2}\Big)\,.
\end{aligned} \tag{4}
$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter. Then

$$
U(x,p) + \frac{v_0}{p^2} = a(p)\,\exp(\frac{p}{c}x) + b(p)\,\exp(-\frac{p}{c}x)\,. \tag{5}
$$

The first term in the solution above would give rise to a wave moving to left, which is prevented by the radiation condition:

$$
\begin{aligned}
&\text{(RC)} \quad a(p) = 0 \\
&\text{(BC)} \quad 0 + \frac{v_0}{p^2} = b(p)\,\exp(0) \\
&U(x,p) = \frac{v_0}{p^2}\Big(\exp(-\frac{p}{c}x) - 1\Big)\,.
\end{aligned} \tag{6}
$$

Inverse Laplace transform yields in turn the displacement $u(x,t)$,

$$
u(x,t) = v_0\Big((t-\frac{x}{c})\,\mathcal{H}(t-\frac{x}{c}) - t\,\mathcal{H}(t)\Big) = \begin{cases} -v_0\,\dfrac{x}{c}, & x < c\,t; \\ -v_0\,t, & x \geq c\,t; \end{cases} \tag{7}
$$

the particle velocity,

$$
\frac{\partial u(x,t)}{\partial t} = \begin{cases} 0, & x < c\,t; \\ -v_0, & x > c\,t; \end{cases} \tag{8}
$$

and the strain,

$$
\frac{\partial u(x,t)}{\partial x} = \begin{cases} -\dfrac{v_0}{c}, & x < c\,t; \\ 0, & x > c\,t\,. \end{cases} \tag{9}
$$

These relations clearly indicate that the mechanical information " the dart head is glued to the wall" propagates to the right along the dart at speed c. At a given time t^*, only points sufficiently close to the wall have received the information, while further points still move with the initial speed $-v_0$. Points behind the wave front undergo compressive straining, while the part of the dart to the right of the wave front is undeformed yet. An observer, located at x^* needs to wait a time $t = x^*/c$ to receive the information: the velocity of this point then stops immediately, and completely.

Exercise I.2: **Laplace transforms of periodic functions**.

Consider the three periodic functions sketched on Fig. I.5.

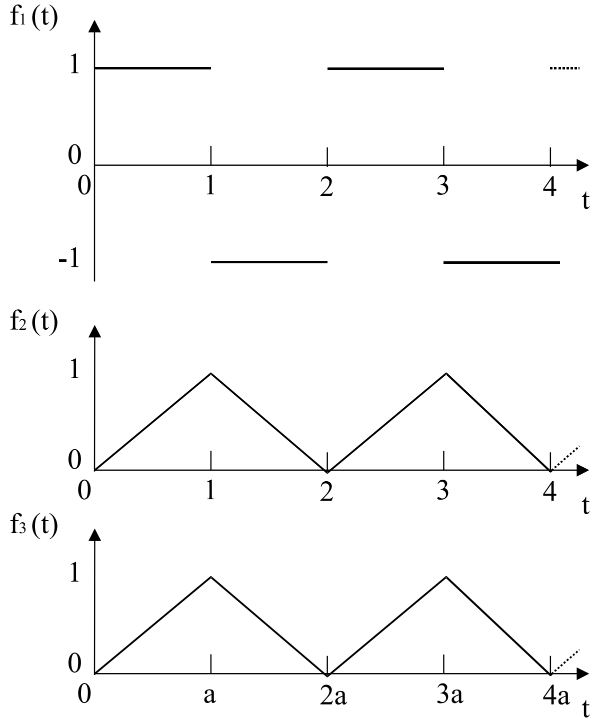

Figure I.5 Periodic functions; for $f_3(t)$, the real $a > 0$ is strictly positive.

Calculate their Laplace transforms:

$$\mathcal{L}\{f_1(t)\}(p) = \frac{1}{p}\,\tanh\Big(\frac{p}{2}\Big);\;\; \mathcal{L}\{f_2(t)\}(p) = \frac{1}{p^2}\,\tanh\Big(\frac{p}{2}\Big);\;\; \mathcal{L}\{f_3(t)\}(p) = \frac{1}{a\,p^2}\,\tanh\Big(\frac{a\,p}{2}\Big)\,. \quad (1)$$

Proof:

1. These functions are periodic, for $t > 0$, with period $T = 2$:

$$\mathcal{L}\{f_1(t)\}(p) = \frac{1}{1-e^{-2p}}\int_0^2 e^{-t\,p}\,f_1(t)\,dt = \frac{1}{1-e^{-2p}}\,\Big(\int_0^1 e^{-t\,p}\,dt + \int_1^2 e^{-t\,p}\,(-1)\,dt\Big)\,. \quad (2)$$

2. The function $f_2(t)$ is the integral of $f_1(t)$:

$$\mathcal{L}\{f_2(t)\}(p) = \mathcal{L}\{\int_0^t f_1(t)\}(p) = \frac{1}{p}\,\mathcal{L}(f_1(t))(p)\,. \quad (3)$$

3. Since $f_3(t) = f_2(b\,t)$, with $b = 1/a$,

$$\mathcal{L}\{f_3(t)\}(p) = \frac{1}{b}\,\mathcal{L}\{f_2(t))(\frac{p}{b})\,. \quad (4)$$

Exercise I.3: **Longitudinal vibrations of a finite bar**.

A bar, of finite length L, is fixed at its left boundary $x = 0$. It is at rest for times $t < 0$. At $t = 0$, it is submitted to a force F_0 at its right boundary $x = L$.

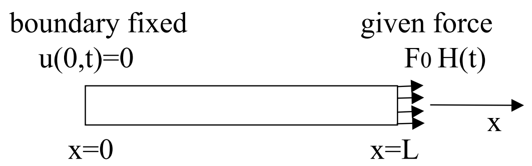

Figure I.6 An elastic bar, fixed at its left boundary, is hit by a sudden load at its right boundary at time t=0.

The bar is made of a linear elastic material, with a Young's modulus E and a section S, and the longitudinal waves propagate at speed c.

The equations of dynamic linear elasticity governing the axial displacement $u(x,t)$ of the points inside the bar are,

$$\begin{aligned} &\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x \in]0, L[; \\ &\text{(IC) initial conditions} \quad u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 0\,; \\ &\text{(BC) boundary conditions} \quad u(0,t) = 0; \quad \frac{\partial u}{\partial x}(L,t) = \frac{F_0}{ES}\,\mathcal{H}(t)\,. \end{aligned} \tag{1}$$

Find the displacement $u(L,t)$ of the right boundary and give a vivid interpretation of the phenomenon.

Solution:

The solution is obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{2}$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{3}$$

Therefore,

$$\begin{aligned} &\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\,U(x,p) - p\,\underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0,\ \text{(IC)}}\Big) = 0 \\ &\frac{d^2}{dx^2}U(x,p) = \frac{p^2}{c^2}\,U(x,p)\,. \end{aligned} \tag{4}$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter. Then

$$U(x,p) = a(p)\,\cosh\big(\frac{p}{c}x\big) + b(p)\,\sinh\big(\frac{p}{c}x\big)\,. \tag{5}$$

The two unknown functions of p are defined by the two boundary conditions,

$$
\begin{aligned}
&(\mathrm{BC})_1 \quad U(x,p) = 0 \Rightarrow a(p) = 0 \\
&(\mathrm{BC})_2 \quad \frac{\partial U}{\partial x}(L,p) = \frac{F_0}{ES}\,\frac{1}{p} = b(p)\,\frac{p}{c}\,\cosh\left(\frac{p}{c}L\right) \\
&U(x,p) = c\,\frac{F_0}{ES}\,\frac{1}{p^2}\,\frac{\sinh\left(\frac{p}{c}x\right)}{\cosh\left(\frac{p}{c}L\right)}\,.
\end{aligned}
\tag{6}
$$

Let $T = L/c$ be the time for the longitudinal wave to travel the length of the bar. Then, the displacement of the right boundary $x = L$ is

$$
u(L,t) = \frac{F_0}{ES}\,c\,\mathcal{L}^{-1}\{\frac{1}{p^2}\,\tanh(T\,p)\}(t) = 2\,L\,\frac{F_0}{ES}\,f_3(t)\,,
\tag{7}
$$

where $f_3(t)$ is a periodic function shown on Fig. I.7. Use has been made of the previous exercise with $a = 2\,T$. Therefore the velocity of the boundary $x = L$ reads,

$$
\frac{\partial u}{\partial t}(L,t) = 2\,L\,\frac{F_0}{ES}\,\underbrace{\frac{df(t)}{dt}}_{\pm 1/2T} = \pm c\,\frac{F_0}{ES}\,.
\tag{8}
$$

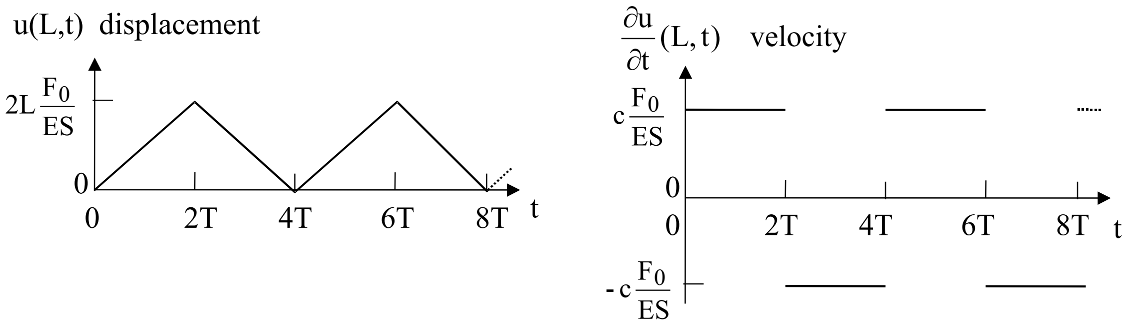

Figure I.7 Following the shock, the ensuing longitudinal wave propagates back and forth, giving rise to a periodic motion with a period $4\,T$ equal to four times the time required for the longitudinal elastic wave to travel the bar.

The interpretation of these longitudinal vibrations is a bit tricky and it goes as follows:

- at time $t = 0$, the right boundary of the bar is hit, and the corresponding signal propagates to the left at speed c;
- it needs a time T to reach the fixed boundary. When the wave comes back, it carries the information that this boundary is fixed;
- this information is known by the right boundary after a time $2\,T$. Since it was not known before, this point was moving to the right, with the positive velocity indicated by (8). Immediately as the information is known, the right boundary stops moving right, and in fact, changes the sign of its velocity, again as indicated by (8);
- the mechanical information "the right boundary is subject to a fixed traction" then travels back to the left, and hits the fixed boundary at time $3\,T$. When the wave comes back, it carries the information that this boundary is fixed. This information reaches the right boundary at time $4\,T$, where in fact its displacement just vanishes;

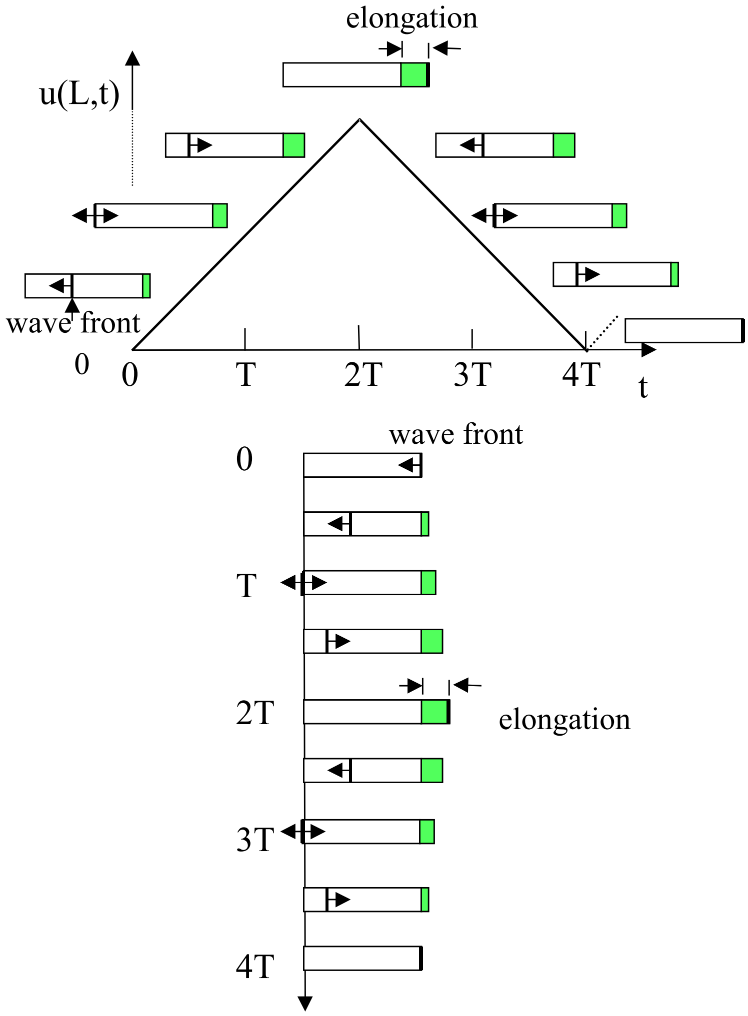

Figure I.8 Two other equivalent illustrations of the back and forth propagation of the wave front in the finite bar with left end fixed and right end subject to a given traction.

- the elongation increases linearly in time from $t = 0$ to reach its maximum at $t = 2T$, and then decreases up to $t = 4T$ where it vanishes;
- this succession of events, with periodicity $4T$, repeats indefinitely.

Figs. I.7 and I.8 display the position of the wave front and elongation at various times within a period.

Exercise I.4: **Transverse vibrations of a beam**.

A beam, of length L, is simply supported at its boundaries $x = 0$ and $x = L$. Consequently , the bending moments, linked to the curvature by the Navier-Bernouilli relation $M = EI\, d^2w/dx^2$, vanish at the boundaries. Here EI is the bending stiffness, and $w(x, t)$ the transverse displacement.

The beam is at rest for times $t < 0$. At time $t = 0$, it is submitted to a transverse shock expressed in terms of the transverse velocity $\partial w/\partial t(x, 0)$.

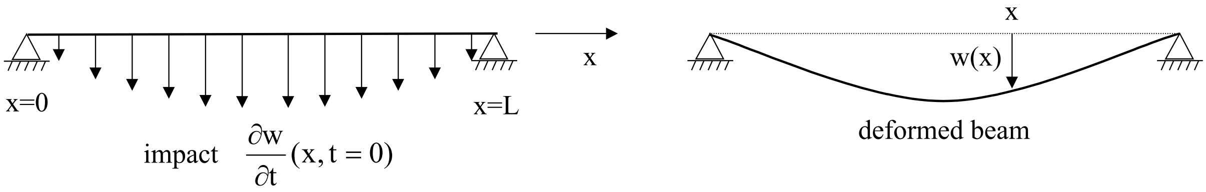

Figure I.9 The supports at the boundaries are bilateral, that is, they prevent up and down vertical motions of the beam.

The equations of dynamic linear elasticity governing the transverse displacement $w(x, t)$ of the beam are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 w}{\partial t^2} + b^2\,\frac{\partial^4 w}{\partial x^4} = 0, \quad t > 0,\ x \in]0, L[;\\
&\text{(IC) initial conditions} \quad w(x, 0) = 0; \quad \frac{\partial w}{\partial t}(x, 0) = V_0 \sin\left(\pi\,\frac{x}{L}\right);\\
&\text{(BC) boundary conditions} \quad w(0, t) = w(L, t) = 0; \quad \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(L, t) = 0\,.
\end{aligned} \tag{1}$$

The coefficient b involved in the field equation is equal to $\sqrt{EI/(\rho\, S)}$, where ρ is the mass density of the material and S the section of the beam, all quantities that we will consider as constants.

The beam remains uncharged. The transverse shock generates transverse vibrations $w(x, t)$. Describe these so-called free vibrations.

Solution:

The solution is obtained through the Laplace transform in time,

$$w(x, t) \to W(x, p) = \mathcal{L}\{w(x, t)\}(p), \tag{2}$$

and we admit that the operators transform and partial derivative in space Laplace commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{w(x, t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x} w(x, t)\}(p)\,. \tag{3}$$

Therefore,

$$\begin{aligned}
\text{(FE)} \quad & b^2\,\frac{\partial^4 W(x, p)}{\partial x^4} + p^2\, W(x, p) - p\,\underbrace{w(x, 0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial w}{\partial t}(x, 0)}_{=V_0 \sin(\pi\, x/L),\ \text{(IC)}} = 0\\
& b^2\,\frac{d^4 W(x, p)}{dx^4} + p^2\, W(x, p) = V_0 \sin(\pi\,\frac{x}{L})\,.
\end{aligned} \tag{4}$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter.

The solution of this nonhomogeneous linear differential equation is equal to the sum of the solution of the homogeneous equation (with zero rhs), and of a particular solution.

The solution of the homogeneous equation is sought in the format $W(w,p) = C(p)\ \exp(\alpha\, x)$, yielding four complex solutions $\alpha = \pm(1 \pm i)\,\beta$, with $\beta(p) = \sqrt{p/(2\,b)}$, and with an priori complex factor $C(p)$. Summing these four solutions, the real part may be rewritten in the format,

$$\begin{aligned} W^{\rm hom}(x,p) &= \Big(c_1(p)\ \cos(\beta\, x) + c_2(p)\ \sin(\beta\, x)\Big)\ \exp(\beta\, x) \\ &+ \Big(c_3(p)\ \cos(\beta\, x) + c_4(p)\ \sin(\beta\, x)\Big)\ \exp(-\beta\, x)\,, \end{aligned} \tag{5}$$

where the $c_i = c_i(p)$, $i = 1-4$, are unknowns to be defined later.

The particular solution is sought in the form of the rhs,

$$W^{\rm par}(x,p) = c_5(p)\ \sin(\pi\,\frac{x}{L}), \quad c_5(p) = \frac{V_0}{p^2 + b^2\,\dfrac{\pi^4}{L^4}}\,. \tag{6}$$

We will need the first two derivatives of the solution,

$$\begin{aligned} W(x,p) &= \Big(c_1(p)\ \cos(\beta\, x) + c_2(p)\ \sin(\beta\, x)\Big)\ \exp(\beta\, x) \\ &+ \Big(c_3(p)\ \cos(\beta\, x) + c_4(p)\ \sin(\beta\, x)\Big)\ \exp(-\beta\, x) \\ &+ c_5(p)\ \sin(\pi\,\frac{x}{L}) \\ \frac{d}{dx}W(x,p) &= \beta\,\Big((c_1(p)+c_2(p))\ \cos(\beta\, x) + (-c_1(p)+c_2(p))\ \sin(\beta\, x)\Big)\ \exp(\beta\, x) \\ &+ \beta\,\Big((-c_3(p)+c_4(p))\ \cos(\beta\, x) - (c_3(p)+c_4(p))\ \sin(\beta\, x)\Big)\ \exp(-\beta\, x) \\ &+ c_5(p)\,\frac{\pi}{L}\ \cos(\pi\,\frac{x}{L}) \\ \frac{d^2}{dx^2}W(x,p) &= 2\,\beta^2\,\Big(c_2(p)\ \cos(\beta\, x) - c_1(p)\ \sin(\beta\, x)\Big)\ \exp(\beta\, x) \\ &+ 2\,\beta^2\,\Big(-c_4(p)\ \cos(\beta\, x) + c_3(p)\ \sin(\beta\, x)\Big)\ \exp(-\beta\, x) \\ &- c_5(p)\,\frac{\pi^2}{L^2}\ \sin(\pi\,\frac{x}{L})\,, \end{aligned} \tag{7}$$

The four boundary conditions are used to obtain the four unknowns,

$$\begin{aligned} & c_1(p) + c_3(p) = 0 \\ & c_2(p) - c_4(p) = 0 \\ & \Big(c_1(p)\ \cos(\beta\, L) + c_2(p)\ \sin(\beta\, L)\Big)\ \exp(\beta\, L) + \Big(c_3(p)\ \cos(\beta\, L) + c_4(p)\ \sin(\beta\, L)\Big)\ \exp(-\beta\, L) = 0 \\ & (c_2(p)\ \cos(\beta\, L) - c_1(p)\ \sin(\beta\, L)\Big)\ \exp(\beta\, L) + (-c_4(p)\ \cos(\beta\, L) + c_3(p)\ \sin(\beta\, L)\Big)\ \exp(-\beta\, L) = 0\,, \end{aligned} \tag{8}$$

yielding $c_1 = -c_3$, $c_2 = c_4$, and a 2×2 linear system for c_1 and c_2,

$$\begin{aligned} \sinh(\beta L)\,\cos(\beta L) \quad c_1(p) \quad &+ \quad \cosh(\beta L)\,\sin(\beta L) \quad c_2(p) \quad = 0, \\ -\cosh(\beta L)\,\sin(\beta L) \quad c_1(p) \quad &+ \quad \sinh(\beta L)\,\cos(\beta L) \quad c_2(p) \quad = 0\,. \end{aligned} \tag{9}$$

The determinant of this system, $4\,(\sinh^2(\beta L) + \sin^2(\beta L))$, does not vanish, and therefore, $c_i = 0$, $i = 1-4$, and finally,

$$W(x,p) = W^{\rm par}(x,p) = \frac{V_0}{p^2 + b^2\,\dfrac{\pi^4}{L^4}}\,\sin(\pi\,\frac{x}{L})\,. \tag{10}$$

Since

$$\mathcal{L}\{\sin(a\,t)\}(p) = \frac{a}{p^2 + a^2}\,, \tag{11}$$

the transverse displacement, and transverse velocity,

$$w(x,t) = V_0\,\frac{L^2}{b\,\pi^2}\,\sin\Big(\frac{b\,\pi^2}{L^2}\,t\Big)\,\sin(\pi\,\frac{x}{L}), \quad \frac{\partial w}{\partial t}(x,t) = V_0\,\cos\Big(\frac{b\,\pi^2}{L^2}\,t\Big)\,\sin(\pi\,\frac{x}{L})\,, \tag{12}$$

are periodic with a frequency,

$$\frac{\pi}{2}\,\frac{b}{L^2}\,, \tag{13}$$

that is inversely proportional to the square of the length of the beam: doubling the length of the beam divides by four its frequency of vibration. On the other hand, the higher the transverse stiffness, the higher the frequency.

Since the solution displays a separation of the space and time variables, all points of the beam vibrate in phase, and the beam keeps its spatial shape for ever.

Exercise I.5: **A finite bar with ends at controlled temperature**.

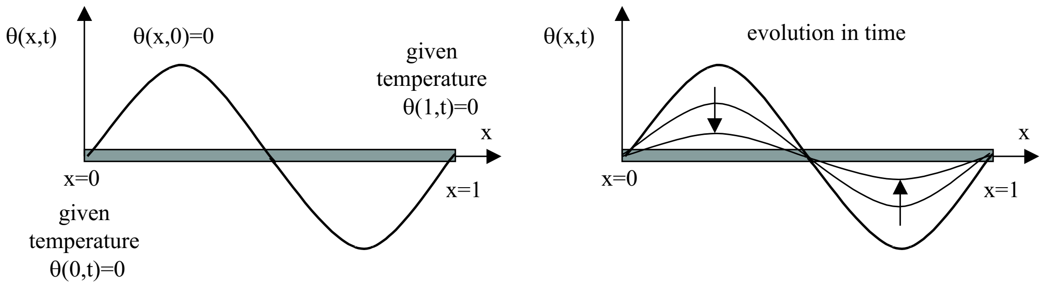

Figure I.10 Heat diffusion in a finite bar subject to given temperature at its ends.

The temperature at the two ends of a finite bar $x \in [0,1]$ is maintained at a given value, say $T(0,t) = T(1,t) = T_0$. The initial temperature along the bar $T(x,0)$=$T(x,0)$ is a function of space, say $T(x,0)$. It is more convenient to work with the field $\theta(x,t) = T(x,t) - T_0$ than with the temperature itself. Moreover, to simplify the notation, scaling of time and space has been used so as to make the thermal conductivity equal to one.

The initial and boundary value problem (IBVP) is thus governed by the following set of equations:

$$\begin{aligned}
&\text{field equation (FE)} \quad \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t} = 0, \quad t > 0,\ x \in]0,1[;\\
&\text{initial condition (IC)} \quad \theta(x,0) = \theta_0 \sin(2\pi x);\\
&\text{boundary conditions (BC)} \quad \theta(0,t) = \theta(1,t) = 0\,.
\end{aligned} \tag{1}$$

Find the temperature $\theta(x,t)$ along the bar at time $t > 0$.

Solution:

The solution is obtained through the Laplace transform

$$\theta(x,t) \to \Theta(x,p) = \mathcal{L}\{\theta(x,t)\}(p)\,. \tag{2}$$

Therefore,

$$\text{(FE)} \ \frac{\partial^2 \Theta(x,p)}{\partial x^2} - \Big(p\,\Theta(x,p) - \underbrace{\theta(x,0)}_{=\theta_0\ \sin(2\pi x),\ \text{(IC)}}\Big) = 0\,. \tag{3}$$

The solution to this nonhomogeneous linear equation is the sum of the solution to the homogeneous equation, and of a particular solution. The latter is sought in the form of the inhomogeneity. Therefore,

$$\Theta^{\text{par}}(x,p) = c(p)\,\sin(2\pi x), \quad c(p) = \frac{\theta_0}{p + 4\pi^2}\,. \tag{4}$$

Therefore,

$$\Theta(x,p) = c_1(p)\,e^{\sqrt{p}\,x} + c_2(p)\,e^{-\sqrt{p}\,x} + \frac{\theta_0}{p + 4\pi^2}\,\sin(2\pi x)\,. \tag{5}$$

The boundary conditions imply the unknowns $c_1(p)$ and $c_2(p)$ to vanish. Therefore

$$\Theta(x,p) = \frac{\theta_0}{p + 4\,\pi^2}\,\sin\,(2\,\pi\,x)\,, \tag{6}$$

and

$$\theta(x,t) = \theta_0\,\sin\,(2\,\pi\,x)\,e^{-4\,\pi^2\,t}\,\mathcal{H}(t)\,. \tag{7}$$

The spatial temperature profile remains identical in time, but its variation with respect to the temperature imposed at the boundaries decreases and ultimately vanishes. In other words, the information imposed at the boundaries penetrates progressively the body.

Since the temperature is imposed at the ends, the heat fluxes $\nabla\theta(x,t)$ at these ends $x = 0, 1$ can be seen as the response of the structure to a constraint.

Exercise I.6: **Relations around the complementary error function** erfc.

1. Show that

$$\mathcal{L}^{-1}(\exp(-a\sqrt{p})(t) = \frac{a}{2\sqrt{\pi\, t^3}}\exp(-\frac{a^2}{4\,t}), \quad a > 0\,, \tag{1}$$

by forming a differential equation.

2. Deduce, using the convolution theorem,

$$\mathcal{L}^{-1}\Big(\frac{1}{p}\exp(-a\sqrt{p})\Big)(t) = \mathrm{erfc}\Big(\frac{a}{2\sqrt{t}}\Big), \quad a > 0\,. \tag{2}$$

Proof:

1.1 The function $\sqrt{p}$ should be made uniform by defining appropriate cuts. One can then calculate the derivatives of $F(p) = \exp(-\sqrt{p})$,

$$F(p) = e^{-\sqrt{p}}, \quad \frac{d}{dp}F(p) = -\frac{e^{-\sqrt{p}}}{2\sqrt{p}}, \quad \frac{d^2}{dp^2}F(p) = \frac{e^{-\sqrt{p}}}{4\,p} + \frac{e^{-\sqrt{p}}}{4\,p\sqrt{p}}\,. \tag{3}$$

Therefore,

$$4\,p\frac{d^2}{dp^2}F(p) + 2\frac{d}{dp}F(p) - F(p) = 0\,. \tag{4}$$

If $f(t)$ has Laplace transform $F(p)$, let us recall the rules,

$$\begin{aligned}
&\mathcal{L}\{t\,f(t)\}(p) = -\frac{d}{dp}F(p),\\
&\mathcal{L}\{t^2 f(t)\}(p) = (-1)^2\frac{d^2}{dp^2}F(p),\\
&\mathcal{L}\{\frac{d}{dt}\Big(t^2 f(t)\Big)\}(p) = p\,\mathcal{L}\{t^2 f(t)\}(p) - \underbrace{\Big(t^2 f(t)\Big)(t=0)}_{=0} = p\frac{d^2}{dp^2}F(p)\,.
\end{aligned} \tag{5}$$

That the second term on the rhs of the last line above is really zero need to be checked, once the solution has been obtained. Collecting these relations, the differential equation in the Laplace domain (4) can be transformed into a differential equation in time,

$$4\frac{d}{dt}\Big(t^2 f(t)\Big) - 2\,t\,f(t) - f(t) = 0\,, \tag{6}$$

which, upon expansion of the first term, becomes,

$$\frac{df}{f} + \Big(\frac{3}{2}\frac{1}{t} - \frac{1}{4}\frac{1}{t^2}\Big)\,dt = 0\,, \tag{7}$$

and thus can be integrated to,

$$f(t) = \frac{c}{t^{3/2}}e^{-1/(4t)}\,. \tag{8}$$

1.2 The constant c can be obtained as follows. We will need a generalized Abel theorem, that we can state as follows:

Assume that, for large t, the two functions $f(t)$ and $g(t)$ are sufficiently close, $f(t) \simeq g(t)$, for $t \gg 1$. Then their Laplace transforms $t \to p$ are also close for small p, namely $F(p) \simeq G(p)$, for $p \sim 0$.

Consider now,

$$t\,f(t) = \frac{c}{t^{1/2}} e^{-1/(4t)}, \quad \mathcal{L}\{t\,f(t)\}(p) = -\frac{d}{dp} F(p) = \frac{e^{-\sqrt{p}}}{2\,\sqrt{p}}\,. \tag{9}$$

Then

$$\begin{aligned} &\text{for } t \gg 1, \quad t\,f(t) \simeq \frac{c}{t^{1/2}} \quad \Rightarrow \quad \mathcal{L}\{t\,f(t)\}(p) \simeq c\,\frac{\Gamma(1/2)}{p^{1/2}}, \\ &\text{for } p \sim 0, \quad \frac{e^{-\sqrt{p}}}{2\,\sqrt{p}} \simeq \frac{1}{2\,\sqrt{p}}\,. \end{aligned} \tag{10}$$

Requiring the transforms in these two lines to be equal as indicated by (9), and by the generalized Abel theorem, yields $c\,\Gamma(1/2) = 1/2$, and therefore $c = 1/(2\sqrt{\pi})$. Therefore

$$\mathcal{L}^{-1}(\exp(-\sqrt{p})(t) = \frac{1}{2\sqrt{\pi\,t^3}} \exp(-\frac{1}{4\,t})\,. \tag{11}$$

The rule, for real $\alpha \geq 0$,

$$\mathcal{L}\{t^\alpha\,\mathcal{H}(t)\}(p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\,, \tag{12}$$

involves the tabulated function Γ, with the properties $\Gamma(n+1) = n!$ for $n \geq 0$ integer and $\Gamma(1/2) = \sqrt{\pi}$.

1.3 Using the rule

$$\mathcal{L}\{f(b\,t)\}(p) = \frac{1}{b}\,\mathcal{L}\{f(t)\}(\frac{p}{b})\,, \tag{13}$$

with $b > 0$ a constant, the result (11) may be generalized to (1), setting $b = 1/a^2$. □

2. Indeed, using the convolution theorem and (1),

$$\begin{aligned} \mathcal{L}^{-1}\Big(\frac{1}{p}\exp(-\sqrt{p}\,a)\Big)(t) &= \int_0^t \mathcal{H}(t-u)\,\frac{a}{2\sqrt{\pi\,u^3}} \exp(-\frac{a^2}{4\,u})\;du \\ &= \frac{2}{\sqrt{\pi}} \int_{a/(2\sqrt{t})}^{\infty} e^{-v^2}\;dv, \quad \text{with} \quad v^2 = a^2/(4\,u)\,. \end{aligned} \tag{14}$$

□

Exercise I.7: **a multipurpose contour integration**.

The purpose is the inversion of the one-sided Laplace transform $Q(x,p)$,

$$Q(x,p) = \frac{\exp(-x\sqrt{p/a})}{p-b}\,, \tag{1}$$

where $a > 0$, $b \geq 0$ are constants, $x \geq 0$ is the space coordinate and p the Laplace variable associated with time t. Show that the inverse reads,

$$q(x,t) = \frac{\exp(b\,t)}{2}\Big(\exp\big(x\sqrt{\frac{b}{a}}\big)\operatorname{erfc}\big(\frac{x}{2\sqrt{at}}+\sqrt{b\,t}\big)+\exp\big(-x\sqrt{\frac{b}{a}}\big)\operatorname{erfc}\big(\frac{x}{2\sqrt{at}}-\sqrt{b\,t}\big)\Big)\,. \tag{2}$$

Proof:

The function is first made uniform by introducing a branch cut along the negative axis $\Re p \leq 0$, and the definitions,

$$p = |p|\,\exp(i\,\theta),\quad \theta \in]-\pi,\pi],\quad \sqrt{p} = \sqrt{|p|}\,\exp(i\,\theta/2)\quad (\Rightarrow \Re\sqrt{p} \geq 0)\,. \tag{3}$$

Assuming known the (inverse) transforms,

$$\mathcal{L}^{-1}\{\exp(-x\sqrt{\frac{p}{a}})\}(u) = \frac{x}{2\sqrt{\pi\,a\,u^3}}\exp(-\frac{x^2}{4\,a\,u}),\quad \mathcal{L}^{-1}\{\frac{1}{p-b}\}(u) = \exp(b\,u)\,\mathcal{H}(u)\,, \tag{4}$$

the function $q(x,t)$ can be written

$$q(x,t) = \exp(b\,t)\,\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{a\,t}}^{\infty}\exp(-v^2-\frac{b\,x^2}{4\,a\,v^2})\,dv\,, \tag{5}$$

since the Laplace transform of a convolution product is equal to the product of the Laplace transforms.

This integral seems out of reach, except if $b = 0$, where

$$q(x,t) = \exp(b\,t)\operatorname{erfc}(\frac{x}{2\sqrt{a\,t}})\,. \tag{6}$$

Still it is provided in explicit form in Abramowitz and Stegun [1964], p. 304, for any b.

Since we want to obtain the result on our own, even for $b \neq 0$, we take a step backward and consider the inversion in the complex plane through an appropriate closed contour $C = C(R,\epsilon)$, that respects the branch cut and includes the pole $p = b$. A preliminary finite contour is shown on Fig. I.11. The residue theorem yields,

$$\begin{aligned}&\frac{1}{2\,i\,\pi}\Big(\int_{c-i\,R}^{c+i\,R}+\int_{C_R}+\int_{C_\epsilon}+\int_{C^+}+\int_{C^-}\Big)\;\exp(tp)\,Q(x,p)\,dp\\ &=\frac{1}{2\,i\,\pi}\oint_C\exp(t\,p)\,Q(x,p)\,dp = \exp(b\,t - x\sqrt{b/a})\,,\end{aligned} \tag{7}$$

where c is an arbitrary real, which is required to be greater than b for a correct definition of the inverse Laplace transform. For vanishingly small radius ϵ, the contour C_ϵ does not contribute as long as $b \neq 0$. If $b = 0$, then the inverse Laplace transfrm is read directly from (5). Moreover,

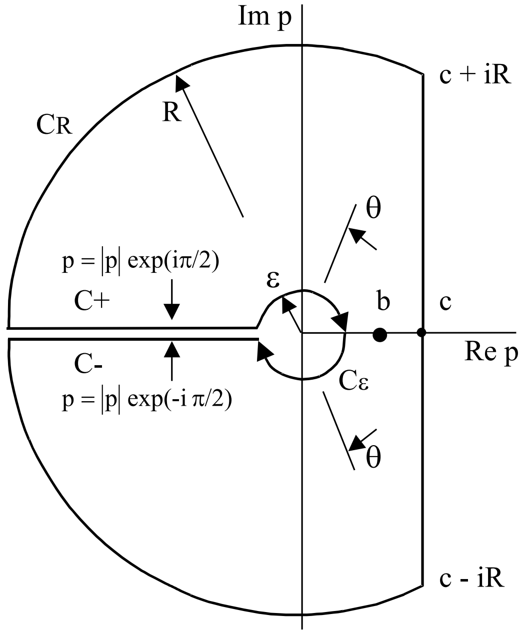

Figure I.11 Contour of integration associated with the integral (7).

since $Q(x,p)$ tends to 0 for large p in view of (3), the contour C_R does not contribute either for large R, in view of Jordan's lemma. Consequently,

$$\begin{aligned} q(x,t) &= \lim_{R\to\infty} \frac{1}{2\,i\,\pi} \int_{c-i\,R}^{c+i\,R} \exp(t\,p)\,Q(x,p)\,dp \\ &= \exp(b\,t - x\,\sqrt{b/a}) - \lim_{R\to\infty,\ \epsilon\to 0} \frac{1}{2\,i\,\pi} \Big(\int_{C+} + \int_{C-} \Big) \exp(t\,p)\,Q(x,p)\,dp\,. \end{aligned} \tag{8}$$

On the branch cut, $p = -r < 0$, but $\sqrt{p}$ is equal to $i\,\sqrt{r}$ on the upper part and to $-i\,\sqrt{r}$ on the lower part. Therefore, the integrals along the branch cut become,

$$\begin{aligned} I &= \frac{1}{2\,i\,\pi} \int_{\infty}^{0} \exp(-t\,r)\,\frac{\exp(-i\,x\,\sqrt{r/a})}{-r-b}\,(-dr) + \frac{1}{2\,i\,\pi} \int_{0}^{\infty} \exp(-t\,r)\,\frac{\exp(i\,x\,\sqrt{r/a})}{-r-b}\,(-dr) \\ &= \frac{1}{2\,i\,\pi} \int_{0}^{\infty} \frac{\exp(-t\,r)}{r+b}\,(\exp(i\,x\,\sqrt{r/a}) - \exp(-i\,x\,\sqrt{r/a}))\,dr \\ &= \frac{1}{i\,\pi} \int_{0}^{\infty} \frac{\exp(-t\,\rho^2)}{\rho^2+b}\,(\exp(i\,x\,\rho/\sqrt{a}) - \exp(-i\,x\,\rho/\sqrt{a}))\,\rho\,d\rho \\ &= \frac{1}{i\,\pi} \int_{-\infty}^{\infty} \exp\big(-t\,\rho^2 + i\,x\,\rho/\sqrt{a}\big)\,\frac{\rho}{\rho^2+b}\,d\rho\,. \end{aligned} \tag{9}$$

In an attempt to see the error function emerging, we make a change of variable that transforms, to within a constant, the argument of the exponential into a square, namely

$$\rho \longrightarrow v = \sqrt{t}\rho - i\,v_0, \quad v_0 \equiv \frac{x}{2\sqrt{at}}\,. \tag{10}$$

Then

$$I = \exp(-(\frac{x}{2\sqrt{at}})^2) \times \frac{1}{2\,i\,\pi} \int_{-\infty-i\,v_0}^{\infty-i\,v_0} \frac{\exp(-v^2)}{v+i\,v^+} + \frac{\exp(-v^2)}{v+i\,v^-}\,dv, \quad v^{\pm} \equiv v_0 \pm \sqrt{b\,t}\,. \tag{11}$$

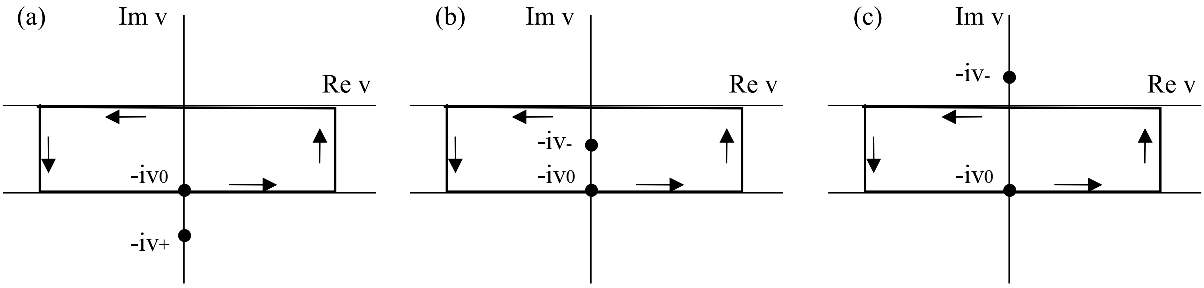

Figure I.12 Contours of integration associated with the integrals (12). Note that (a) v^+ is always larger than $v_0 > 0$, while v^- is (b) either between 0 and v^0 or (c) negative.

First observe that, by an appropriate choice of integration path, Fig. I.12, and application of the residue theorem,

$$\frac{1}{2\,i\,\pi}\int_{-\infty-i\,v_0}^{\infty-i\,v_0}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv = \frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv + \begin{cases} 0 & \text{for } v^\pm = v^+ \text{ or } v^- < 0, \\ \exp((v^-)^2) & \text{for } v^\pm = v^- > 0, \end{cases} \tag{12}$$

we are left with integrals on the real line. The basic idea is to insert the following identity,

$$\int_0^\infty \exp(-u\,X^2)\,du = \frac{1}{X^2}, \quad X \neq 0\,, \tag{13}$$

in the integrals to be estimated and to exchange the order of integration. With this idea in mind, the following straightforward transformations are performed,

$$\begin{aligned}
& \frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv \\
= \;& -\frac{v^\pm}{\pi}\int_0^\infty \frac{\exp(-v^2)}{v^2+(v^\pm)^2}\,dv \quad (\text{use (13) }) \\
= \;& -\frac{v^\pm}{\pi}\int_0^\infty \exp\big(-v^2-u\,(v^2+(v^\pm)^2)\big)\,du\,dv, \quad (du\,dv \to dv\,du, \quad v \to v\,\sqrt{1+u}) \qquad (14) \\
= \;& -\frac{v^\pm}{2\sqrt{\pi}}\int_0^\infty \frac{\exp(-u\,(v^\pm)^2)}{\sqrt{1+u}}\,du \quad (u \to (\sinh\,v)^2, \quad |v^\pm|\,\cosh\,v \to w) \\
= \;& -\frac{\mathrm{sgn}(v^\pm)}{2}\exp\big((v^\pm)^2\big)\,\frac{2}{\sqrt{\pi}}\int_{|v^\pm|}^\infty \exp(-w^2)\,dw\,,
\end{aligned}$$

and therefore

$$\frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv = -\frac{\mathrm{sgn}(v^\pm)}{2}\exp\big((v^\pm)^2\big)\,\mathrm{erfc}(|v^\pm|)\,. \tag{15}$$

Collecting the results (12) and (15), and using the property $(\text{I.3.17})_3$,

$$\frac{1}{2\,i\,\pi}\int_{-\infty-i\,v_0}^{\infty-i\,v_0}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv = \begin{cases} -\frac{1}{2}\exp\big((v^+)^2\big)\,\mathrm{erfc}(v^+) & \text{for } v^\pm = v^+, \\ -\frac{1}{2}\exp\big((v^-)^2\big)\,\mathrm{erfc}(v^-) + \exp\big((v^-)^2\big) & \text{for } v^\pm = v^-, \end{cases} \tag{16}$$

Finally, the inversion formula deduces from (8) and (16).

Chapter II

Solving IBVPs with Fourier transforms

The exponential (complex) Fourier transform is well adapted to solve IBVPs in *infinite* bodies, while the real Fourier transforms are better suited to address IBVPs in *semi-infinite* bodies. The choice between the sine and cosine Fourier transforms will be shown to depend on the boundary conditions.

When it can be obtained, the response to a point load, so called-Green function, is instrumental to build the response to arbitrary loading. [1]

II.1 Exponential Fourier transform for diffusion problems

We consider a diffusion phenomenon in an infinite bar,

$$-\infty \quad \cdots \overline{\overline{\qquad\qquad\qquad}} \cdots \quad +\infty \tag{II.1.1}$$

aligned with the x-axis, and endowed with a diffusion coefficient $D > 0$. For definiteness, the unknown field $u = u(x,t)$ may be interpreted as a temperature. The initial temperature $u = u(x,0)$ is a known function of space. The radiation condition imposes the temperature to vanish at infinity. The governing equations are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial u}{\partial t} - D\,\frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ x \in]-\infty,\infty[;\\
&\text{(IC) initial condition} \quad u(x,0) = h(x), \quad x \in]-\infty,\infty[\,;\\
&\text{(RC) radiation conditions} \quad u(|x| \to \infty, t) = 0\,.
\end{aligned} \tag{II.1.2}$$

To motivate the use of the Fourier transform, we observe that the space variable x varies from $-\infty$ to ∞, and we choose the transform

$$u(x,t) \to U(\alpha,t) = \mathcal{F}\{u(x,t)\}(\alpha)\,, \tag{II.1.3}$$

and we admit that the operators Fourier transform and partial derivative in time commute,

$$\frac{\partial}{\partial t}\mathcal{F}\{u(x,t)\}(\alpha) = \mathcal{F}\{\frac{\partial}{\partial t}u(x,t)\}(\alpha)\,. \tag{II.1.4}$$

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[1] Posted, November 29, 2008; updated, April 03, 2009

The field equation becomes a ODE (ordinary differential equation) for $U(\alpha, t)$ where the Fourier variable α is seen as a parameter,

$$\begin{aligned} &\text{(FE)} \quad \frac{dU}{dt}(\alpha,t) + D\,\alpha^2\,U(\alpha,t) = 0, \quad t > 0, \\ &\text{(CI)} \quad U(\alpha,0) = H(\alpha)\,, \end{aligned} \tag{II.1.5}$$

that solves to

$$U(\alpha,t) = H(\alpha)\,e^{-\alpha^2\,D\,t}\,. \tag{II.1.6}$$

The inverse is

$$\begin{aligned} u(x,t) &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\,\alpha\,x-\alpha^2\,D\,t}\,H(\alpha)\,d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} \cos\,(\alpha(x-\xi))\,e^{-\alpha^2\,D\,t}\,d\alpha\,h(\xi)\,d\xi\,. \end{aligned} \tag{II.1.7}$$

The imaginary part has disappeared due to the fact that the integrand is even with respect to the variable α. This expression may be simplified by the use of a Green function.

II.1.1 The Green function as the solution to a point source at the origin

For a point source at the origin,

$$\text{(IC)} \quad u_\delta(x,0) = h(x) = \delta(x)\,, \tag{II.1.8}$$

the solution u_δ can be expressed in explicit form,

$$u_\delta(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\,(\alpha\,x)\,e^{-\alpha^2\,D\,t}\,d\alpha = \frac{1}{2\sqrt{\pi\,D\,t}}\,\exp\Big(-\frac{x^2}{4\,D\,t}\Big)\,. \tag{II.1.9}$$

The proof of (II.1.9) is detailed in exercise II.6.

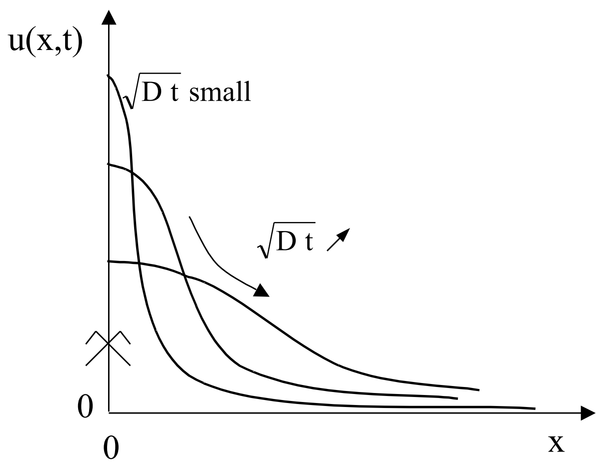


Figure II.1 An infinite bar subjected to a point heat source at the origin. Evolution of the spatial profile in time.

To check that the initial condition (IC) is satisfied, we use the fact, established in the Chapter on distributions, that, in the sense of distributions,

$$\lim_{\epsilon\to 0^+} \frac{1}{2\sqrt{\pi\,\epsilon}} \exp\Big(-\frac{x^2}{4\,\epsilon}\Big) = \delta(x)\,. \tag{II.1.10}$$

The formula (II.1.9) also yields the additional result, for $\kappa > 0$,

$$\mathcal{F}\Big\{\exp\Big(-\frac{x^2}{4\kappa}\Big)\Big\}(\alpha) = 2\sqrt{\pi\,\kappa}\exp(-\kappa\,\alpha^2), \quad \mathcal{F}^{-1}\{\exp(-\kappa\,\alpha^2)\}(x) = \frac{1}{2\sqrt{\pi\,\kappa}}\exp\Big(-\frac{x^2}{4\kappa}\Big)\,. \tag{II.1.11}$$

II.1.2 Point source at an arbitrary location

The solution to a point source $h(x) = \delta(x-\xi)$ at ξ is easily deduced as

$$\frac{1}{2\sqrt{\pi\,D\,t}} \exp\Big(-\frac{(x-\xi)^2}{4\,D\,t}\Big)\,. \tag{II.1.12}$$

II.1.3 Response to an arbitrary source via the Green function

For an arbitrary source, the solution may be built starting from the Green function u_δ. Indeed, since the Fourier transform of $\delta(x)$ is 1,

$$\begin{aligned} U_\delta(\alpha,t) &= e^{-\alpha^2\,D\,t}, \\ U(\alpha,t) &= H(\alpha)\,e^{-\alpha^2\,D\,t}, \end{aligned} \tag{II.1.13}$$

in view of (II.1.6), and therefore,

$$U(\alpha,t) = U_\delta(\alpha,t)\,H(\alpha)\,, \tag{II.1.14}$$

so that the inverse is a convolution product where the Green function appears as the kernel,

$$u(x,t) = \frac{1}{2\sqrt{\pi\,D\,t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-\xi)^2}{4\,D\,t}\Big)\,h(\xi)\,d\xi\,. \tag{II.1.15}$$

II.2 Exponential Fourier transform for the inhomogeneous wave equation

II.3 Sine and cosine Fourier transforms

Fourier transforms in sine and cosine are well adapted tools to solve PDEs over semi-infinite bodies, e.g.

$$0\ \boxed{}\cdots \quad \infty \qquad \text{(II.3.1)}$$

They stem from the integral Fourier theorem,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\int_{-\infty}^{\infty} e^{-i\alpha\xi} f(\xi)\,d\xi\;d\alpha\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\cos(\alpha(x-\xi)) + i\,\overbrace{\sin(\alpha(x-\xi))}^{\text{odd in }\alpha}\Big) f(\xi)\,d\xi\;d\alpha\\
&= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\cos(\alpha(x-\xi))\,f(\xi)\,d\xi\;d\alpha\\
&= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\Big(\cos(\alpha x)\,\cos(\alpha\xi) + \sin(\alpha x)\,\sin(\alpha\xi)\Big) f(\xi)\,d\xi\;d\alpha\\
&= \frac{1}{\pi}\int_{0}^{\infty}\cos(\alpha x)\int_{-\infty}^{\infty}\cos(\alpha\xi)\;f(\xi)\,d\xi\;d\alpha\\
&+ \frac{1}{\pi}\int_{0}^{\infty}\sin(\alpha x)\int_{-\infty}^{\infty}\sin(\alpha\xi)\;f(\xi)\,d\xi\;d\alpha\,.
\end{aligned} \qquad \text{(II.3.2)}$$

II.3.1 Sine Fourier transforms for odd $f(x)$ in $]-\infty,\infty[$

If the function $f(x)$ is odd over $]-\infty,\infty[$, or, if it is initially defined over $[0,\infty[$, and extended to an odd function over $]-\infty,\infty[$, then (II.3.2) yields,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin(\alpha x)\int_0^{\infty}\sin(\alpha\xi)\;f(\xi)\,d\xi\;d\alpha\,, \qquad \text{(II.3.3)}$$

expression which motivates the definition of a sine transform, and by the same token, of its inverse,

$$F_S(\alpha) = \int_0^{\infty}\sin(\alpha x)\,f(x)\,dx, \quad f(x) = \frac{2}{\pi}\int_0^{\infty}\sin(\alpha x)\;F_S(\alpha)\,d\alpha\,, \qquad \text{(II.3.4)}$$

II.3.2 Cosine Fourier transforms for even $f(x)$ in $]-\infty,\infty[$

If the function $f(x)$ is even over $]-\infty,\infty[$, or, if it is initially defined over $[0,\infty[$, and extended to an even function over $]-\infty,\infty[$, then (II.3.2) yields,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos(\alpha x)\int_0^{\infty}\cos(\alpha\xi)\;f(\xi)\,d\xi\;d\alpha\,, \qquad \text{(II.3.5)}$$

expression which motivates the definition of a cosine transform, and by the same token, of its inverse,

$$F_C(\alpha) = \int_0^{\infty}\cos(\alpha x)\,f(x)\,dx, \quad f(x) = \frac{2}{\pi}\int_0^{\infty}\cos(\alpha x)\;F_C(\alpha)\,d\alpha\,. \qquad \text{(II.3.6)}$$

Which of these two transforms is more appropriate to solve PDEs over semi-infinite bodies? The answer depends on the boundary conditions, as will be seen now.

II.3.3 Rules for derivatives

The following transforms of a derivative are easily established by simple integration by parts, accounting that $f(x)$ tends to 0 at $\pm\infty$,

$$\begin{aligned}
\mathcal{F}_C\Big\{\frac{\partial f(x)}{\partial x}\Big\}(\alpha) &= \alpha\,\mathcal{F}_S\{f(x)\}(\alpha) - f(0),\\
\mathcal{F}_S\Big\{\frac{\partial f(x)}{\partial x}\Big\}(\alpha) &= -\alpha\,\mathcal{F}_C\{f(x)\}(\alpha),\\
\mathcal{F}_C\Big\{\frac{\partial^2 f(x)}{\partial x^2}\Big\}(\alpha) &= -\alpha^2\,\mathcal{F}_C\{f(x)\}(\alpha) - \frac{\partial f}{\partial x}(0),\\
\mathcal{F}_S\Big\{\frac{\partial^2 f(x)}{\partial x^2}\Big\}(\alpha) &= -\alpha^2\,\mathcal{F}_S\{f(x)\}(\alpha) + \alpha\, f(0)\,.
\end{aligned} \tag{II.3.7}$$

Therefore, since diffusion equations involve a second order derivative wrt space,

- a boundary condition at $x = 0$ in $\partial f(0)/\partial x$ is accounted for by the cosine transform;
- a boundary condition at $x = 0$ in $f(0)$ is accounted for by the sine transform.

II.4 Sine and cosine Fourier transforms to solve IBVPs in semi-infinite bodies

We consider a diffusion phenomenon in a semi-infinite bar, aligned with the x-axis, and endowed with a diffusion coefficient $D > 0$. For definiteness, the unknown field may be interpreted as a temperature. The initial temperature is a known function of space. The radiation condition imposes the temperature to vanish at infinity. The governing equations

$$\begin{aligned}
&\text{field equation (FE)} \quad \frac{\partial u}{\partial t} - D\,\frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ x > 0;\\
&\text{initial condition (IC)} \quad u(x,0) = h(x), \quad 0 \leq x < \infty\,;\\
&\text{radiation condition (RC)} \quad u(x \to \infty, t) = 0\,,
\end{aligned} \tag{II.4.1}$$

will be completed by two types of boundary conditions.

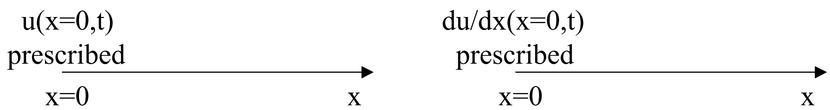


Figure II.2 The boundary conditions considered consist in prescribing either the primary unknown or its space derivative, interpreted as a flux.

II.4.1 Prescribed flux at $x = 0$

We first consider a situation where the flux is prescribed,

$$\text{boundary condition (BC)} \quad \frac{\partial u}{\partial x}(x = 0, t) = f(t), \quad t > 0\,. \tag{II.4.2}$$

The solution,

$$u(x,t) = \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\, h(y)\, dy - \sqrt{\frac{D}{\pi}} \int_0^t \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big) \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau\,, \tag{II.4.3}$$

where the function $h(y>0)$ has been extended to $y<0$ as an **even** function, clearly evidences the contributions of the initial condition and boundary condition. Incidentally, note that these contributions simply sum, since the problem is linear.

Proof:

The proof is a bit lengthy and tedious, but otherwise straightforward. Since the boundary condition prescribes a flux, the problem is solved through the cosine Fourier transform,

$$u(x,t) \to U_c(\alpha,t) = \mathcal{F}_C\{u(x,t)\}(\alpha,t)\,. \tag{II.4.4}$$

Therefore,

$$\begin{aligned} &(\text{FE})\ \frac{\partial U_C}{\partial t}(\alpha,t) + D\,\alpha^2\, U_C(\alpha,t) = -D\,\frac{\partial u(0,t)}{\partial x} \overset{(\text{BC})}{=} -D\, f(t)\,, \quad t>0,\\ &(\text{CI})\ U_C(\alpha,t) = H_C(\alpha)\,, \end{aligned} \tag{II.4.5}$$

which, switching to total derivative in time with the Fourier variable being seen as a parameter, is easily integrated to

$$\begin{aligned} &e^{-\alpha^2\,D\,t}\,\frac{d}{dt}(e^{\alpha^2\,D\,t}\,U_C(\alpha,t)) = -D\,f(t)\,, \quad t>0,\\ &U_C(\alpha,t) = H_C(\alpha)\,e^{-\alpha^2\,D\,t} - D\int_0^t e^{-D\alpha^2\,(t-\tau)}\, f(\tau)\,d\tau\,, \end{aligned} \tag{II.4.6}$$

The inverse is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \cos(\alpha\,x)\, H_C(\alpha)\, e^{-\alpha^2\,D\,t}\, d\alpha - \frac{2}{\pi}\,D\int_0^\infty \cos(\alpha\,x) \int_0^t e^{-D\alpha^2\,(t-\tau)}\, f(\tau)\, d\tau\, d\alpha\,. \tag{II.4.7}$$

The first term can be manipulated as follows,

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty \cos(\alpha\,x)\, H_C(\alpha)\, e^{-\alpha^2\,D\,t}\, d\alpha\\ &= \frac{2}{\pi}\int_0^\infty \cos(\alpha\,x)\, e^{-\alpha^2\,D\,t} \int_0^\infty \cos(\alpha\,y)\, h(y)\, dy\, d\alpha\\ &= \int_0^\infty h(y)\,\frac{1}{\pi}\int_0^\infty e^{-\alpha^2\,D\,t}\Big(\cos\,(\alpha\,(y-x)) + \cos\,(\alpha\,(y+x))\Big)\, d\alpha\, dy\\ &= \frac{1}{2\sqrt{\pi\,D\,t}}\int_0^\infty h(y)\Big(\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big) + \exp\Big(-\frac{(y+x)^2}{4\,D\,t}\Big)\Big)\, dy \quad \text{using } (II.1.9)\\ &= \frac{1}{2\sqrt{\pi\,D\,t}}\Big(\int_0^\infty h(y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\, dy + \overbrace{\int_{-\infty}^0 h(-y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\, dy}^{y\to -y}\Big),\\ &= \frac{1}{2\sqrt{\pi\,D\,t}}\int_{-\infty}^\infty h(y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\, dy\,, \end{aligned} \tag{II.4.8}$$

where the function $h(y>0)$ has been extended to $y<0$ as an even function.

The second term can be transformed as well,

$$\begin{aligned} & -\frac{2}{\pi} D \int_0^\infty \cos(\alpha\, x) \int_0^t e^{-D\alpha^2\,(t-\tau)}\, f(\tau)\, d\tau\, d\alpha \\ & = -\sqrt{\frac{D}{\pi}} \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}} \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big)\, d\tau\,, \end{aligned} \tag{II.4.9}$$

using (II.1.9). □

II.4.2 Prescribed temperature at $x=0$

We consider now a situation where the temperature is prescribed,

$$\text{boundary condition (BC)} \quad u(x=0,t) = g(t), \quad t>0\,. \tag{II.4.10}$$

The solution,

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\, h(y)\, dy \\ &+ \frac{x}{2\sqrt{D\,\pi}} \int_0^t \exp\Big(-\frac{x^2}{4\,D\,(t-\tau)}\Big) \frac{g(\tau)}{(t-\tau)^{3/2}}\, d\tau\,, \end{aligned} \tag{II.4.11}$$

may also be written in a format that highlights a relation with the previous problem,

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\, h(y)\, dy \\ &+ \Big(\frac{\partial}{\partial x}\Big)\Big(-\sqrt{\frac{D}{\pi}} \int_0^t \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big) \frac{g(\tau)}{\sqrt{t-\tau}}\, d\tau\Big)\,, \end{aligned} \tag{II.4.12}$$

where the function $h(y>0)$ has been extended to $y<0$ as an **odd** function.

Proof:

Since the boundary condition prescribes the primary unknown, the problem is solved through the sine Fourier transform,

$$u(x,t) \to U_S(\alpha,t) = \mathcal{F}_S\{u(x,t)\}(\alpha,t)\,. \tag{II.4.13}$$

Therefore,

$$\begin{aligned} & \text{(FE)}\ \frac{\partial U_S}{\partial t}(\alpha,t) + D\,\alpha^2\, U_S(\alpha,t) = D\,\alpha\, u(0,t) \overset{\text{(BC)}}{=} D\,\alpha\, g(t)\,, \quad t>0, \\ & \text{(CI)}\ U_S(\alpha,t) = H_S(\alpha)\,, \end{aligned} \tag{II.4.14}$$

which, switching to total derivative in time with the Fourier variable being seen as a parameter, is easily integrated to

$$U_S(\alpha,t) = H_S(\alpha)\, e^{-\alpha^2\, D\, t} + D\,\alpha \int_0^t e^{-D\alpha^2\,(t-\tau)}\, g(\tau)\, d\tau\,, \quad t>0\,. \tag{II.4.15}$$

The inverse is

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \sin(\alpha\, x)\, H_S(\alpha)\, e^{-\alpha^2 D\, t}\, d\alpha + \frac{2}{\pi}\, D \int_0^\infty \sin(\alpha\, x) \int_0^t e^{-D\alpha^2 (t-\tau)}\, g(\tau)\, d\tau\, \alpha\, d\alpha\,. \tag{II.4.16}$$

The first term can be manipulated as follows,

$$\begin{aligned}
&\frac{2}{\pi} \int_0^\infty \sin(\alpha\, x)\, H_S(\alpha)\, e^{-\alpha^2 D\, t}\, d\alpha \\
&= \int_0^\infty h(y)\, \frac{1}{\pi} \int_0^\infty e^{-\alpha^2 D\, t} \Big(-\cos\,(\alpha\,(y+x)) + \cos\,(\alpha\,(y-x)) \Big)\, d\alpha\, dy \\
&= \frac{1}{2\sqrt{\pi\, D\, t}} \Big(\int_0^\infty h(y)\, \exp\Big(-\frac{(y-x)^2}{4\, D\, t} \Big)\, dy - \int_{-\infty}^0 h(-y)\, \exp\Big(-\frac{(y-x)^2}{4\, D\, t} \Big)\, dy \Big) \\
&= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^\infty h(y)\, \exp\Big(-\frac{(y-x)^2}{4\, D\, t} \Big)\, dy\,,
\end{aligned} \tag{II.4.17}$$

where the function $h(y>0)$ has been extended to $y<0$ as an odd function. Use has been made of (II.1.9) to go from the 2nd line to the 3rd line.

The second term can be transformed as well,

$$\begin{aligned}
&\frac{2}{\pi}\, D \int_0^\infty \sin(\alpha\, x) \int_0^t e^{-D\alpha^2 (t-\tau)}\, g(\tau)\, d\tau\, \alpha\, d\alpha \\
&= \frac{2}{\pi}\, D \int_0^t g(\tau) \Big(-\frac{\partial}{\partial x} \Big) \Big(\int_0^\infty \cos(\alpha\, x)\, e^{-D\alpha^2 (t-\tau)}\, d\alpha \Big)\, d\tau \\
&= \Big(\frac{\partial}{\partial x} \Big) \Big(-\sqrt{\frac{D}{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\, \exp\Big(\frac{-x^2}{4\, D\,(t-\tau)} \Big)\, d\tau \Big) \\
&= \frac{x}{2\,\sqrt{\pi\, D}} \int_0^t \frac{g(\tau)}{(t-\tau)^{3/2}}\, \exp\Big(-\frac{x^2}{4\, D\,(t-\tau)} \Big)\, d\tau\,.
\end{aligned} \tag{II.4.18}$$

Use has been made of (II.1.9) to go from the 2nd line to the 3rd line. The latter line indicates that the second term in this section could have been guessed by applying the operator $\partial/\partial x$ to the result of the previous section. □

Particular case:

$$\begin{aligned}
&\text{initial condition (IC)} \quad u(x,0) = h(x) = 0, \quad 0 \le x < \infty\,; \\
&\text{boundary condition (BC)} \quad u(x=0,t) = g(t) = u_0, \quad t > 0\,.
\end{aligned} \tag{II.4.19}$$

With the change of variable,

$$\tau \;\rightarrow \eta = \frac{x}{2\,\sqrt{D\,(t-\tau)}} \quad \Rightarrow \quad \frac{\partial \eta}{\partial \tau} = \frac{x}{2\,\sqrt{D}\,(t-\tau)^{3/2}}\,, \tag{II.4.20}$$

the solution (II.4.11) becomes

$$u(x,t) = u_0\, \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{D\, t}}^\infty e^{-\eta^2}\, d\eta = u_0\, \mathrm{erfc}\Big(\frac{x}{2\sqrt{D\, t}} \Big)\,. \tag{II.4.21}$$

Note the derivative

$$\frac{\partial u(x,t)}{\partial x} = \frac{-u_0}{\sqrt{\pi\, D\, t}}\, \exp\Big(-\frac{x^2}{4\, D\, t} \Big)\,. \tag{II.4.22}$$

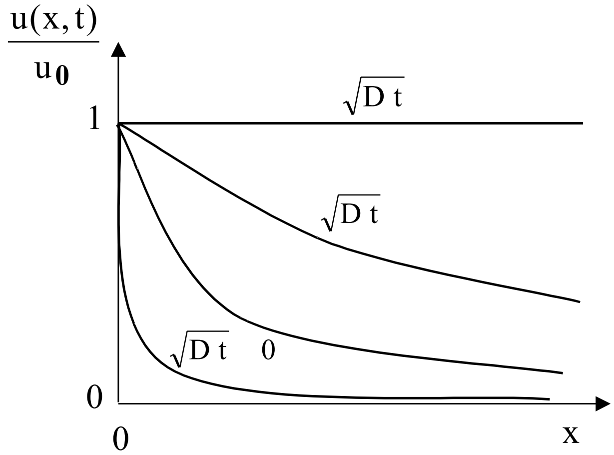


Figure II.3 A semi-infinite bar is subject to a heat shock at its boundary $x = 0$ at time $t = 0$. Spatial profile of the temperature at various times.

II.5 Two general algebraic relations with physical relevance

The following algebraic relations have far-reaching consequences in mathematical physics. Still, we will be content to show a mere academic application [2].

II.5.1 Parseval identities

Parseval identities indicate that the scalar products in the original and transformed spaces are conserved.

Under suitable conditions for the real functions $f(x)$ and $g(x)$, and with the standard notation for their complex transforms $F(\alpha)$ and $G(\alpha)$, cosine transforms $F_C(\alpha)$ and $G_C(\alpha)$, and sine transforms $F_S(\alpha)$ and $G_S(\alpha)$, respectively, Parseval identities can be cast in the formats,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\, g(x)\, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, \overline{G}(\alpha)\, d\alpha \\
\int_{0}^{\infty} f(x)\, g(x)\, dx &= \frac{2}{\pi} \int_{0}^{\infty} F_C(\alpha)\, G_C(\alpha)\, d\alpha \\
\int_{0}^{\infty} f(x)\, g(x)\, dx &= \frac{2}{\pi} \int_{0}^{\infty} F_S(\alpha)\, G_S(\alpha)\, d\alpha
\end{aligned} \tag{II.5.1}$$

When $f = g$, Parseval relations can be interpreted as indicating that

energy is invariant under Fourier transforms

We may offer a simple algebraic consequence of these relations, to calculate a 'difficult' integral from a simple one. For example, with the preliminary transform, with $a > 0$,

$$f(x) = e^{-a,x}\, \mathcal{H}(x) \to F(\alpha) = \frac{1}{a + i\,\alpha}\,, \tag{II.5.2}$$

[2] This section should be part of the basics of Fourier analysis. It is here because it uses the three Fourier transforms at once. Please skip the section for now.

results

$$\int_{-\infty}^{\infty} \frac{1}{|a+i\,\alpha|^2}\,d\alpha = 2\,\pi \int_0^{\infty} e^{-2ax}\,d\alpha = \frac{\pi}{a}\,, \tag{II.5.3}$$

which can be easily checked,

$$\int_{-\infty}^{\infty} \frac{1}{a^2+\alpha^2}\,d\alpha = \frac{1}{a}\left[\tan^{-1}\left(\frac{\alpha}{a}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{a}\ ! \tag{II.5.4}$$

Similarly, using the real Fourier transforms of the very same function, Exercise II.4,

$$\mathcal{F}_C\{e^{-a\,x}\}(\alpha) = \frac{a}{a^2+\alpha^2}\,, \quad \mathcal{F}_S\{e^{-a\,x}\}(\alpha) = \frac{\alpha}{a^2+\alpha^2}\,, \tag{II.5.5}$$

we can estimate two further ‘difficult’ integrals,

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{a^2+\alpha^2}\right)^2 d\alpha &= \frac{\pi}{2a^2}\int_0^{\infty} \left(e^{-ax}\right)^2 d\alpha = \frac{\pi}{4\,a^3} \\ \int_0^{\infty} \left(\frac{\alpha}{a^2+\alpha^2}\right)^2 d\alpha &= \frac{\pi}{2}\int_0^{\infty} \left(e^{-ax}\right)^2 d\alpha = \frac{\pi}{4\,a}\,. \end{aligned} \tag{II.5.6}$$

Note that, as they should, the two relations (II.5.6) associated with the cosine (real part) and sine (imaginary part) Fourier transforms, imply (II.5.4), associated with the complex Fourier transform.

II.5.2 Heisenberg uncertainty principle

The basic idea here is that

the smaller the support of a function,
the larger the support of its Fourier transforms,
and conversely

Perhaps, the most conspicuous example is the Dirac distribution. This property is coined ‘Heisenberg uncertainty principle’ as an analogy to the fact that one can not estimate with the same accuracy position and momentum of a particle. Improving on one side implies worsening on the other side.

For a function $f(x)$ with Fourier transform $F(\alpha)$, the relation is given the following algebraic expression,

$$W_x\,W_\alpha \geq 1\,, \tag{II.5.7}$$

with

$$W_x = 2\,\frac{\displaystyle\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx}{\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\,dx}, \quad W_\alpha = 2\,\frac{\displaystyle\int_{-\infty}^{\infty} \alpha^2\,|F(\alpha)|^2\,d\alpha}{\displaystyle\int_{-\infty}^{\infty} |F(\alpha)|^2\,d\alpha}\,. \tag{II.5.8}$$

The proof goes as follows. It starts from Cauchy-Schwarz inequality,

$$X^2 \equiv |\int_{-\infty}^{\infty} x\,f(x)\,f'(x)\,dx|^2 \leq \int_{-\infty}^{\infty} |x\,f(x)|^2\,dx \int_{-\infty}^{\infty} |f'(x)|^2\,dx\,. \tag{II.5.9}$$

Now, by integration by part, assuming $f(x)$ to be decrease sufficiently at infinity, and using Parseval relation,

$$\begin{aligned} X = \int_{-\infty}^{\infty} x\, f(x)\, f'(x)\, dx &= -\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx + \frac{1}{2}\overbrace{\Big[x\, f(x)^2\Big]_{-\infty}^{\infty}}^{=0} \\ &= \overbrace{-\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx}^{A} \\ &\overset{\text{Parseval}}{=} \overbrace{-\frac{1}{4\,\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha}^{B}\, . \end{aligned} \tag{II.5.10}$$

Using again Parseval relation, and the rule of the transform of a derivative,

$$\begin{aligned} \int_{-\infty}^{\infty} |f'(x)|^2\, dx &\overset{\text{Parseval}}{=} \frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\mathcal{F}\{f'(x)\}(\alpha)|^2\, d\alpha \\ &\overset{f'(x)\to i\alpha\, F(\alpha)}{=} \overbrace{\frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\alpha\, F(\alpha)|^2\, d\alpha}^{C}\, . \end{aligned} \tag{II.5.11}$$

The inequality is finally deduced by inserting the previous relations into (II.5.9),

$$\begin{aligned} X^2 &= \overbrace{\Big(-\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx\Big)}^{A}\ \overbrace{\Big(-\frac{1}{4\,\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha\Big)}^{B} \\ &\overset{\text{(II.5.9)}}{\leq} \int_{-\infty}^{\infty} |x\, f(x)|^2\, dx\ \overbrace{\frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\alpha\, F(\alpha)|^2\, d\alpha}^{C}\, . \end{aligned} \tag{II.5.12}$$

□

II.6 Some basic information on plane strain elasticity

This section serves as a brief introduction to Exercice 5, that addresses a problem of plane strain elasticity in the half-plane.

Let me list first the basic equations of plane strain elasticity, before introducing the Airy stress function.

Static equilibrium with vanishing body forces

Static equilibrium expresses in terms of the Cauchy stress $\boldsymbol{\sigma}$ with components σ_{xx}, σ_{yy}, and $\sigma_{xy} = \sigma_{yx}$, in the cartesian axes (x, y). Cauchy stress satisfies, at each point inside the body Ω, the field equation $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$, namely componentwise,

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{II.6.1}$$

Plane strain in the plane (x, y)

The infinitesimal strain $\boldsymbol{\epsilon}$ with components ϵ_{xx}, ϵ_{yy}, and $\epsilon_{xy} = \epsilon_{yx}$, in the cartesian axes (x, y), expresses in terms of the displacement field $\mathbf{u} = (u_x, u_y)$, namely $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, or componentwise,

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad 2\,\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{II.6.2}$$

Compatibility condition

Since there are only two displacement components, and three strain components, the latter obey a relation,

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}. \tag{II.6.3}$$

Plane strain elasticity

For a compressible isotropic elastic body, the strain and stress tensors are linked by a one-to-one relation, phrased in terms of the Young's modulus $E > 0$, and Poisson's ratio $\nu \in]-1, 1/2[$,

$$\begin{aligned} E\,\epsilon_{xx} &= \sigma_{xx} - \nu\,(\sigma_{yy} + \sigma_{zz}) \\ E\,\epsilon_{yy} &= \sigma_{yy} - \nu\,(\sigma_{xx} + \sigma_{zz}) \\ 2\,E\,\epsilon_{xy} &= 2\,(1+\nu)\,\sigma_{xy}\,. \end{aligned} \tag{II.6.4}$$

Since the out-of-plane strain component ϵ_{zz} vanishes,

$$E\,\epsilon_{zz} = \sigma_{zz} - \nu\,(\sigma_{xx} + \sigma_{yy}) = 0\,, \tag{II.6.5}$$

the out-of-plane stress component σ_{zz} does *not* vanish. Substituting the resulting value in the constitutive equations (II.6.4), and inserting the strain components in the compatibility relation (II.6.3) yields the field equation,

$$\Delta(\sigma_{xx} + \sigma_{yy}) = 0\,. \tag{II.6.6}$$

Airy stress function

The static equilibrium is automatically satisfied if the stress components are expressed in terms of the Airy stress function $\phi(x, y)$,

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{II.6.7}$$

The sole condition to be satisfied is the compatibility condition (II.6.6), which in fact becomes a **biharmonic equation** for the Airy stress function,

$$\Delta\Delta\phi(x, y) = 0, \quad (x, y) \in \Omega\,. \tag{II.6.8}$$

Exercise II.1: **drainage of an infinite porous medium, an elliptic PDE**.

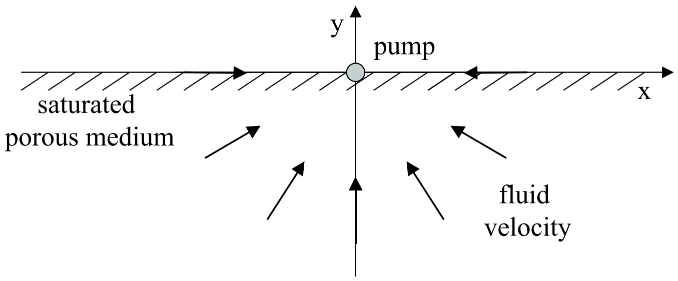


Figure II.4 A half-space is drained by a linear pump located on the free surface.

The lower half-space $y < 0$ is constituted by a porous medium, which is saturated by water. Seepage is induced by a rectilinear drain, aligned with the axis z, that pumps water at a given constant flow rate Q.

The issue is to derive the velocity $\mathbf{v}$ of the fluid in the lower half-space. Since the drain is infinite in the z-direction, the velocity does not depend on the out-of-plane coordinate z.

Water is assumed to be incompressible, so that its velocity $\mathbf{v}$ is divergence free,

$$\operatorname{div}\mathbf{v} = 0\,. \tag{1}$$

Seepage is governed by Darcy law that relates the fluid velocity to the gradient of a scalar potential ϕ,

$$\mathbf{v} = \nabla\phi\,, \tag{2}$$

which is contributed by the fluid pressure and the potential energy. The coordinates have been scaled so as to absorb the hydraulic conductivity.

The equations governing the seepage problem are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \operatorname{div}\nabla\phi = \Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad x \in]-\infty,\infty[,\ y<0;\\
&\text{(BC)}_1 \text{ boundary condition} \quad \mathbf{v} = \nabla\phi \to 0, \quad \text{as} \quad x^2+y^2 \to \infty\,;\\
&\text{(BC)}_2 \text{ boundary condition} \quad v_y = \frac{\partial\phi}{\partial y}(x,0) = Q\,\delta(x)\,.
\end{aligned} \tag{3}$$

Derive the potential $\phi(x,y)$ and show that the fluid velocity is purely radial. Check that the solution satisfies the boundary conditions.

Solution:

Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \to \alpha$,

$$\phi(x,y) \to \Phi(\alpha,y) = \mathcal{F}\{\phi(x,y)\}(\alpha)\,, \tag{4}$$

and we admit that the operators Fourier transform and partial derivative wrt y commute,

$$\frac{\partial}{\partial y}\mathcal{F}\{\phi(x,y)\}(\alpha) = \mathcal{F}\{\frac{\partial}{\partial y}\phi(x,y)\}(\alpha)\,. \tag{5}$$

The transforms of the field equation and boundary conditions,

$$\begin{aligned}
&(\text{FE}) \ \frac{\partial^2\Phi}{\partial y^2}(\alpha,y) - |\alpha|^2\,\Phi(\alpha,y) = 0, \quad y<0,\\
&(\text{BC})_1 \ \frac{\partial\Phi}{\partial y}(\alpha, y\to\infty) = 0,\\
&(\text{BC})_2 \ \frac{\partial\Phi}{\partial y}(\alpha, y=0) = Q\,,
\end{aligned} \tag{6}$$

solve to

$$\Phi(\alpha,y) = Q\,\frac{e^{|\alpha|\,y}}{|\alpha|}\,. \tag{7}$$

Note the trick used to introduce the absolute value in $(6)_1$.

The inverse Fourier transform can be easily manipulated,

$$\begin{aligned}
\phi(x,y) &= \frac{Q}{2\,\pi}\int_{-\infty}^{+\infty}\frac{e^{i\alpha\,x+|\alpha|\,y}}{|\alpha|}\,d\alpha\\
&= \frac{Q}{2\,\pi}\Big(\int_{-\infty}^{0}\frac{e^{i\alpha\,x-\alpha\,y}}{(-\alpha)}\,d\alpha + \int_{0}^{+\infty}\frac{e^{i\alpha\,x+\alpha\,y}}{\alpha}\,d\alpha\Big)\\
&= \frac{Q}{\pi}\int_{0}^{+\infty}\frac{e^{\alpha\,y}}{\alpha}\,\cos\,(\alpha\,x)\;d\alpha\,.
\end{aligned} \tag{8}$$

The latter integral can be estimated by first taking the derivative wrt y,

$$\begin{aligned}
\frac{\partial\phi}{\partial y}(x,y) &= \frac{Q}{\pi}\int_{0}^{+\infty} e^{\alpha\,y}\,\cos\,(\alpha\,x)\;d\alpha\\
&= \frac{Q}{2\,\pi}\int_{0}^{+\infty}(e^{i\,\alpha\,x+\alpha\,y}+e^{-i\,\alpha\,x+\alpha\,y})\,d\alpha\\
&= -\frac{Q}{2\,\pi}\Big(\frac{1}{i\,x+y}+\frac{1}{-i\,x+y}\Big)\\
&= -\frac{Q}{\pi}\,\frac{y}{r^2},
\end{aligned} \tag{9}$$

with $r^2=x^2+y^2$. Integration wrt y yields finally

$$\phi(x,y) = -\frac{Q}{\pi}\,\text{Ln}\,r, \tag{10}$$

to within a constant, and

$$v_r = \frac{\partial\phi}{\partial r} = -\frac{Q}{\pi}\,\frac{1}{r}\,. \tag{11}$$

Note that, for pumping $Q>0$, the velocity vector points to the pump as expected. Moreover, it has been shown in the Chapter devoted to distributions that, in the sense of distributions,

$$\lim_{\epsilon\to 0^+}\,\frac{1}{\pi}\,\frac{\epsilon}{x^2+\epsilon^2} = \delta(x)\,, \tag{12}$$

Therefore, setting here $\epsilon=-y>0$, the solution (9) is seen to satisfy the boundary condition $(\text{BC})_2$.

Exercise II.2: **potential flow in the upper half-plane**.

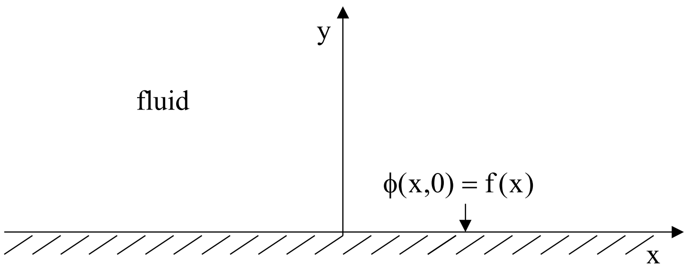


Figure II.5 The flow potential in the upper half-plane is prescribed along the x-axis.

A fluid fills the upper half-plane $y > 0$. The equations governing the flow potential of the fluid are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad x \in]-\infty, \infty[,\ y > 0; \\
&\text{(BC) boundary condition} \quad \phi(x, y = 0) = f(x), \quad x \in]-\infty, \infty[\,; \\
&\text{(RC) radiation condition} \quad \phi(x, y) \text{ bounded}, \quad x^2 + y^2 \to \infty\,.
\end{aligned} \tag{1}$$

Derive the flow potential, sometimes referred to as a Poisson's formula for the half-plane,

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)}{y^2 + (x - \xi)^2}\, d\xi\,. \tag{2}$$

A sort of inverse problem is proposed in Exercise II.4: given the potential $\phi(x, y)$, the issue is to identify the boundary data $f(x)$.

Solution:

Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \to \alpha$, and the operators Fourier transform and partial derivative wrt y are assumed to commute.

The transforms of the field equation and boundary conditions take the form,

$$\begin{aligned}
&\text{(FE)} \ \frac{\partial^2 \Phi}{\partial y^2}(\alpha, y) - |\alpha|^2\, \Phi(\alpha, y) = 0, \quad y > 0, \\
&\text{(BC)} \ \Phi(\alpha, y = 0) = F(\alpha) \\
&\text{(RC)} \ \Phi(\alpha, y) \text{ bounded } \forall y \geq 0\,.
\end{aligned} \tag{3}$$

Instead of using directly the radiation condition, the Fourier transform itself is required to remain bounded at infinity.

The problem solves to

$$\Phi(\alpha, y) = e^{-|\alpha|\, y}\, F(\alpha)\,. \tag{4}$$

Note the trick used to introduce the absolute value in $(3)_1$.

The inverse Fourier transform can be easily manipulated,

$$\begin{aligned}
\phi(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha\, x-|\alpha|\, y}\int_{-\infty}^{+\infty} e^{-i\alpha\,\xi}\, f(\xi)\, d\xi\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\Big(\int_{-\infty}^{0} e^{(y+i,(x-\xi))\,\alpha}\; d\alpha + \int_{0}^{+\infty} e^{(-y+i,(x-\xi))\,\alpha}\; d\alpha\Big)\; d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\Big(\frac{1}{y+i\,(x-\xi)}+\frac{1}{y-i\,(x-\xi)}\Big)\; d\xi \\
&= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)}{y^2+(x-\xi)^2}\; d\xi\,.
\end{aligned} \tag{5}$$

Exercise II.3: **potential flow in a semi-infinite strip of finite width**.

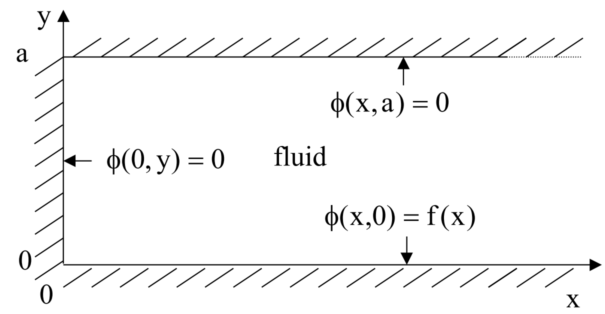


Figure II.6 Flow potential in a semi-infinite strip of finite width with prescribed data along the boundaries.

A fluid fills a semi-infinite strip Ω of finite width a,

$$\Omega = \{x \geq 0; \quad 0 \leq y \leq a\}\,. \tag{1}$$

The equations governing the flow potential $\phi(x, y)$ of the fluid are,

$$\begin{aligned}
&\text{(FE) field equation} && \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad x > 0; \quad 0 < y < a;\\
&\text{(BC) boundary conditions} && \phi(x, y = 0) = f(x), \quad x \in [0, \infty[\\
& && \phi(x, y = a) = 0, \quad x \in [0, \infty[\\
& && \phi(x = 0, y) = 0, \quad y \in [0, a]\,;\\
&\text{(RC) radiation condition} && \phi(x, y) \text{ bounded}, \quad \forall (x, y) \in \Omega\,.
\end{aligned} \tag{2}$$

Show that, when the width a is very large, the flow potential can be cast in the format,

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\text{sgn}(\xi) f(|\xi|)}{y^2 + (x - \xi)^2}\, d\xi\,. \tag{3}$$

Solution:

Since the space variable x varies between 0 and $+\infty$, and the unknown is prescribed on the boundary, the problem will be solved via the sine Fourier transform $x \to \alpha$, and the operators Fourier transform and partial derivative wrt y are assumed to commute.

The transforms of the field equation and boundary conditions take the form,

$$\begin{aligned}
&\text{(FE)} && \frac{\partial^2 \Phi_S}{\partial y^2}(\alpha, y) - \alpha^2\, \Phi_S(\alpha, y) + \alpha\, \overbrace{\phi(0, y)}^{=0,\text{(BC)}} = 0, \quad 0 < y < a,\\
&\text{(BC)} && \Phi_S(\alpha, y = 0) = F_S(\alpha)\\
& && \Phi_S(\alpha, y = a) = 0\,.
\end{aligned} \tag{4}$$

The problem solves to (charitable reminder $\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$, $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$),

$$\Phi_S(\alpha, y) = F_S(\alpha)\, \frac{\sinh(\alpha(a-y))}{\sinh(\alpha\, a)}\,, \tag{5}$$

with inverse

$$\phi(x,y) = \frac{2}{\pi} \int_0^\infty \sin(\alpha\, x) \int_0^\infty \sin(\alpha\, \xi)\, f(\xi)\, d\xi\, \frac{\sinh(\alpha(a-y))}{\sinh(\alpha\, a)}\, d\alpha\,. \tag{6}$$

This integral wrt to α can be integrated in the complex plane. But we shall be content to consider the limit case where the width a tends to infinity. Then

$$\Phi_S(\alpha, y) = F_S(\alpha)\, e^{-\alpha\, y}\,, \tag{7}$$

with inverse

$$\phi(x,y) = \frac{2}{\pi} \int_0^\infty \sin(\alpha\, x) \int_0^\infty \sin(\alpha\, \xi)\, f(\xi)\, d\xi\; e^{-\alpha\, y}\, d\alpha\,. \tag{8}$$

Now

$$\begin{aligned}
& \int_0^\infty 2\sin(\alpha\, x)\, \sin(\alpha\, \xi)\; e^{-\alpha\, y}\, d\alpha \\
= & \int_0^\infty \Big(\cos(\alpha\,(x-\xi)) - \cos(\alpha\,(x+\xi))\Big)\; e^{-\alpha\, y}\, d\alpha \\
= & \int_0^\infty \frac{1}{2}\Big(e^{i\,\alpha\,(x-\xi)} + e^{-i\,\alpha\,(x-\xi)} - e^{i\,\alpha\,(x+\xi)} - e^{-i\,\alpha\,(x+\xi)}\Big)\; e^{-\alpha\, y}\, d\alpha \\
= & \frac{1}{2}\Big[\frac{e^{-\alpha\, y + i\,\alpha\,(x-\xi)}}{-y + i\,(x-\xi)} + \frac{e^{-\alpha\, y - i\,\alpha\,(x-\xi)}}{-y - i\,(x-\xi)} - \frac{e^{-\alpha\, y + i\,\alpha\,(x+\xi)}}{-y + i\,(x+\xi)} - \frac{e^{-\alpha\, y - i\,\alpha\,(x+\xi)}}{-y - i\,(x+\xi)}\Big]_0^\infty \\
= & \frac{y}{y^2 + (x-\xi)^2} - \frac{y}{y^2 + (x+\xi)^2}\,.
\end{aligned} \tag{9}$$

Consequently,

$$\begin{aligned}
\phi(x,y) &= \frac{y}{\pi} \int_0^\infty \Big(\frac{y}{y^2 + (x-\xi)^2} - \frac{y}{y^2 + (x+\xi)^2}\Big)\, f(\xi)\, d\xi \\
&= \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\operatorname{sgn}(\xi) f(|\xi|)}{y^2 + (x-\xi)^2}\, d\xi\,.
\end{aligned} \tag{10}$$

The latter writing can be seen as the solution for the half-plane $y \geq 0$ with odd data $f(x)$.

Exercise II.4: **an integral equation and an inverse problem**.

Prove the following formulas, for $x \geq 0$, $b > 0$:

1.
$$\mathcal{F}_C\{e^{-b\,x}\}(\alpha) = \frac{b}{b^2+\alpha^2}\,, \quad \mathcal{F}_S\{e^{-b\,x}\}(\alpha) = \frac{\alpha}{b^2+\alpha^2}\,. \tag{1}$$

2.
$$\int_0^\infty \frac{\cos(v\,x)}{b^2+x^2}\,dv = \frac{\pi}{2\,b}\,e^{-b\,x}\,. \tag{2}$$

3. Solve the integral equation,

$$\int_{-\infty}^\infty \frac{f(u)}{a^2+(x-u)^2}\,du = \frac{1}{b^2+x^2}\,, \quad 0<a<b\,, \tag{3}$$

for $f(x)$, $x \in]-\infty,\infty[$, and show that the solution reads,

$$f(x) = \frac{1}{\pi}\,\frac{a}{b}\,\frac{b-a}{(b-a)^2+x^2}\,. \tag{4}$$

Proof and solution:

1.
$$\mathcal{F}_C\{e^{-b\,x}\}(\alpha) + i\,\mathcal{F}_S\{e^{-b\,x}\}(\alpha) = \int_0^\infty e^{i\,\alpha\,x-b\,x}\,dx = \frac{-1}{i\,\alpha-b} = \frac{b+i\,\alpha}{b^2+\alpha^2}\,. \tag{5}$$

2. Simply apply the inverse cosine Fourier transform to the first relation,

$$e^{-b\,x} = \frac{2}{\pi}\int_0^\infty \cos(v\,x)\,\frac{b}{b^2+v^2}\,dv\,. \tag{6}$$

3. First,

$$\mathcal{F}\{\frac{1}{b^2+x^2}\}(\alpha) = \int_{-\infty}^\infty \frac{e^{-i\,\alpha\,x}}{b^2+x^2}\,dx = 2\int_0^\infty \frac{\overbrace{\cos(\alpha\,x)}^{\alpha=|\alpha|}}{b^2+x^2}\,dx \overset{\text{eqn }(2)}{=} \frac{\pi}{b}\,e^{-b|\alpha|}\,. \tag{7}$$

Repeated use of this formula yields,

$$\mathcal{F}\{\Big(f(u)*\frac{1}{a^2+u^2}\Big)(x)\}(\alpha) = \begin{cases} \mathcal{F}\{f(x)\}(\alpha)\,\dfrac{\pi}{a}\,e^{-a|\alpha|} & \text{by the convolution theorem} \\ \mathcal{F}\{\displaystyle\int_{-\infty}^\infty \frac{f(u)}{a^2+(x-u)^2}\,du\}(\alpha) & \\ \overset{\text{hyp.}}{=} \mathcal{F}\{\dfrac{1}{b^2+x^2}\}(\alpha) = \dfrac{\pi}{b}\,e^{-b|\alpha|} & \end{cases} \tag{8}$$

Therefore, equating the two lines of the rhs of the latter equation yields,

$$\mathcal{F}\{f(x)\}(\alpha) = \frac{a}{b}\,e^{-(b-a)|\alpha|}\,, \tag{9}$$

whose Fourier inverse is the sought function f,

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\,\alpha\,x}\,\frac{a}{b}\,e^{-(b-a)|\alpha|}\,d\alpha \\
&= \frac{1}{\pi}\,\frac{a}{b}\int_{0}^{\infty} \cos\left(\alpha\,x\right)e^{-(b-a)\alpha}\,d\alpha \\
&= \frac{1}{2\pi}\,\frac{a}{b}\int_{0}^{\infty} e^{i\,\alpha\,x-(b-a)\alpha}+e^{-i\,\alpha\,x-(b-a)\alpha}\,d\alpha \\
&= \frac{1}{2\pi}\,\frac{a}{b}\left[\frac{1}{-i\,x+(b-a)}+\frac{1}{i\,x+(b-a)}\right].
\end{aligned}
\tag{10}
$$

Exercise II.5: **an elastic half-plane under normal surface load**.

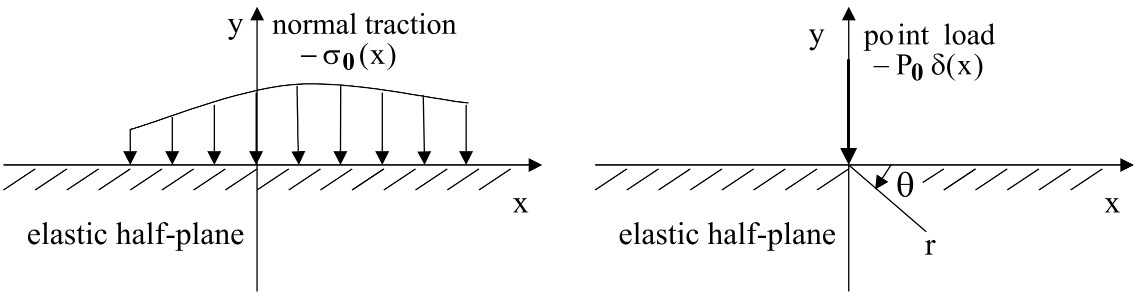


Figure II.7 A half-plane is loaded by a continuous load density, or by a point load, which are normal to the surface.

Some background of plane elasticity has been sketched in Sect. II.6. The whole procedure consists in deriving first the Airy potential $\phi(x, y)$, from which other fields, namely stress and displacement, can be deduced. For the lower half-plane loaded *on* the surface, the Airy potential is governed by the field equation and boundary conditions,

$$\begin{array}{lll}
\text{(FE) field equation} & \Delta\Delta\phi = 0, \quad x \in]-\infty, \infty[,\ y < 0; & \\
\text{(BC) boundary conditions} & \sigma_{yy}(x,0) = \dfrac{\partial^2\phi}{\partial x^2}(x,0) = -\sigma_0(x), & \\
& \sigma_{xy}(x,0) = -\dfrac{\partial^2\phi}{\partial x\partial y}(x,0) = 0; & (1)\\
\text{(RC) radiation condition} & \phi(x,y) \to 0, \quad \text{as} \quad x^2 + y^2 \to \infty\,. &
\end{array}$$

1. Derive the potential $\phi(x, y)$,

$$\phi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\,x + |\alpha|\,y}\,(1 - |\alpha|\,y)\,\frac{\Sigma_0(\alpha)}{\alpha^2}\,d\alpha\,, \tag{2}$$

and deduce the stress components,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \partial^2\phi/\partial y^2 \\ \partial^2\phi/\partial x^2 \\ -\partial^2\phi/\partial x\partial y \end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\,x + |\alpha|\,y}\,\Sigma_0(\alpha) \begin{bmatrix} -1 - |\alpha|\,y \\ -1 + |\alpha|\,y \\ i\,\alpha\,y \end{bmatrix} d\alpha\,, \tag{3}$$

where

$$\Sigma_0(\alpha) = \int_{-\infty}^{\infty} e^{-i\,\alpha\,x}\,\sigma_0(x)\,dx\,. \tag{4}$$

2. If the load of magnitude P_0 is applied at the point $x = 0$, show that the stresses are

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{2\,P_0}{\pi\,r^4} \begin{bmatrix} x^2\,y \\ y^3 \\ x\,y^2 \end{bmatrix} = \frac{2\,P_0}{\pi\,r} \begin{bmatrix} \cos^2\theta\,\sin\theta \\ \sin^3\theta \\ \cos\theta\,\sin^2\theta \end{bmatrix}. \tag{5}$$

Check that the solution satisfies the boundary conditions.

Solution:

1. Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \rightarrow \alpha$, and we admit that the Fourier transform and partial derivative wrt y operators commute.

The transform of the field equation and boundary conditions,

$$\text{(FE)} \quad \Big(\frac{\partial^2}{\partial y^2} - |\alpha|^2\Big)^2 \, \Phi(\alpha, y) = 0, \quad y < 0, \tag{6}$$

solves to

$$\Phi(\alpha, y) = (A(\alpha) + B(\alpha)\, y)\, e^{-|\alpha|\, y} + (C(\alpha) + D(\alpha)\, y)\, e^{|\alpha|\, y} \,. \tag{7}$$

In order to ensure the radiation condition, the Fourier transform itself is required to remain finite at large y, so that $A = B = 0$. The remaining unknowns are defined by the boundary conditions,

$$\begin{aligned} \text{(BC)} \quad & -\alpha^2\, \Phi(\alpha, y = 0) = -\Sigma_0(\alpha) = -\alpha^2\, C(\alpha), \\ & -i\, \alpha\, \frac{\partial \Phi}{\partial y}(\alpha, y = 0) = 0 = -(D + |\alpha|\, C(\alpha)) \,, \end{aligned} \tag{8}$$

so that,

$$\Phi(\alpha, y) = e^{|\alpha|\, y}\, (1 - |\alpha|\, y)\, \frac{\Sigma_0(\alpha)}{\alpha^2} \,. \tag{9}$$

Remark

The solution (7) would require some justification. It can be obtained by solving first,

$$\Big(\frac{\partial^2}{\partial y^2} - a^2\Big) \Big(\frac{\partial^2}{\partial y^2} - b^2\Big)\, \Phi(\alpha, y) = 0 \,, \tag{10}$$

and next by making $b \rightarrow a$.

2. For $\sigma_0(x) = P_0\, \delta(x)$,

$$\Sigma_0(\alpha) = \int_{-\infty}^{\infty} e^{-i\, \alpha\, x}\, \sigma_0(x)\, dx = P_0 \,. \tag{11}$$

Therefore (3) simplifies to

$$\begin{aligned} \sigma_{xx} &= \frac{P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\, \alpha\, x + |\alpha|\, y}\, (-1 - |\alpha|\, y)\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, (1 + \alpha\, y)\, d\alpha \\ \sigma_{yy} &= \frac{P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\, \alpha\, x + |\alpha|\, y}\, (-1 + |\alpha|\, y)\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, (1 - \alpha\, y)\, d\alpha \\ \sigma_{xy} &= \frac{i\, P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\, \alpha\, x + |\alpha|\, y}\, \alpha\, y\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \sin(\alpha\, x)\, \alpha\, y\, d\alpha \end{aligned} \tag{12}$$

The integrals can be evaluated by using the first relations of Exercice 4, namely here

$$\int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{-y}{x^2 + y^2} \,, \tag{13}$$

and therefore

$$\begin{aligned} \int_0^{\infty} e^{\alpha\, y}\, \alpha\, \cos(\alpha\, x)\, d\alpha &= \frac{d}{dy} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{d}{dy}\Big(\frac{-y}{x^2 + y^2}\Big), \\ \int_0^{\infty} e^{\alpha\, y}\, \alpha\, \sin(\alpha\, x)\, d\alpha &= -\frac{d}{dx} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{d}{dx}\Big(\frac{y}{x^2 + y^2}\Big) \,. \end{aligned} \tag{14}$$

The expressions (5) follow for $x = r\cos\theta$, $y = r\sin\theta$.

Moreover, it has been shown in the Chapter devoted to distributions that, in the sense of distributions,

$$\lim_{\epsilon\to 0^+} \frac{2}{\pi}\,\frac{\epsilon^3}{(x^2+\epsilon^2)^2} = \delta(x), \quad \lim_{\epsilon\to 0^+} \frac{2}{\pi}\,\frac{x\,\epsilon^2}{(x^2+\epsilon^2)^2} = 0\,. \tag{15}$$

Therefore, setting here $\epsilon = -y > 0$, the solution (5) is seen to satisfy the boundary conditions $\sigma_{yy}(x,0) = -P_0\,\delta(x)$, and $\sigma_{xy}(x,0) = 0$.

Exercise II.6: **an explicit formula for the Green function**.

Prove the explicit expression of the Green function (II.1.9),

$$u_\delta(x,t) = \frac{1}{\pi}\int_0^\infty \cos(\alpha\,x)\,e^{-\alpha^2\,D\,t}\,d\alpha = \frac{1}{2\sqrt{\pi\,D\,t}}\,\exp\Big(-\frac{x^2}{4\,D\,t}\Big)\,. \tag{1}$$

Proof:

The proof goes as follows. Let us first introduce the change of variable,

$$\alpha \to z = \sqrt{D\,t}\,\alpha\,. \tag{2}$$

Then

$$u_\delta(x,t) = \frac{1}{\pi\,\sqrt{D\,t}}\,I\Big(\frac{x}{\sqrt{D\,t}}\Big)\,, \tag{3}$$

where

$$I(\mu) \equiv \int_0^\infty e^{-z^2}\,\cos(\mu\,z)\,dz\,. \tag{4}$$

The intermediate integral I is obtained by forming a differential equation,

$$\begin{aligned} \frac{dI}{d\mu} &= \int_0^\infty (-z\,e^{-z^2})\,\sin(\mu\,z)\,dz \\ &= \frac{1}{2}\Big[e^{-z^2}\,\sin(\mu\,z)\Big]_0^\infty - \frac{\mu}{2}\int_0^\infty e^{-z^2}\,\cos(\mu\,z)\,dz = -\frac{\mu}{2}\,I(\mu)\,, \end{aligned} \tag{5}$$

that solves to

$$I(\mu) = \frac{\sqrt{\pi}}{2}\,\exp\Big(-\frac{\mu^2}{4}\Big)\,, \tag{6}$$

admitting $\int_0^\infty e^{-z^2}\,dz = \sqrt{\pi}/2$. □

Exercise II.7: **Inhomogeneous waves over an infinite domain**.

Consider the initial value problem governing the axial displacement $u(x,t)$,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = h(x,t), \quad t>0,\ x \in]-\infty,\infty[;\\
&\text{(IC) initial conditions} \quad u(x,0)=f(x); \quad \frac{\partial u}{\partial t}(x,0)=g(x)\,;\\
&\text{(BC) boundary conditions} \quad u(x\to\pm\infty,t)=0\,,
\end{aligned} \tag{1}$$

in an infinite elastic bar, subject to prescribed initial displacement and velocity fields, $f=f(x)$ and $g=g(x)$ respectively. Here c is speed of elastic waves.

Show that the solution reads,

$$u(x,t) = \frac{1}{2}\Big(f(x-ct)+f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} h(y,\tau)\;dy\;d\tau\,. \tag{2}$$

As an alternative to the Fourier transform to be used here, Exercise IV.1 will exploit the method of characteristics.

Chapter III

Classification of partial differential equations into elliptic, parabolic and hyperbolic types

The previous chapters have displayed examples of partial differential equations in various fields of mathematical physics. Attention has been paid to the interpretation of these equations in the specific contexts they were presented. [1]

In fact, we have delineated three types of field equations, namely hyperbolic, parabolic and elliptic. The basic idea that the mathematical nature of these equations was fundamental to their physical significance has been creeping throughout.

Still, the formats in which these three types were presented correspond to their canonical forms, that is, a form that one recognizes at first glance. Such is not the general case. For example, it is not obvious (to this author at least!) that the following second order equation,

$$2\,\frac{\partial^2 u}{\partial x^2} - 4\,\frac{\partial^2 u}{\partial x \partial t} - 6\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} = 0\,,$$

is of hyperbolic type. In other words, it shares essential physical properties with the wave equation,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0\,.$$

Indeed, this is the aim of the present chapter to show that all equations of mathematical physics can be recast in these three fundamental types. By the same token, we introduce a new notion, that of a characteristic curve. A method to solve IBVPs based on characteristics will be exposed in the next chapter.

The terminology used to coin the three types of PDEs borrows from geometry, as the criterion will be seen to rely on the nature of the roots of quadratic equations.

We envisage in turn first of order equations, sets of first order equations, and second order equations. The use of a common terminology to class first and second order equations is challenged by the fact that a set of two first order equation may be transformed into a second

[1] Posted, December 05, 2008; updated, December 12, 2008

order equation, and conversely. The point will not be developed throughout, but rather treated via examples.

Since we are concerned in this chapter with the nature of partial differential equtions, we will not specify the domain in which they assume to hold. On the other hand, the issue surfaces when we intend to solve IBVPs, as considered in Chapters I, II and IV.

III.1 First order partial differential equations

III.1.1 A single equation

We consider first a single first order partial differential equation for the unknown function $u = u(x, y)$,

$$\begin{aligned} &u = u(x,y) \quad \text{unknown,} \\ &(x,y) \quad\quad \text{variables}\,, \end{aligned} \tag{III.1.1}$$

that can be cast in the format,

$$a\,\frac{\partial u}{\partial x} + b\,\frac{\partial u}{\partial y} + c = 0\,. \tag{III.1.2}$$

This equation is said to be (please think a little bit to this terminology),

- *linear* if $a = a(x, y)$, $b = b(x, y)$, and c constant;
- *quasi-linear* if these coefficients depend in addition on the unknown u;
- *nonlinear* if these coefficients depend further on the derivatives of the unknown u.

Let

$$\mathbf{s} = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{III.1.3}$$

be the unit vector that makes it possible to recast the PDE (III.1.2) into the format,

$$\mathbf{s} \cdot \nabla u + d = 0\,, \tag{III.1.4}$$

with $d = c/\sqrt{a^2 + b^2}$.

The curves, starting from an initial curve I_0, and with a slope,

$$\frac{dy}{dx} = \frac{b}{a}\,, \tag{III.1.5}$$

are called **characteristic curves**. A point on these curves is reckoned by the curvilinear abscissa σ,

$$(d\sigma)^2 = (dx)^2 + (dy)^2\,. \tag{III.1.6}$$

Typically, σ is set to 0 on the initial curve I_0.

Then

$$\mathbf{s} = \begin{bmatrix} dx/d\sigma \\ dy/d\sigma \end{bmatrix}, \tag{III.1.7}$$

and **the partial differential equation (PDE)** (III.1.4) for $u(x, y)$,

$$\frac{\partial u}{\partial x}\,\frac{dx}{d\sigma} + \frac{\partial u}{\partial y}\,\frac{dy}{d\sigma} + d = \frac{du}{d\sigma} + d = 0\,, \tag{III.1.8}$$

magically becomes **an ordinary differential equation (ODE)** for $u(\sigma)$ along a characteristic $dy/dx = b/a$. Hum··· puzzling, how is that possible? There should be a trick here ··· My mum warned me, "my little boy, nothing comes for free in this world, except AIDS perhaps". Indeed, there is a price to pay, and the price is to find the characteristic curves, which are not known beforehand.

Taking a step backward, the transformation of a PDE to an ODE is a phenomenon that we have already encountered. Indeed, this is in fact the basic principle of Laplace or Fourier transforms. The initial PDE is transformed into an ODE where the variable associated to the transform is temporarily seen as a parameter. The price to pay here is the inverse transformation.

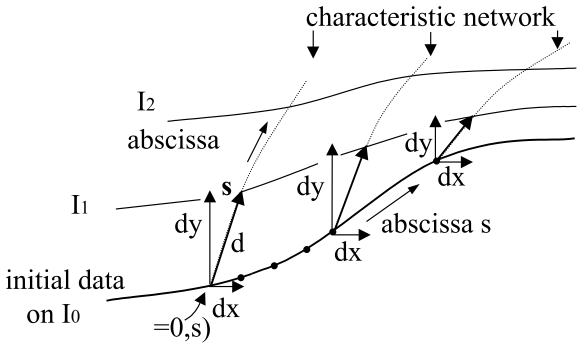


Figure III.1 Given data on a non characteristic initial curve I_0, the characteristic network and solution are built simultaneously, step by step. Each characteristic is endowed with a curvilinear abscissa σ, while points on the initial curve I_0 are reckoned by a curvilinear abscissa s.

Analytical and/or Numerical solution

The above observations provide the basics to a method for solving a partial differential equation.

If the PDE is linear, then

- the characteristics and curvilinear abscissa are obtained by (III.1.5) and (III.1.6);
- the solution u is deduced from (III.1.8).

If the PDE is quasi-linear, a numerical scheme is developed to solve simultaneously (III.1.5) and (III.1.8):

- assume u to be known along a curve I_0, which is required not to be a characteristic;
- at each point of I_0, one may obtain and draw the characteristic using (III.1.5), which provides also $d\sigma$ by (III.1.6);
- du results from (III.1.8), whence the solution on the new curve I_1;
- the three steps above are repeated, starting from I_1, and so on.

It is now clear why the initial curve I_0 should not be a characteristic. Indeed, otherwise, the subsequent curves $I_1 \cdots$ would be I_0 itself, so that the solution could not be obtained at points (x, y) other than on I_0.

III.1.2 A system of quasi-linear equations

The concept of a characteristic curve is now extended to a quasi-linear system of first order partial differential equations for the n unknown functions $u's$,

$$\begin{aligned} &u_j = u_j(x,t), \quad j \in [1,n], \quad \text{unknowns}, \\ &(x,t) \quad \text{variables}, \end{aligned} \tag{III.1.9}$$

that can be cast in the format,

$$\begin{aligned}\mathcal{L}\cdot\mathbf{U} &= \mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial t} + \mathbf{b}\cdot\frac{\partial\mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0}\\ \mathcal{L}_{ij}\,u_j &= a_{ij}\,\frac{\partial u_j}{\partial t} + b_{ij}\,\frac{\partial u_j}{\partial x} + c_i = 0\,,\quad i\in[1,n]\,,\end{aligned} \tag{III.1.10}$$

where the coefficient matrices $\mathbf{a} = (a_{ij})$ and $\mathbf{b} = (b_{ij})$ with $(i,j) \in [1,n]^2$, and vector $\mathbf{c} = (c_i)$, with $i \in [1,n]$, may depend on the variables and unknowns, but not of their derivatives.

In order to form an *ordinary* differential equation in terms of a (yet) unknown curvilinear abscissa σ, we devise a linear combination of these n partial differential equations, namely,

$$\begin{aligned}\boldsymbol{\lambda}\cdot\mathcal{L}\cdot\mathbf{u} &= \mathbf{p}\cdot\frac{d\mathbf{u}}{d\sigma} + r = 0\\ \lambda_i\mathcal{L}_{ij}\,u_j &= p_j\,\frac{du_j}{d\sigma} + r = 0\,.\end{aligned} \tag{III.1.11}$$

The vector $\boldsymbol{\lambda}$ will appear to be a left eigenvector of the matrix $\mathbf{a}\,dx/dt - \mathbf{b}$, namely

$$\begin{aligned}\boldsymbol{\lambda}\cdot(\mathbf{a}\frac{dx}{dt} - \mathbf{b}) &= \mathbf{0},\\ \lambda_i\;(a_{ij}\frac{dx}{dt} - b_{ij}) &= 0,\quad j\in[1,n]\,.\end{aligned} \tag{III.1.12}$$

To prove this property, we pre-multiply (III.1.10) by $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda}\cdot\mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\lambda}\cdot\mathbf{b}\cdot\frac{\partial\mathbf{u}}{\partial x} + \boldsymbol{\lambda}\cdot\mathbf{c} = 0\,, \tag{III.1.13}$$

which can be of the form (III.1.11) only if

$$\frac{\boldsymbol{\lambda}\cdot\mathbf{a}}{dt} = \frac{\boldsymbol{\lambda}\cdot\mathbf{b}}{dx} = \frac{\mathbf{p}}{d\sigma}\,. \tag{III.1.14}$$

Elimination of the vector $\mathbf{p}$ in this relation yields the generalized eigenvalue problem (III.1.12). For the eigenvector $\boldsymbol{\lambda}$ not to vanish, the associated coefficient matrix should be singular,

$$\det\,(\mathbf{a}\frac{dx}{dt} - \mathbf{b}) = 0\,. \tag{III.1.15}$$

This characteristic equation should be seen as a polynomial equation of degree n for dx/dt. The classification of first order partial differential equations is based on the above spectral analysis.

Classification of first order linear PDEs

- if the nb of real eigenvalues is 0, the system is said **elliptic**;
- if the eigenvalues are real and distinct, or
  if the eigenvalues are real and the system is not defective, the system is said **hyperbolic**;
- if the eigenvalues are real, but the system is defective, the system is said to be **parabolic**.

Let us recall that a system of size n is said non defective if its eigenvectors generate $\mathbb{R}^n$, that is, the algebraic and geometric multiplicities of each eigenvector are identical.

Characteristic curves and Riemann invariants

Each eigenvalue dx/dt defines a curve in the plane (x,t) called characteristic. To each characteristic is associated a curvilinear abscissa σ, defined by its differential,

$$\begin{aligned}\frac{d}{d\sigma} &= \frac{dt}{d\sigma}\,\frac{\partial}{\partial t} + \frac{dx}{d\sigma}\,\frac{\partial}{\partial x}\\ &= \frac{dt}{d\sigma}\Big(\frac{\partial}{\partial t} + \frac{dx}{dt}\,\frac{\partial}{\partial x}\Big)\,.\end{aligned} \tag{III.1.16}$$

Inserting (III.1.12) into (III.1.13) yields,

$$\boldsymbol{\lambda} \cdot \mathbf{a} \cdot \frac{d\mathbf{u}}{d\sigma} + \frac{dt}{d\sigma} \boldsymbol{\lambda} \cdot \mathbf{c} = 0\,. \tag{III.1.17}$$

Quantities that are constant along a characteristic are called Riemann invariants.

A simple, but subtle and tricky issue

1. Please remind that the left and right eigenvalues of an arbitrary square matrix are identical, but the left and right eigenvectors do not, if the matrix is not symmetric. The left eigenvectors of a matrix $\mathbf{a}$ are the right eigenvectors of its transpose $\mathbf{a}^T$.
2. The generalized (left) eigenvalue problem $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{b}) = \mathbf{0}$ becomes a standard (left) eigenvalue problem when $\mathbf{b} = \mathbf{I}$, i.e. $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{I}) = \mathbf{0}$. The left eigenvectors of the pencil $(\mathbf{a}, \mathbf{b})$ are also the right eigenvectors of the pencil $(\mathbf{a}^T, \mathbf{b}^T)$.
3. Note the subtle interplay between the sets of matrices $(\mathbf{a}, \mathbf{b})$, and the variables (t, x). The above writing has made use of the ratio dx/dt, and not of dt/dx: we have broken symmetry without care. That temerity might not be without consequence. Indeed, an immediate question comes to mind: are the eigenvalue problems $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{b}) = \mathbf{0}$ and $\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\, dt/dx) = \mathbf{0}$ equivalent? The answer is not so straightforward, as will be illustrated in Exercise III.2.

Some further terminology

If the system of PDEs,

$$\mathbf{a} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{b} \cdot \frac{\partial \mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0}\,, \tag{III.1.18}$$

can be cast in the format,

$$\frac{\partial \mathbf{F}(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial x} = \mathbf{0}\,, \tag{III.1.19}$$

it is said to be of **divergence type**. In the special case where the system can be cast in the format,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial x} = \mathbf{0}\,, \tag{III.1.20}$$

it is termed a **conservation law**.

III.2 Second order partial differential equations

The analysis addresses a single equation, delineating the case of constant coefficients from that of variable coefficients.

III.2.1 A single equation with constant coefficients

Let us start with an example. For the homogeneous wave equation,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0\,, \tag{III.2.1}$$

the change of coordinates,

$$\xi = x - c\,t, \quad \eta = x + c\,t\,, \tag{III.2.2}$$

transforms the *canonical form* (III.2.1) into another *canonical form*,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial \xi \partial \eta} = 0\,. \tag{III.2.3}$$

Therefore, the solution expresses in terms of two arbitrary functions,

$$u(\xi,\eta) = f(\xi) + g(\eta)\,, \tag{III.2.4}$$

which should be prescribed along a non characteristic curve.

But where are the characteristics here? Well, simply, they are the lines ξ constant and η constant.

Let us try to generalize this result to a second order partial differential equation for the unknown $u(x, y)$,

$$\begin{aligned} &u = u(x,y) \quad \text{unknown,} \\ &(x,y) \quad \text{variables}\,, \end{aligned} \tag{III.2.5}$$

with constant coefficients,

$$\mathcal{L}u = A\,\frac{\partial^2 u}{\partial x^2} + 2\,B\,\frac{\partial^2 u}{\partial x \partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + F\,u + G = 0\,. \tag{III.2.6}$$

The question is the following: can we find characteristic curves, so as to cast this PDE into an ODE? The answer was positive for the wave equation. What do we get in this more general case?

Well, we are on a moving ground here. To be safe, we should keep some degrees of freedom. So we bet on a change of coordinates,

$$\xi = -\alpha_1\,x + y, \quad \eta = -\alpha_2\,x + y\,, \tag{III.2.7}$$

where the coefficients α_1 and α_1 are left free, that is, they are to be discovered.

Now come some tedious algebras,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -\alpha_1\,\frac{\partial u}{\partial \xi} - \alpha_2\,\frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\,, \end{aligned} \tag{III.2.8}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \alpha_1^2\,\frac{\partial^2 u}{\partial \xi^2} + 2\,\alpha_1\,\alpha_2\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \alpha_2^2\,\frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2} + 2\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial x \partial y} &= -\alpha_1\,\frac{\partial^2 u}{\partial \xi^2} - (\alpha_1 + \alpha_2)\,\frac{\partial^2 u}{\partial \xi \partial \eta} - \alpha_2\,\frac{\partial^2 u}{\partial \eta^2}\,. \end{aligned} \tag{III.2.9}$$

Inserting these relations into (III.2.6) yields the PDE in terms of the new coordinates,

$$\begin{aligned} \mathcal{L}u = {} & (A\,\alpha_1^2 - 2\,B\,\alpha_1 + C)\,\frac{\partial^2 u}{\partial \xi^2} + (A\,\alpha_2^2 - 2\,B\,\alpha_2 + C)\,\frac{\partial^2 u}{\partial \eta^2} \\ & +2\,(\alpha_1\,\alpha_2\,A - (\alpha_1 + \alpha_2)\,B + C)\,\frac{\partial^2 u}{\partial \xi \partial \eta} \\ & +(-\alpha_1\,D + E)\,\frac{\partial u}{\partial \xi} + (-\alpha_2\,D + E)\,\frac{\partial u}{\partial \eta} + F\,u + G = 0\,. \end{aligned} \tag{III.2.10}$$

Let us choose the coefficients α to be the roots of

$$A\,\alpha^2 - 2\,B\,\alpha + C = 0, \tag{III.2.11}$$

namely,

$$\alpha_{1,2} = \frac{B}{A} \pm \frac{1}{A}\,\sqrt{B^2 - A\,C}\,. \tag{III.2.12}$$

Therefore we are led to distinguish three cases, depending on the nature of these roots. But before we enter this classification, we can make a very important observation:

the nature of the equation depends only on the coefficients of the second order terms. First order terms and zero order terms do not play a role here.

III.2.1.1 Hyperbolic equation $B^2 - A\,C > 0$, e.g. the wave equation

If the discriminant of the quadratic equation (III.2.11) is strictly positive, the two roots are real distinct, and the equation is said hyperbolic. The coefficient of the mixed second derivative of the equation does not vanish,

$$2\,(\alpha_1\,\alpha_2\,A - (\alpha_1 + \alpha_2)\,B + C) = -\frac{4}{A}\sqrt{B^2 - A\,C} \neq 0\,. \tag{III.2.13}$$

The equation can then be cast in the canonical form,

$$\text{(H)}\quad \frac{\partial^2 u}{\partial\xi\partial\eta} + D'\,\frac{\partial u}{\partial\xi} + E'\,\frac{\partial u}{\partial\eta} + F'\,u + G' = 0\,, \tag{III.2.14}$$

where the superscript ' indicates that the original coefficients have been divided by the non zero term (III.2.13).

Another equivalent canonical form,

$$\text{(H)}\quad \frac{\partial^2 u}{\partial\sigma^2} - \frac{\partial^2 u}{\partial\tau^2} + D''\,\frac{\partial u}{\partial\sigma} + E''\,\frac{\partial u}{\partial\tau} + F''\,u + G'' = 0\,, \tag{III.2.15}$$

is obtained by the new set of coordinates,

$$\sigma = \frac{1}{2}\,(\xi + \eta), \quad \tau = \frac{1}{2}\,(\xi - \eta)\,. \tag{III.2.16}$$

The superscript $''$ in (III.2.15) indicates another modification of the original coefficients.

III.2.1.2 Parabolic equation $B^2 - A\,C = 0$, e.g. heat diffusion

A single family of characteristics exists, defined by

$$\alpha_1 = \alpha_2 = \frac{B}{A}\,. \tag{III.2.17}$$

A second arbitrary coordinate is introduced,

$$\xi = -\alpha\,x + y, \quad \eta = -\beta\,x + y, \quad \beta \neq \alpha\,, \tag{III.2.18}$$

which allows to cast the equation in the canonical form,

$$\text{(P)}\quad \frac{\partial^2 u}{\partial\eta^2} + D'\,\frac{\partial u}{\partial\xi} + E'\,\frac{\partial u}{\partial\eta} + F'\,u + G' = 0\,, \tag{III.2.19}$$

where the superscript ' indicates a modification of the original coefficients.

III.2.1.3 Elliptic equation $B^2 - A\,C < 0$, e.g. the laplacian

There are no real characteristics. Still, one may introduce the real coordinates,

$$\sigma = \frac{1}{2}\,(\xi + \eta) = -a\,x + y, \quad \tau = \frac{1}{2i}\,(\xi - \eta) = -b\,x\,, \tag{III.2.20}$$

with real coefficients a and b,

$$\alpha_{1,2} = a \pm i\,b, \quad a = \frac{B}{A}, \quad b = \frac{B}{A}\sqrt{A\,C - B^2}\,, \tag{III.2.21}$$

so as to cast the equation in the canonical form,

$$\text{(E)} \quad \frac{\partial^2 u}{\partial \sigma^2} + \frac{\partial^2 u}{\partial \tau^2} + D''\,\frac{\partial u}{\partial \sigma} + E''\,\frac{\partial u}{\partial \tau} + F''\,u + G'' = 0\,, \tag{III.2.22}$$

where the superscript ” indicates yet another modification of the original coefficients.

III.2.2 A single equation with variable coefficients

When the coefficients of the second order equation are variable, the analysis becomes more complex, but, fortunately, the main features of the constant case remain. Moreover, **the analysis below shows that this nature relies entirely on the sign of $B^2 - A\,C$, like in the constant coefficient equation**.

The characteristics are sought in the more general format,

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)\,, \tag{III.2.23}$$

whence

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\,\frac{\partial \eta}{\partial x} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\,\frac{\partial \eta}{\partial y}\,, \end{aligned} \tag{III.2.24}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\,\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial x}\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x^2}\,\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x^2}\,\frac{\partial u}{\partial \eta} \\ \frac{\partial^2 u}{\partial y^2} &= \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\,\frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial y}\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial y^2}\,\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial y^2}\,\frac{\partial u}{\partial \eta} \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial \xi}{\partial x}\,\frac{\partial \xi}{\partial y}\,\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial y}\right) \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\,\frac{\partial \eta}{\partial y}\,\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x \partial y}\,\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x \partial y}\,\frac{\partial u}{\partial \eta}\,, \end{aligned} \tag{III.2.25}$$

yielding finally,

$$\mathcal{L}u = A'\,\frac{\partial^2 u}{\partial \xi^2} + 2\,B'\,\frac{\partial^2 u}{\partial \xi \partial \eta} + C'\,\frac{\partial^2 u}{\partial \eta^2} + D'\,\frac{\partial u}{\partial \xi} + E'\,\frac{\partial u}{\partial \eta} + F'\,u + G' = 0\,. \tag{III.2.26}$$

The coefficients of higher order,

$$A' = Q(\xi, \xi), \quad B' = Q(\xi, \eta), \quad C' = Q(\eta, \eta)\,, \tag{III.2.27}$$

are defined via the bilinear form Q,

$$Q(\xi,\eta) = A\,\frac{\partial\xi}{\partial x}\,\frac{\partial\eta}{\partial x} + B\,\Big(\frac{\partial\xi}{\partial x}\,\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\,\frac{\partial\eta}{\partial x}\Big) + C\,\frac{\partial\xi}{\partial y}\,\frac{\partial\eta}{\partial y}\,. \tag{III.2.28}$$

The remaining coefficients are,

$$\begin{aligned} D' &= D\,\frac{\partial\xi}{\partial x} + E\,\frac{\partial\xi}{\partial y} + A\,\frac{\partial^2\xi}{\partial x^2} + 2\,B\,\frac{\partial^2\xi}{\partial x\partial y} + C\,\frac{\partial^2\xi}{\partial y^2}\\ E' &= D\,\frac{\partial\eta}{\partial x} + E\,\frac{\partial\eta}{\partial y} + A\,\frac{\partial^2\eta}{\partial x^2} + 2\,B\,\frac{\partial^2\eta}{\partial x\partial y} + C\,\frac{\partial^2\eta}{\partial y^2}\\ F' &= F\\ G' &= G\,. \end{aligned} \tag{III.2.29}$$

Crucially,

$$B'^2 - A'C' = (B^2 - A\,C)\,\Big(\frac{\partial\xi}{\partial x}\,\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\,\frac{\partial\eta}{\partial x}\Big)^2\,. \tag{III.2.30}$$

Therefore, like for the constant coefficient equation, we are led to distinguish three cases, depending on the nature of the roots of a quadratic equation.

III.2.2.1 Hyperbolic equation $B^2 - A\,C > 0$, two real characteristics defined by $A' = C' = 0$, $B' \neq 0$

If $A = C = 0$, the original equation is already in the canonical format. Let us therefore consider the case $A \neq 0$.

The roots $f = \xi$ and η of $A' = 0$ and $C' = 0$ are,

$$\frac{\partial f/\partial x}{\partial f/\partial y} = a \pm b, \quad a = -\frac{B}{A}, \quad b = \frac{1}{A}\sqrt{B^2 - A\,C} \neq 0\,, \tag{III.2.31}$$

and, along the curves of slope

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -(a \pm b)\,, \tag{III.2.32}$$

f is constant,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0\,. \tag{III.2.33}$$

Consequently, these curves are the characteristics we were looking for.

III.2.2.2 Parabolic equation $B^2 - A\,C = 0$, one real characteristic defined by $A' = B' = 0$, $C' \neq 0$

A single family of characteristics exists, defined by $A' = 0$,

$$\frac{\partial\xi/\partial x}{\partial\xi/\partial y} = a = -\frac{B}{A}\,. \tag{III.2.34}$$

A second family of curves $\eta(x,y)$ is introduced, arbitrary but not parallel to the curves ξ constant,

$$\frac{\partial\eta/\partial x}{\partial\eta/\partial y} \neq \frac{\partial\xi/\partial x}{\partial\xi/\partial y} = a\,. \tag{III.2.35}$$

On the other hand, since $B/A = C/B = -a$, B' defined by eqns (III.2.27)-(III.2.28),

$$
\begin{aligned}
B' &= \underbrace{\Big(A\,\frac{\partial\xi}{\partial x} + B\,\frac{\partial\xi}{\partial y}\Big)}_{B/A=-a}\,\frac{\partial\eta}{\partial x} + \underbrace{\Big(B\,\frac{\partial\xi}{\partial x} + C\,\frac{\partial\xi}{\partial y}\Big)}_{C/B=-a}\,\frac{\partial\eta}{\partial y} \\
&= A\,\underbrace{\Big(\frac{\partial\xi}{\partial x} - a\,\frac{\partial\xi}{\partial y}\Big)}_{=0}\,\frac{\partial\eta}{\partial x} + \frac{AC}{B}\,\underbrace{\Big(\frac{\partial\xi}{\partial x} - a\,\frac{\partial\xi}{\partial y}\Big)}_{=0}\,\frac{\partial\eta}{\partial y}\,,
\end{aligned}
\tag{III.2.36}
$$

vanishes, due to (III.2.34), but, as a consequence of the inequality (III.2.35),

$$
C' = \Big(\frac{\partial\eta}{\partial x} + \frac{C}{B}\,\frac{\partial\eta}{\partial y}\Big)\,\Big(A\,\frac{\partial\eta}{\partial x} + B\,\frac{\partial\eta}{\partial y}\Big) \neq 0\,, \tag{III.2.37}
$$

does not vanish.

III.2.2.3 Elliptic equation $B^2 - A\,C < 0$, two complex characteristics defined by $B' = 0$, $A' = C' \neq 0$

There are no real characteristics. The roots of $Q(f, f) = 0$ are complex,

$$
\frac{\partial\xi/\partial x}{\partial\xi/\partial y} = a + i\,b, \quad a = \frac{B}{A}, \quad b = \frac{B}{A}\sqrt{A\,C - B^2}\,. \tag{III.2.38}
$$

Still, one may introduce the real coordinates,

$$
\sigma = \frac{1}{2}\,(\xi + \eta), \quad \tau = \frac{1}{2i}\,(\xi - \eta)\,. \tag{III.2.39}
$$

Inserting

$$
\xi = \sigma + i\,\tau, \quad \eta = \sigma - i\,\tau\,, \tag{III.2.40}
$$

into B', eqns (III.2.27)-(III.2.28), yields,

$$
Q(\sigma, \sigma) - Q(\tau, \tau) + 2\,i\,Q(\sigma, \tau) = 0\,, \tag{III.2.41}
$$

and therefore,

$$
A' = Q(\sigma, \sigma) = Q(\tau, \tau) = C' \neq 0, \quad B' = Q(\sigma, \tau) = 0\,. \tag{III.2.42}
$$

The reals $A' = C'$ do not vanish because the roots ξ and η of $Q(f, f) = 0$ are complex.

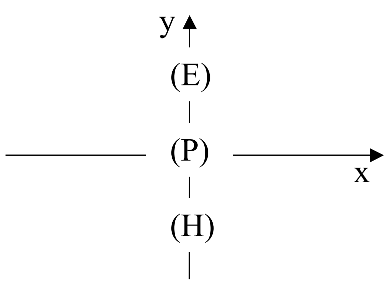


Figure III.2 Tricomi equation of transonic flow provides a conspicuous example of second order equation with variable coefficients where the type varies pointwise.

Remark 1: the Tricomi equation of fluid dynamics

The type of a nonlinear equation may change pointwise. The prototype that illustrates best this issue is the Tricomi equation of transonic flow,

$$\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 u}{\partial y^2} = 0\,, \tag{III.2.43}$$

corresponding to $A = 1$, $B = 0$, $C = y$, so that the nature of the equation depends on $B^2 - A\,C = -y$, whence the types displayed on Fig. III.2.

At this point, we should emit a warning. Even if, in this equation, the boundary in the plane (x, y) between the (H) and (E) types is of (P) type, this is by no means a general situation.

Remark 2: are the classifications of first and second order equations compatible?

Note that we have used the same terminology to class the types of equations, whether first order or second order. This was perhaps a bit too presumptuous. Indeed, for example, a second order equation can be written in the format of two first order equations, and conversely. Examples are provided in Exercises III.1, and III.6. Therefore, we have defined two classifications for second order equations, that of Sect. III.1, and that associated to the set of two first order equations exposed in Sect. III.2.

As they say in French, we looked for the stick to be beaten. However, no worry, man, we are safe! This is because the classifications were built on physical grounds, that is, on the interpretations exposed at length in the previous chapters. Whether written in one form or the other, equations convey the same physical information.

As an illustration, a set of two first order hyperbolic equations is considered in Exercise III.1. The associated second order equation turns out to propagate disturbances at the same speed as the first order set, and it is therefore hyperbolic as well!

III.2.3 Properties of real characteristics

III.2.3.1 The equation of the characteristics

In the previous section, we have shown that the existence of real characteristics corresponds to either $A' = 0$, or $C' = 0$, or both. Let $f = f(x, y) =$ constant be the analytical expression of such a real characteristic. Thus, along a characteristic,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0\,, \tag{III.2.44}$$

and, therefore,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}\,. \tag{III.2.45}$$

Inserting (III.2.45) into $Q(f, f)$ defined by (III.2.28) yields the equation that provides the slope(s) of the real characteristic(s),

$$A\,(dy)^2 - 2\,B\,dy\,dx + C\,(dx)^2 = 0\,. \tag{III.2.46}$$

Please pay attention to the sign in front of the mixed term.

III.2.3.2 Indeterminacy of the Cauchy problem

The characteristics may be given another definition:

these are the curves along which the Cauchy problem is indeterminate or impossible

The issue is the following:

- consider a function u that satisfies the equation (III.2.6);
- prescribe u, $\partial u/\partial x$, and $\partial u/\partial y$;
- obtain the three second order derivatives of u in terms of u, $\partial u/\partial x$, and $\partial u/\partial y$.

The 3×3 linear system to be solved is,

$$\begin{bmatrix} A & 2\,B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \partial^2 u/\partial x^2 \\ \partial^2 u/\partial x \partial y \\ \partial^2 u/\partial y^2 \end{bmatrix} = \begin{bmatrix} -D\,\partial u/\partial x - E\,\partial u/\partial y - F\,u - G \\ d(\partial u/\partial x) \\ d(\partial u/\partial y) \end{bmatrix}. \tag{III.2.47}$$

That the matrix displayed here is singular along the characteristics curves defined by (III.2.46) is easily checked.

Another way to express the indeterminacy of the Cauchy problem is to state that characteristics are the sole curves along which discontinuities can be propagated.

III.3 Extension to more than two variables

The classification can be extended to equations of order higher than 2, and depending on more than 2 variables. For example, let us consider the second order equation depending on n variables,

$$\sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i\,\frac{\partial u}{\partial x_i} + c\,u + g = 0\,. \tag{III.3.1}$$

The coefficient matrix in (III.3.1) should be symmetrized because we have tacitly assumed the partial derivatives to commute, namely $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$, for any i and j in $[1, n]$.

The classification is as follows:

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det \mathbf{a} = 0$)
- (E) for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- (ultraH) for ($Z = 0$ and $1 < P < n - 1$)

where

- Z: nb. of zero eigenvalues of $\mathbf{a}$
- P: nb. of strictly positive eigenvalues of $\mathbf{a}$

The alternatives in the (H) and (P) definitions are due to the fact that multiplication by -1 of the equation leaves it unchanged.

The canonical forms in the characteristic coordinates $\xi' s$ generalize the previous expressions for two variables:

$$\begin{aligned}
&(\mathrm{H}) \quad \frac{\partial^2 u}{\partial \xi_1^2} - \sum_{i=2}^{n} \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{P}) \quad \sum_{i=1}^{n-Z} \pm \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{E}) \quad \sum_{i=1}^{n} \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{uH}) \quad \sum_{i=1}^{P} \frac{\partial^2 u}{\partial \xi_i^2} - \sum_{i=P+1}^{n} \frac{\partial^2 u}{\partial \xi_i^2}
\end{aligned} \tag{III.3.2}$$

To make the link with the analysis of the previous section, set

$$n = 2, \quad \begin{bmatrix} A = a_{11} & B = a_{12} \\ B = a_{21} & C = a_{22} \end{bmatrix}, \tag{III.3.3}$$

whence,

$$\begin{aligned}
&\det(\mathbf{a} - \lambda\,\mathbf{I}) = \lambda^2 - (A + C)\,\lambda + A\,C - B^2 = 0 \\
&\Leftrightarrow \begin{cases} \lambda_1\,\lambda_2 < 0 & \Leftrightarrow \quad A\,C - B^2 < 0 \quad (\mathrm{H}) \\ \lambda_1\,\lambda_2 > 0 & \Leftrightarrow \quad A\,C - B^2 > 0 \quad (\mathrm{E}) \\ \lambda = 0 & \Leftrightarrow \quad A\,C - B^2 = 0 \quad (\mathrm{P}) \end{cases}
\end{aligned} \tag{III.3.4}$$

Example: consider the second order equation for the unknown $u(x, y, z)$,

$$3\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4\,\frac{\partial^2 u}{\partial y \partial z} + 4\,\frac{\partial^2 u}{\partial z^2} = 0\,. \tag{III.3.5}$$

Its nature is obtained by inspecting the spectral properties of the symmetric matrix

$$\mathbf{a} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{a} - \lambda\,\mathbf{I}) = (3 - \lambda)\,\lambda\,(\lambda - 5)\,, \tag{III.3.6}$$

which turns out to have a zero eigenvalue so that the equation is parabolic.

Exercise III.1: **1D-waves in shallow water**.

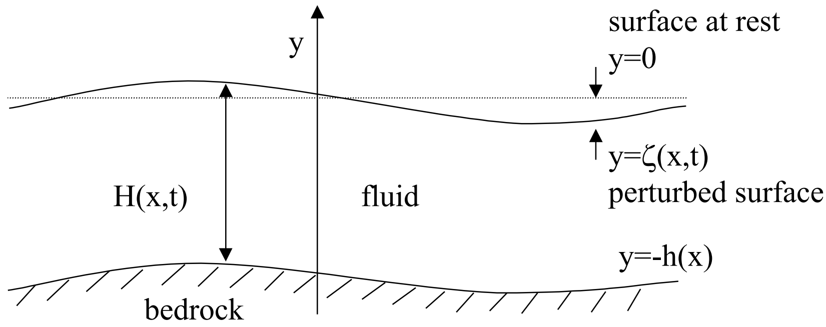


Figure III.3 In shallow water channels, horizontal wavelengths are longer than the depth.

Disturbances at the surface of a fluid surface may give rise to waves because gravity tends to restore equilibrium. In a shallow channel, filled with an incompressible fluid with low viscosity, horizontal wavelengths are much larger than the depth, and water flows essentially in the horizontal directions. In this circumstance, the equations that govern the motion of the fluid take a simplified form. To simplify further the problem, the horizontal flow is restricted to one direction along the x-axis.

Let $u(x,t)$ be the horizontal particle velocity, $\zeta(x,t)$ the position of the perturbed free surface, and $h(x)$ the vertical position of the fixed bedrock. Then

$$H(x,t) = h(x) + \zeta(x,t)\,, \tag{1}$$

is the water height. Mass conservation,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (uH)}{\partial x} = 0\,, \tag{2}$$

and horizontal balance of momentum, involving the gravitational acceleration g,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + g\,\frac{\partial \zeta}{\partial x} = 0, \tag{3}$$

are the two coupled nonlinear equations governing the unknown velocity $u(x,t)$ and fluid height $H(x,t)$.

If we were interested in solving completely the associated IBVP, we should prescribe boundary conditions and initial conditions. However, here, we are only interested in checking the nature of the field equations (FE).

1. In the case of an horizontal bedrock $h(x) = h$ constant, show that the system of equations remains coupled, and find its nature.
2. Give an interpretation to your finding. Hint: linearize the equations.
3. Define the Riemann invariants.
4. Show that the set of the two equations is a conservation law, in the sense of (III.1.20).

Solution:

The system of equations is first cast into the standard format (III.1.10),

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} H \\ u \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} u & H \\ g & u \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} H \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4}$$

The resulting eigenvalue problem (III.1.12),

$$\boldsymbol{\lambda} \cdot (\mathbf{a}\frac{dx}{dt} - \mathbf{b}) = \mathbf{0}\,, \tag{5}$$

yields two real distinct eigenvalues, and associated independent eigenvectors,

$$\frac{dx_+}{dt} = u + \sqrt{gH}, \quad \boldsymbol{\lambda}_+ = \begin{bmatrix} g \\ \sqrt{gH} \end{bmatrix}; \quad \frac{dx_-}{dt} = u - \sqrt{gH}, \quad \boldsymbol{\lambda}_- = \begin{bmatrix} g \\ -\sqrt{gH} \end{bmatrix}, \tag{6}$$

so that the system is hyperbolic, that is, it is expected to be able to propagate disturbances at finite speed.

2. Can we define this speed? To clarify this issue, let us linearize the equations around $u = 0$, $\zeta = 0$,

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + H\,\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + g\,\frac{\partial \zeta}{\partial x} &= 0\,. \end{aligned} \tag{7}$$

Applying the operator $-g\,\partial/\partial x$ to the first line, and $\partial/\partial t$ to the second line, and adding the results yields the second order wave equation,

$$\frac{\partial^2 u}{\partial t^2} - (\sqrt{gH})^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{8}$$

which shows that

$$c = \sqrt{gH}\,, \tag{9}$$

is the wave-speed at which infinitesimal second order disturbances propagate.

Well, that is fine, but we are not totally satisfied because we started from first order equations. Indeed, let us seek if first order waves of the form,

$$\begin{aligned} u(x,t) &= u_+(x + c\,t) + u_-(x - c\,t) \\ \zeta(x,t) &= \zeta_+(x + c\,t) + \zeta_-(x - c\,t)\,, \end{aligned} \tag{10}$$

can propagate to the right and to the left at the very same speed c, as second order waves. Inserting the expressions (10) in (7) shows that indeed these waves can propagate for arbitrary u_+ and u_- and for

$$\zeta_+(x + c\,t) = -\frac{c}{g}\,u_+(x + c\,t), \quad \zeta_-(x - c\,t) = \frac{c}{g}\,u_-(x - c\,t)\,. \tag{11}$$

3. We now come back to the nonlinear system. For each characteristic, the Riemann invariants are defined via (III.1.17), which specializes here to,

$$\boldsymbol{\lambda} \cdot \frac{d\mathbf{u}}{d\sigma} = 0\,. \tag{12}$$

Substituting c for $H = c^2/g$ yields

$$\frac{d}{d\sigma_\pm}(u \pm 2\,c) = 0 \quad \text{along the characteristic} \quad \frac{dx}{dt} = u \pm c\,. \tag{13}$$

The interpretation is as follows: $u + 2c$ is constant along the characteristic $dx/dt = u + c$, and $u - 2c$ is constant along the characteristic $dx/dt = u - c$.

4. Indeed, the system can be recast into the format (III.1.20),

$$\frac{\partial}{\partial t}\overbrace{\begin{bmatrix} H \\ u \end{bmatrix}}^{\mathbf{u}} + \frac{\partial}{\partial x}\overbrace{\begin{bmatrix} u\,H \\ g\,H + u^2/2 \end{bmatrix}}^{\mathbf{G}(\mathbf{u})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{14}$$

For those who want to know more.

Mass conservation is obtained by considering a vertical column of height H and constant horizontal area S, and requiring the time rate of change of its mass $M = \rho SH$ to be equal to the flux $M\mathbf{v}$ traversing the column,

$$\frac{\partial M}{\partial t} + \mathrm{div}(M\mathbf{v}) = 0\,. \tag{15}$$

Since the density ρ is constant, mass conservation,

$$\frac{\partial \zeta}{\partial t} + \mathrm{div}(H\mathbf{v}) = 0\,, \tag{16}$$

simplifies to equation (2) since the particle velocity $\mathbf{v}$ is essentially horizontal.

Momentum balance expresses in terms of the gradient of pressure ∇p, vertical gravitational acceleration $\mathbf{g}$, and particle acceleration $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}$,

$$-\nabla p + \rho\left(\mathbf{g} - \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{0}\,. \tag{17}$$

For shallow waters, the vertical component of the momentum balance is dominated by the pressure gradient and gravity terms, yielding the hydrostatic pressure $p = \rho\,g\,(\zeta - y)$. Inserting this expression in the horizontal component of the momentum balance yields equation (3), once again under the assumption of an essentially horizontal flow.

Exercise III.2: **Switching from first and to second order equations, and conversely**.

1. Define the nature of the set of first order equations,

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0\,, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0\,. \tag{1}$$

Obtain the equivalent second order equation. Was the nature of the system unexpected?

2. Consider the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{2}$$

which is the prototype of a parabolic equation. Obtain an equivalent set of two first order equations. Analyze its nature.

3. Consider the telegraph equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a\,\frac{\partial u}{\partial t} + b\,u = 0\,, \tag{3}$$

where a, b and c are positive quantities, possibly dependent on position. Define the nature of this equation. Obtain an equivalent first order system of partial differential equations.

Solution:

1. Of course, we recognize the Cauchy Riemann equations. The system of equations can be cast into the standard format (III.1.10),

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial x} \overbrace{\begin{bmatrix} u \\ v \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

The resulting eigenvalue problem (III.1.12) implies $\det(\mathbf{a}\,\lambda - \mathbf{b}) = \lambda^2 + 1 = 0$ with $\lambda = dy/dx$, so that the eigenvalues are complex, and the system is therefore elliptic, according to the terminology of Sect. III.1.2.

Now, a basic manipulation of the equations,

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) = \Delta u = 0\,, \tag{5}$$

shows, assuming sufficient smoothness, that u is harmonic, and therefore solution of an elliptic second order equation. So is v, namely $\Delta v = 0$, due to the fact that, if the set $((x, y), (u, v))$ satisfies the Cauchy Riemann equations, so does the set $((y, x), (v, u))$.

2. For example, we may set $v = \partial u/\partial x$. The first order equivalent system becomes,

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} u \\ v \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ -v \end{bmatrix}}^{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

The resulting eigenvalue problem, with $\lambda = dx/dt$, implies $\det(\mathbf{a}\,\lambda - \mathbf{b}) = 1 \cdots$ strange $\cdots$ Never mind! We should not be deterred at the first difficulty. Let us change the angle of attack, and consider the eigenvalue problem, $\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\,\lambda) = \mathbf{0}$, with associated characteristic polynomial,

$$\det\,(\mathbf{a} - \mathbf{b}\,\lambda) = \lambda^2\,, \tag{7}$$

where now $\lambda = dt/dx$. Therefore, $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2. The associated eigenspace, generated by the vectors $\boldsymbol{\lambda}$ such that,

$$\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\,\lambda) = \boldsymbol{\lambda} \cdot \mathbf{a} = \mathbf{0}\,, \tag{8}$$

is in fact spanned by the sole eigenvector $\boldsymbol{\lambda} = [0, 1]$. Therefore, the generalized eigenvalue problem (8) is defective, and the set of the two first order equations is parabolic, according to the terminology of Sect. III.1.2. Thus we have another example where the terminologies used to class first and second order equations are consistent.

3. The telegraph equation is clearly hyperbolic, and $c > 0$ is the wave speed.

With $u_1 = u$, $u_2 = \partial u/\partial x$ and $u_3 = \partial u/\partial t$, the telegraph equation may be equivalently expressed as a first order system of PDEs,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -c^2 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \overbrace{\begin{bmatrix} -u_3 \\ 0 \\ c^2\,(a\,u_3 + b\,u_1) \end{bmatrix}}^{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

The generalized eigenvalues $dx/dt = 0, \pm\, c$ are real and distinct.

Exercise III.3: **Air compressibility in high-speed aerodynamics**.

Air compressibility can not be neglected in high-speed aerodynamics. If air is assumed to be a perfect gas, its pressure p and density ρ are linked by the constitutive relation $p/p_0 = (\rho/\rho_0)^\gamma$, (p_0, ρ_0) being reference values and $\gamma > 0$ is a material constant, equal to the ratio of heat capacities. It is instrumental to define a quantity c, that can be interpreted as a wave-speed,

$$c^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0^\gamma} \rho^{\gamma-1} = \gamma \frac{p}{\rho}, \tag{1}$$

and therefore,

$$dp = c^2 \, d\rho, \quad d\rho = \frac{2}{\gamma - 1} \frac{\rho}{c} \, dc. \tag{2}$$

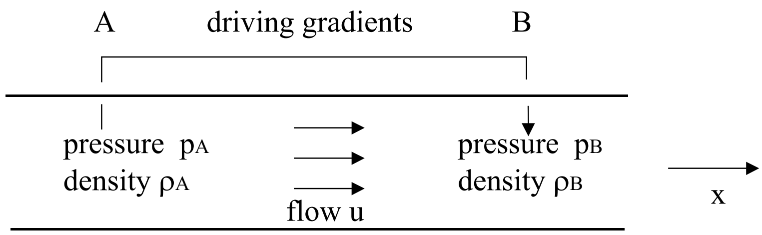


Figure III.4 The flow of ideal gas in a tube is triggered by differences of pressure and density at the ends of the tube.

The one-dimensional flow u of an ideal gas in a tube of axis x is triggered by gradients of pressure $p > 0$ and density $\rho > 0$. The field equations governing these three unknown functions of space x and time t are the three coupled nonlinear partial differential equations,

$$\begin{aligned}
\text{mass conservation :} \quad & \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\
\text{momentum conservation :} \quad & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \\
\text{constitutive equation :} \quad & \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0 .
\end{aligned} \tag{3}$$

1. Define the nature of this set of first order equations, the eigenvalues and eigenvectors associated to the characteristic problem, and the Riemann invariants along each characteristic.
2. Show that this system is of divergence type along the definition (III.1.19).

Solution:

1. The system of equations can be cast into the standard format (III.1.10),

$$\begin{array}{c} (a) \\ (b) \\ (c) \end{array} \quad \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

The characteristic equation,

$$\det(\mathbf{a}\,\lambda - \mathbf{b}) = (\lambda - u)\,\Big((\lambda - u)^2 - c^2\Big) = 0\,, \tag{5}$$

provides three real distinct eigenvalues and independent (left) eigenvectors,

$$\lambda_1 = u\,, \quad \boldsymbol{\lambda}_1 = \begin{cases} 1 \\ 0 \\ 0 \end{cases} ; \quad \lambda_{2,3} = u \pm c\,, \quad \boldsymbol{\lambda}_{2,3} = \begin{cases} 0 \\ 1 \\ \pm 1/(\rho c) \end{cases} , \tag{6}$$

so that the system is hyperbolic.

The Riemann invariants along each characteristic are obtained by (III.1.17), namely, $d\rho = 0$ along the first characteristic, and, along the second and third characteristics,

$$d\Big(u \pm \frac{2}{\gamma - 1}\,c\Big) = 0\,. \tag{7}$$

2. Indeed, the system can be cast in the format,

$$\begin{matrix} (a)' \\ (b)' \\ (c)' \end{matrix} \quad \frac{\partial}{\partial t}\overbrace{\begin{bmatrix} \rho \\ \rho u \\ p + (\gamma - 1)\,\dfrac{\rho u^2}{2} \end{bmatrix}}^{\mathbf{F}(\mathbf{u})} + \frac{\partial}{\partial x}\overbrace{\begin{bmatrix} \rho u \\ p + \rho u^2 \\ \Big(p + (\gamma - 1)\,(p + \dfrac{\rho u^2}{2})\Big)\,u \end{bmatrix}}^{\mathbf{G}(\mathbf{u})} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \tag{8}$$

The proof is as follows:
2.1 (a)'=(a).
2.2 (b)'=$\rho\times$(b)+$u\times$(a).
2.3

$$\begin{aligned} (c)' &= \frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\,\frac{\partial}{\partial x}(\rho\,u^2) \\ &= \frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\,u^2\,\frac{\partial \rho}{\partial x} + (\gamma - 1)\,\rho\,u\,\frac{\partial u}{\partial x} \\ &= \overbrace{-u\,\frac{\partial p}{\partial x} - \gamma p\frac{\partial u}{\partial x}}^{(c)} - \overbrace{\frac{\gamma - 1}{2}\,u^2\,\frac{\partial(\rho u)}{\partial x}}^{(a)} - \overbrace{(\gamma - 1)\,\rho\,u\,(u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x})}^{(b)} \\ &= -\gamma\frac{\partial(up)}{\partial x} - (\gamma - 1)\,\frac{\partial}{\partial x}(\rho\,\frac{u^3}{2}) \end{aligned} \tag{9}$$

$\Box$.

Exercise III.4: **Second order equations**.

1. Define the nature of the second order equation,

$$3\,\frac{\partial^2 u}{\partial x^2}+2\,\frac{\partial^2 u}{\partial x\partial y}+5\,\frac{\partial^2 u}{\partial y^2}x\,+2\,\frac{\partial u}{\partial y}=0\,. \tag{1}$$

2. Consider the wave equation $c^2\,\partial^2 u/\partial t^2-\partial^2 u/\partial x^2=0$, the heat equation $\partial u/\partial t-\partial^2 u/\partial x^2=0$, and the Laplacian $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=0$. Define the respective characteristic curves.

3. Indicate the nature, define the characteristics and cast into canonical form the second order equation,

$$e^{2x}\,\frac{\partial^2 u}{\partial x^2}+2\,e^{x+y}\,\frac{\partial^2 u}{\partial x\partial y}+e^{2y}\,\frac{\partial^2 u}{\partial y^2}=0\,. \tag{2}$$

4. Same questions for the second order equation,

$$2\,\frac{\partial^2 u}{\partial x^2}-4\,\frac{\partial^2 u}{\partial x\partial y}-6\,\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}=0\,. \tag{3}$$

Solution:

1.1 With the method developed in Sect. III.2, we identify $A=3$, $B=1$ $C=5$, so that $B^2-A\,C=-14<0$, and therefore the equation is elliptic.

1.2 With the method exposed in Sect. III.3, the symmetric matrix $\mathbf{a}$ is identified as,

$$\mathbf{a}=\begin{bmatrix}3 & 1\\ 1 & 5\end{bmatrix}. \tag{4}$$

The characteristic equation becomes $\det(\mathbf{a}-\lambda\,\mathbf{I})=\lambda^2-8\,\lambda+14=0$, whose roots are real, positive and distinct, so that the conclusion above is retrieved!

2. Equation (III.2.46) gives the slope of the real characteristics of a second order equation. Thus, the characteristics are, respectively for the wave equation,

$$\left.\begin{array}{l} c^2\,\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial t^2}=0\\ c^2\,(dt)^2-\,(dx)^2=0\end{array}\right\}\quad x\pm a\,t=\text{constant}; \tag{5}$$

for the heat equation,

$$\left.\begin{array}{l} D\,\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial u}{\partial t}=0\\ D\,(dt)^2=0\end{array}\right\}\quad t=\text{constant}; \tag{6}$$

for the Laplacian,

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0\\ (dx)^2+\,(dy)^2=0\end{array}\right\}\quad \text{no real characteristics}\,. \tag{7}$$

3. Along (III.2.46), the slope(s) of the characteristic(s) is(are) defined by the equation,

$$e^{2x}\,(dy)^2-2\,e^{x+y}\,dy\,dx+e^{2y}\,(dx)^2=(e^x\,dy-e^y\,dx)^2=0\,, \tag{8}$$

and therefore there is a single characteristic $\xi(x, y) = e^{-x}\, dx - e^{-y}\, dy$ = constant, and the equation is parabolic. A second arbitrary coordinate may be defined, e.g. $\eta(x, y) = x \neq \xi(x, y)$. Using the relations (III.2.25) between partial derivatives, the equation becomes

$$e^{2\,\eta}\, \frac{\partial^2 u}{\partial \xi^2} + \cdots \frac{\partial u}{\partial \eta} = 0\,, \tag{9}$$

in terms of the coordinates (ξ, η).

4. Along (III.2.46), the slope(s) of the characteristic(s) is(are) defined by the equation,

$$2\,(dy)^2 + 4\,dy\,dx - 6\,(dx)^2 = 2\,(dy + 3\,dx)\,(dy - dx) = 0\,, \tag{10}$$

and therefore there are two characteristic $\xi(x, y) = -x + y$ constant, $\eta(x, y) = 3\,x + y$ constant, and the equation is hyperbolic. Using the relations (III.2.25) between partial derivatives, the equation becomes

$$-32\, \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial u}{\partial \xi} + 3\, \frac{\partial u}{\partial \eta} = 0\,, \tag{11}$$

in terms of the coordinates (ξ, η).

Exercise III.5: **Normal form of a hyperbolic system**.

The unknown vector $\mathbf{u} = \mathbf{u}(x,t)$, of size n, obeys the first order differential system,

$$\mathbf{a} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{b} \cdot \frac{\partial \mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0}, \tag{1}$$

where $\mathbf{a}$, $\mathbf{b}$ are non singular constant matrices and $\mathbf{c}$ is a vector.

1-a The system is said under *normal form* if the matrix $\mathbf{a}$ can be decomposed into a product of a diagonal matrix $\mathbf{d}$ times the matrix $\mathbf{b}$,

$$\mathbf{a} = \mathbf{d} \cdot \mathbf{b}\,. \tag{2}$$

Show that a normal system is hyperbolic.

1-b Conversely, if one admits that, for any non singular matrices $\mathbf{a}$ and $\mathbf{b}$, there exist a non singular matrix $\mathbf{t}$ and a non singular diagonal matrix $\mathbf{d}$, such that,

$$\mathbf{t} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{t} \cdot \mathbf{b}\,, \tag{3}$$

show that any hyperbolic system can be written in normal form.

2-a Consider now the particular matrices and vectors,

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 2 & -2 \\ 1 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -v \\ -u \end{bmatrix}. \tag{4}$$

Show that the system is hyperbolic, define the characteristics, and write it in normal form.

2-b Consider now the above homogeneous system, i.e. with $\mathbf{c} = \mathbf{0}$. Given initial data, namely $u(x, t = 0) = u_0(x)$, $v(x, t = 0) = v_0(x)$, solve the system of partial differential equations.

Solution:

1-a. The nature of the system is defined by the spectral properties of the pencil $(\mathbf{a}, \mathbf{b})$, with characteristic polynomial,

$$\det\,(\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = \det\,(\mathbf{d} - \mathbf{I}\frac{dt}{dx})\det\,\mathrm{b} = 0\,, \tag{5}$$

so that the eigenspace generates $\mathbb{R}^n$, and the i-th vector is associated to the eigenvalue $(dt/dx)_i = d_i$, $i \in [1, n]$. Note that since $\mathbf{a}$ and $\mathbf{b}$ are non singular, so is $\mathbf{d}$.

1-b. Pre-multiplication of (6) by $\mathbf{t}$ yields,

$$\begin{aligned} \mathbf{t} \cdot \mathbf{a} \cdot \frac{\partial \mathbf{u}}{\partial t} &+ \mathbf{t} \cdot \mathbf{b} \cdot \frac{\partial \mathbf{u}}{\partial x} + \mathbf{t} \cdot \mathbf{c} = \mathbf{0} \\ \mathbf{d} \cdot \tilde{\mathbf{b}} \cdot \frac{\partial \mathbf{u}}{\partial t} &+ \tilde{\mathbf{b}} \cdot \frac{\partial \mathbf{u}}{\partial x} + \tilde{\mathbf{c}} = \mathbf{0}, \end{aligned} \tag{6}$$

with $\tilde{\mathbf{b}} = \mathbf{t} \cdot \mathbf{b}$, $\tilde{\mathbf{c}} = \mathbf{t} \cdot \mathbf{c}$.

2-a. The eigenvalues,

$$\det\,(\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = -\,(2 + \frac{dt}{dx})\,(3 - \frac{dt}{dx}) = 0\,, \tag{7}$$

are real and distinct. The characteristics are the lines,

$$\xi = t + 2\,x \text{ const}, \quad \eta = t - 3\,x \text{ const}. \tag{8}$$

Let $\mathbf{d} = \mathbf{diag}[-2, 3]$ be the diagonal matrix of the eigenvalues. Then the matrix $\mathbf{t}$ has to satisfy the equations,

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{d} \cdot \mathbf{b} \quad \Leftrightarrow \quad \begin{bmatrix} 2\,t_{11} + t_{12} & -2\,t_{11} - 4\,t_{12} \\ 2\,t_{21} + t_{22} & -2\,t_{21} - 4\,t_{22} \end{bmatrix} = \begin{bmatrix} -2\,t_{11} & 6\,t_{11} - 2\,t_{12} \\ 3\,t_{21} & -9\,t_{21} + 3\,t_{22} \end{bmatrix}. \tag{9}$$

The matrix $\mathbf{t}$ can be defined to within two arbitrary degrees of freedom, namely $t_{12} = -4\,t_{11}$, $t_{21} = t_{22}$. One may take,

$$\mathbf{t} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}, \tag{10}$$

and the system (6) then writes,

$$\begin{bmatrix} -2 & 14 \\ 3 & -6 \end{bmatrix} \cdot \frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 & -7 \\ 1 & -2 \end{bmatrix} \cdot \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 4\,u - v \\ -u - v \end{bmatrix} = \mathbf{0}, \tag{11}$$

or equivalently,

$$\begin{aligned} \Big(-2\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \quad (u - 7\,v) \quad +4\,u - v \quad &= \quad 0 \\ \Big(3\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \quad (u - 2\,v) \quad -u - v \quad &= \quad 0\,, \end{aligned} \tag{12}$$

or in terms of the coordinates (ξ, η),

$$\begin{aligned} -5\,\frac{\partial}{\partial \eta}\,(u - 7\,v) \quad +4\,u - v \quad &= \quad 0 \\ 5\,\frac{\partial}{\partial \xi}\,(u - 2\,v) \quad -u - v \quad &= \quad 0\,. \end{aligned} \tag{13}$$

2-b. The homogeneous system,

$$\frac{\partial}{\partial \eta}\,(u - 7\,v) = 0, \quad \frac{\partial}{\partial \xi}\,(u - 2\,v) = 0\,, \tag{14}$$

displays the Riemann invariants in explicit form,

$$\begin{aligned} u - 7\,v &= (u - 7\,v)(E_0) \text{ constant along the characteristic } \eta = \text{const}\,, \\ u - 2\,v &= (u - 2\,v)(X_0) \text{ constant along the characteristic } \xi = \text{const}\,. \end{aligned} \tag{15}$$

Let $P(x, t)$ an arbitrary point, and $X_0(x + t/2, 0)$, $E_0(x - t/3, 0)$ the points of the x-axis from which the characteristics that meet at point P emanates. The solution at an arbitrary point (x, t) reads,

$$\begin{aligned} u(x, t) &= \frac{7}{5}(u_0 - 2\,v_0)(X_0) - \frac{2}{5}(u_0 - 7\,v_0)(E_0) \\ v(x, t) &= \frac{1}{5}(u_0 - 2\,v_0)(X_0) - \frac{1}{5}(u_0 - 7\,v_0)(E_0)\,. \end{aligned} \tag{16}$$

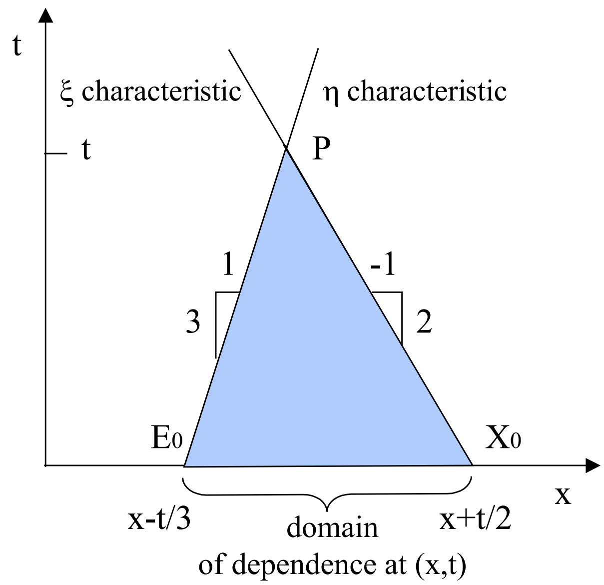


Figure III.5 Given initial data on the x-axis, the solution is built from the characteristic network, playing with the Riemann invariants. The region E_0PX_0 is referred to as the **domain of dependence** of the solution at point $P(x,t)$: indeed, change of the initial conditions outside the interval E_0X_0 will not modify this solution.

Exercise III.6: **Transmission lines and the telegraph equation**.

An electrical circuit representing a transmission line is shown on Fig. III.6. It involves an inductance $L = L(x)$; a resistance $R = R(x)$; a capacitance to ground $C = C(x)$; a conductance to ground $G = G(x)$. In a first step, all these material properties are assumed to be strictly positive. They may vary along the line.

The variation of the potential dV over a segment of length dx is due to the resistance $R\,dx\,I$ and to the inductance $L\,dx\,dI/dt$. Let $q = C\,dx\,V$ be the charge across the capacitor. The variation of the current dI is due to the capacitance $C\,dx\,dV/dt$ and to the conductance $G\,dx\,V$.

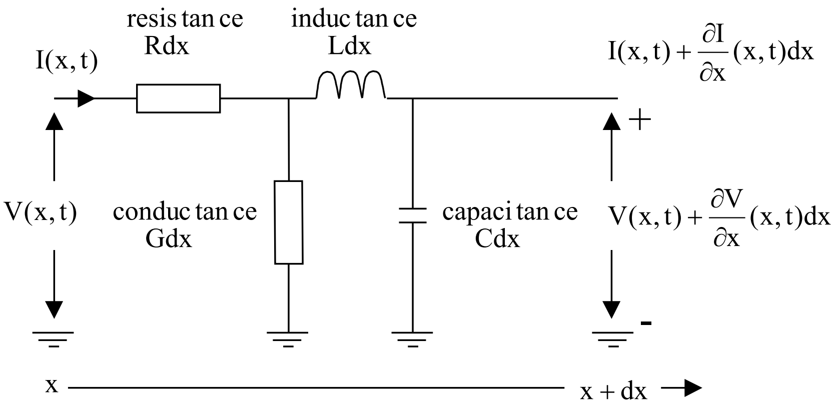


Figure III.6 Elementary circuit of length dx used as a model of the transmission lines.

1-a Therefore, the equations governing the current $I(x,t)$ and potential $V(x,t)$ in a transmission line of axis x can be cast in the format of a linear system of two partial differential equations,

$$\overbrace{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} I \\ V \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} RI \\ GE \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

Show that the system is hyperbolic, and define its characteristics.

The line properties are henceforth assumed to be uniform in space.

1-b Write the system in normal form.

2-a. To substantiate the nature of the system, obtain the equivalent second order equation that the unknowns I and V satisfy. Observe that this second order equation involves a single variable. Therefore, we have obtained a *decoupled* system, at the price of a higher order operator. This equation is referred to as *telegraph equation.* Find the nature of this system, and comment.

2-b. Consider now a *distortionless* line $RC = LG$. Show that $I(t)\,e^{tR/L}$ and $V(t)\,e^{tR/L}$ satisfy a canonical form of the wave equation, with wave speed $1/\sqrt{LC}$.

When $RC \neq LG$, find a modified function in the same mood as above that satisfies the non homogeneous wave equation.

2-c. So far we have manipulated the equations assuming all line coefficients to be different from zero. Consider now Heaviside's ideal line with $L = G = 0$. What is its nature?

Solution:

1-a. The resulting eigenvalue problem (III.1.12),

$$\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = \mathbf{0}\,, \tag{2}$$

yields two real and distinct eigenvalues dt/dx, and associated independent eigenvectors $\boldsymbol{\lambda}$,

$$\frac{dt_+}{dx} = \sqrt{LC}, \quad \boldsymbol{\lambda}_+ = \begin{bmatrix} \sqrt{C} \\ \sqrt{L} \end{bmatrix}; \quad \frac{dt_-}{dx} = -\sqrt{LC}, \quad \boldsymbol{\lambda}_- = \begin{bmatrix} \sqrt{C} \\ -\sqrt{L} \end{bmatrix}, \tag{3}$$

so that the system is hyperbolic, that is, it is expected to be able to propagate disturbances at finite speed.

1-b Let $\mathbf{d} = \mathrm{diag}[\sqrt{LC}, -\sqrt{LC}]$ be the diagonal matrix of the eigenvalues. We look for a matrix $\mathbf{t}$ such that $\mathbf{t} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{t} \cdot \mathbf{b}$, as explained in Exercise III.5. The matrix $\mathbf{t}$ is of course not unique, and in fact there is a double indeterminacy, $t_{12} = t_{11}\sqrt{L/C}$, $t_{21} = -t_{22}\sqrt{C/L}$. We take $t_{11} = 1/(2L\sqrt{C})$, $t_{22} = -1/(2C\sqrt{L})$, and therefore

$$\mathbf{t} = \frac{1}{2\,LC} \begin{bmatrix} \sqrt{C} & \sqrt{L} \\ \sqrt{C} & -\sqrt{L} \end{bmatrix}, \quad \mathbf{t} \cdot \mathbf{a} = \frac{1}{2\sqrt{LC}} \begin{bmatrix} \sqrt{L} & \sqrt{C} \\ \sqrt{L} & -\sqrt{C} \end{bmatrix}. \tag{4}$$

Let us introduce the new unknowns,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{t} \cdot \mathbf{a} \begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{2\sqrt{LC}} \begin{bmatrix} \sqrt{L} & \sqrt{C} \\ \sqrt{L} & -\sqrt{C} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}, \quad \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} \sqrt{C} & \sqrt{C} \\ \sqrt{L} & -\sqrt{L} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{5}$$

Upon pre-multiplication by $\mathbf{t}$, the system (1) becomes

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} + \frac{1}{\sqrt{LC}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} RC+LG & RC-LG \\ RC-LG & RC+LG \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

2-a. Applying the operator $-C\partial/\partial t$ to the first line of (1), and to the second line the operator $\partial/\partial x$, adding the results and using again the first line to eliminate the undesirable unknown, we get the telegraph equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (RC+LG)\,\frac{\partial}{\partial t} + GR\Big)\, X = 0, \quad X = I, V\,. \tag{7}$$

2-b. Let $Y(t) = X(t)\,e^{\alpha\,t}$ with α an unknown exponent. The function $Y(t)$ satisfies the equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (RC+LG-2\,\alpha\,LC)\,\frac{\partial}{\partial t} + (\alpha^2\,LC - (RC+LG)\,\alpha + GR)\Big)\, Y = 0\,. \tag{8}$$

The coefficients of the zero and first order terms vanish simultaneously only if $RC = LG$ and then $\alpha = R/L$, and $Y(t)$ satisfies the wave equation,

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{1}{(\sqrt{LC})^2}\frac{\partial^2}{\partial x^2}\Big)\, Y = 0\,, \tag{9}$$

where $c = 1/\sqrt{LC}$ appears clearly as the wave speed. A typical value is 3×10^8 m/s. This second order analysis is of course consistent with the hyperbolic nature of the initial first order system.

More generally, if $\alpha = (RC + LG)/(2\,LC)$, the first order term vanishes, and we have an inhomogeneous wave equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{(RC - LG)^2}{4\,LC}\Big)\,Y = 0\,. \tag{10}$$

The solution is then a wave followed by a residual wave due to the source term.

2-c. When $L = G = 0$, the telegraph equation (7) is still valid, but it looses its hyperbolic character and becomes a diffusion equation,

$$\Big(-\frac{\partial^2}{\partial x^2} + RC\,\frac{\partial}{\partial t}\Big)\,X = 0, \quad X = I,\,V\,, \tag{11}$$

with a diffusion coefficient equal to $1/(RC)$. Therefore in these circumstances, the mode of propagation of the electrical signal is quite different from the general analysis above. For a voltage shock V_0 applied at the end of the line, one might define qualitatively a *beginning of arrival time* at a point x when the voltage is equal to say 10% of V_0, and an *arrival time* when the voltage is say 50% of V_0. As indicated in Chapter I, the solution has the form of the complementary error function, and the characteristic time is in proportion to $RC\,x^2$.

Chapter IV

Solving linear and nonlinear partial differential equations by the method of characteristics

Chapter III has brought to light the notion of characteristic curves and their significance in the process of classification of partial differential equations.

Emphasis will be laid here on the role of characteristics to guide the propagation of information within hyperbolic equations. As a tool to solve PDEs, the method of characteristics requires, and provides, an understanding of the structure and key aspects of the equations addressed. It is particularly useful to inspect the effects of initial conditions, and/or boundary conditions.

While the method of characteristics may be used as an alternative to methods based on transform techniques to solve linear PDEs, it can also address PDEs which we call quasi-linear (but that one usually coins as nonlinear). In that context, it provides a unique tool to handle special nonlinear features, that arise along shock curves or expansion zones.

As a model problem, the method of characteristics is first applied to solve the wave equation due to disturbances over infinite domains so as to avoid reflections. The situation is more complex in semi-infinite or finite bodies where waves get reflected at the boundaries. The issue is examined in Exercise IV.2.

Basic features of scalar conservation laws are next addressed with emphasis on under- and over-determined characteristic network, associated with expansion zone and shock curves.

Finally some guidelines to solve PDEs via the method of characteristics are provided. Unlike transform methods, the method is not automatic, is a bit tricky and requires some experience. [1]

IV.1 Waves generated by initial disturbances

IV.1.1 An initial value problem in an infinite body

For an infinite elastic bar, aligned with the axis x,

$$-\infty \quad \cdots \overline{\underline{\hspace{6em}}} \cdots \quad +\infty \qquad \text{(IV.1.1)}$$

[1]Posted, December 12, 2008; Updated, April 24, 2009

the field equation describing the homogeneous wave equation,

$$\text{(FE)} \quad \frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = 0\,, \quad -\infty < x < \infty, \quad t > 0\,, \tag{IV.1.2}$$

with

$$\begin{array}{ll} u = u(x,t) & \text{unknown axial displacement,} \\ (x,t) & \text{variables}\,, \end{array} \tag{IV.1.3}$$

is complemented by Cauchy initial conditions,

$$\begin{array}{ll} (\mathrm{CI})_1 & u(x,t=0) = f(x), \quad -\infty < x < \infty \\ (\mathrm{CI})_2 & \dfrac{\partial u}{\partial t}(x,t=0) = g(x), \quad -\infty < x < \infty\,, \end{array} \tag{IV.1.4}$$

and conditions at infinity,

$$\begin{array}{ll} (\mathrm{CL})_1 & u(x \to \pm\infty,\, t) = 0 \\ (\mathrm{CL})_2 & \dfrac{\partial u}{\partial t}(x \to \pm\infty,\, t) = 0\,. \end{array} \tag{IV.1.5}$$

Written in terms of the characteristic coordinates,

$$\xi = x - c\,t, \quad \eta = x + c\,t\,, \tag{IV.1.6}$$

the *canonical form* (IV.1.2) transforms into the *canonical form*,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0\,, \quad -\infty < x < \infty, \quad t > 0\,, \tag{IV.1.7}$$

whose general solution,

$$\begin{array}{rcl} u(\xi,\eta) & = & \phi(\xi) + \psi(\eta) \\ & = & \phi(x - c\,t) + \psi(x + c\,t)\,, \end{array} \tag{IV.1.8}$$

expresses in terms of two arbitrary functions to be defined.

One could not stress enough the interpretation of these results.

Along a characteristic $\xi = x - c\,t$ constant, the solution $\phi(\xi)$ keeps constant: for an observer moving to the right at speed c, the initial profile $\phi(\xi)$ keeps identical. A similar interpretation holds for the part of the solution contained in $\psi(\eta)$ which propagates to the left.

We now will consider the effects of the initial conditions, so as to obtain the two unknown functions ϕ and ψ.

IV.1.2 D'Alembert solution

The initial conditions,

$$\begin{array}{ll} (\mathrm{CI})_1 & u(x,t=0) = f(x) = \phi(x) + \psi(x), \quad -\infty < x < \infty \\ (\mathrm{CI})_2 & \dfrac{\partial u}{\partial t}(x,t=0) = g(x) = -c\,\dfrac{d\phi}{dx} + c\,\dfrac{d\psi}{dx}, \quad -\infty < x < \infty\,, \end{array} \tag{IV.1.9}$$

imply

$$-\phi(x) + \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(y)\,dy - A\,, \tag{IV.1.10}$$

where x_0 and A are arbitrary constants, and therefore,

$$\begin{aligned}\phi(x) &= \frac{1}{2}(f(x)+A) - \frac{1}{2c}\int_{x_0}^{x} g(y)\,dy\\ \psi(x) &= \frac{1}{2}(f(x)-A) + \frac{1}{2c}\int_{x_0}^{x} g(y)\,dy\,.\end{aligned} \tag{IV.1.11}$$

Substituting x for $x+ct$ in ϕ and x for $x-ct$ in ψ, the final expressions of the displacement and velocity,

$$\begin{aligned}u(x,t) &= \frac{1}{2}\Big(f(x-ct)+f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy\\ \frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f'(x-ct)+f'(x+ct)\Big) + \frac{1}{2}\Big(g(x-ct)+g(x+ct)\Big)\,,\end{aligned} \tag{IV.1.12}$$

highlight the influences of an initial displacement $f(x)$ and of an initial velocity $g(x)$.

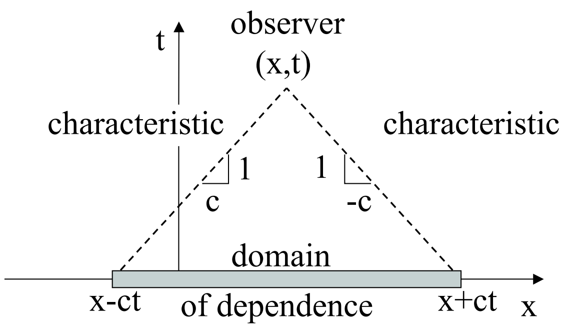


Figure IV.1 Sketch illustrating the notion of domain of dependence of the solution of the wave equation at a point (x,t).

IV.1.2.1 Domain of dependence

An observer sitting at point (x,t) sees two characteristics coming to him, $x-ct$ and $x+ct$ respectively. These characteristics bring the effects

- of an initial displacement f at $x-ct$ and $x+ct$ *only*;
- of an initial velocity g *all along the interval* $[x-ct, x+ct]$.

Furthermore, the velocity at point (x,t) is effected only by f' and g at $x-ct$ and $x+ct$. It is important to realize that the data outside this interval do not effect the solution at (x,t).

IV.1.2.2 Zone of influence

Conversely, it is also of interest to consider the domain of the (x,t)-plane that data at the point $(x_0, t=0)$ influence. In fact, this domain is a triangular zone delimited by the characteristics $\xi = x_0 - ct$ and $\eta = x_0 + ct$.

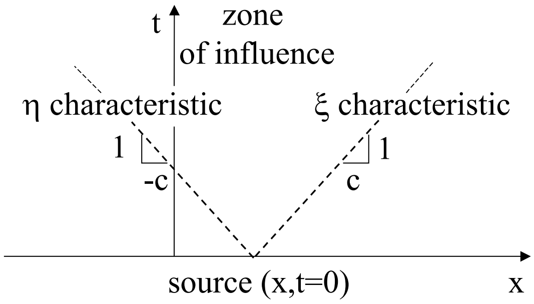


Figure IV.2 Sketch illustrating the notion of zone of influence of the initial data.

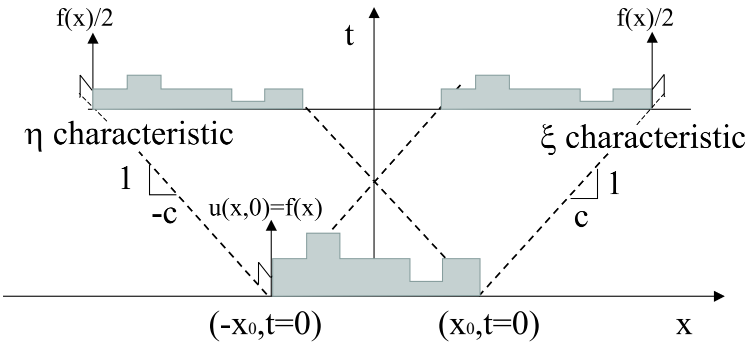


Figure IV.3 Sketch illustrating how an initial displacement is propagated form-invariant, but with half magnitude along each of the two characteristics.

IV.1.2.3 Effects of an initial displacement

The effect of an initial displacement can be illustrated by considering the special data,

$$f(x) = \begin{cases} f_0(x) & |x| \le x_0 \\ 0\,, & |x| > x_0, \end{cases} \quad ; \quad g(x) = 0, \quad -\infty < x < \infty\,. \tag{IV.1.13}$$

The solution (IV.1.12),

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f_0(x-ct) + f_0(x+ct)\Big) \\ \frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f_0'(x-ct) + f_0'(x+ct)\Big)\,, \end{aligned} \tag{IV.1.14}$$

indicates that that the initial disturbance $f_0(x)$ propagates *without alteration* along the two characteristics $\xi = x - ct$ and $\eta = x + ct$, but scaled by a factor $1/2$.

IV.1.2.4 Effects of an initial velocity

The effect of an initial velocity,

$$f(x) = 0, \quad -\infty < x < \infty; \quad g(x) = \begin{cases} g_0(x) & |x| \le x_0 \\ 0\,, & |x| > x_0, \end{cases} \tag{IV.1.15}$$

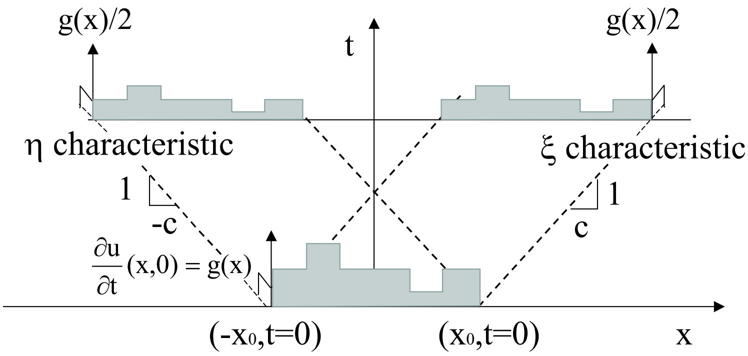


Figure IV.4 Sketch illustrating how an initial velocity is propagated form-invariant, but with half magnitude along each of the two characteristics.

on the displacement field can also be inspected via (IV.1.12),

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y)\,dy \\ \frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\Big(g_0(x-ct)+g_0(x+ct)\Big)\,, \end{aligned} \tag{IV.1.16}$$

The effect on the velocity is simpler to address. In fact, this effect is similar to that of an initial displacement on the displacement field, as described above.

IV.1.3 The inhomogeneous wave equation

The additional effect of a volume source is considered in Exercise IV.1.

IV.1.4 A semi-infinite body. Reflection at boundaries

Thus far, we have been concerned with an infinite body. The idea was to avoid reflection of signals impinging boundaries located at finite distance.

With the basic presentation in mind, we can now address this phenomenon. This is the aim of Exercise IV.2.

IV.2 Conservation law and shock

Most field equations in engineering stem from balance statements. Matter or energy may be transported in space. Matter may undergo physical changes, like phase transform, aggregation, erosion $\cdots$. Energy may be used by various physical processes, or even change nature, from electrical or chemical turned mechanical.

Still in all these processes some entity is conserved, typically mass, momentum or energy. We explore here the basic mathematical structure of conservation laws, and the consequences in the solution of PDEs via the method of characteristics.

IV.2.1 Conservation law

A scalar (one-dimensional) conservation law is a partial differential equation of the form,

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{dq}{du}\frac{\partial u}{\partial x} = 0\,, \tag{IV.2.1}$$

where

- $u = u(x,t)$ is the primary unknown, representing for example, the density of particles along a line, or the density of vehicles along the segment of a road devoid of entrances and exits;
- $q(x,t)$ is the flux of particles, vehicles $\cdots$ crossing the position x at time t. This flux is linked to the primary unknown u, by a **constitutive relation** $q = q(u)$ that characterizes the flow.

Perhaps the simplest conservation law is

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{1}{2}u^2\Big) = 0, \quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,. \tag{IV.2.2}$$

The proof of the conservation law goes as follows. Let us consider a segment $[a,b]$,

- along which all particles move with some non zero velocity;
- such that all particles that enter at $x = a$ exit at $x = b$, and conversely.

One defines

- the particle density as $u(x,t)$= nb of particles per unit length;
- the flux as $q(x,t)$= nb. of particles crossing the position x per unit time.

The conservation of particles in the section $[a,b]$ can be stated as follows: the variation of the nb of particles in this section is equal to the difference between the fluxes at a and b:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + q(b,t) - q(a,t) = 0\,. \tag{IV.2.3}$$

Given that a and b are fixed positions, this relation can be rewritten,

$$\int_a^b \frac{\partial u(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x}\,dx = 0\,, \tag{IV.2.4}$$

whence the partial differential relation (IV.2.1), given that a and b are arbitrary.

IV.2.2 Shock and the jump relation

IV.2.2.1 Under and overdetermined characteristic network

Let us first consider a simple initial value problem (IVP), motivated by the sketch displayed in Fig. IV.5. We would like to solve the following problem for $u = u(x,t)$,

$$\begin{aligned} &\text{(FE)} \quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} &=&\; 0, \quad -\infty < x < \infty,\; t > 0 \\ &\text{(IC)} \quad\quad u(x,0) &=&\; \begin{cases} A, & x < 0 \\ B, & x \geq 0\,. \end{cases} \end{aligned} \tag{IV.2.5}$$

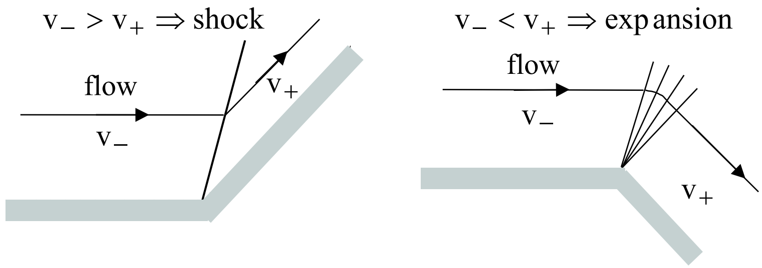


Figure IV.5 Qualitative sketch illustrating the shock-expansion theory. The flow velocity may decrease over a concave corner, or increase over a convex corner.

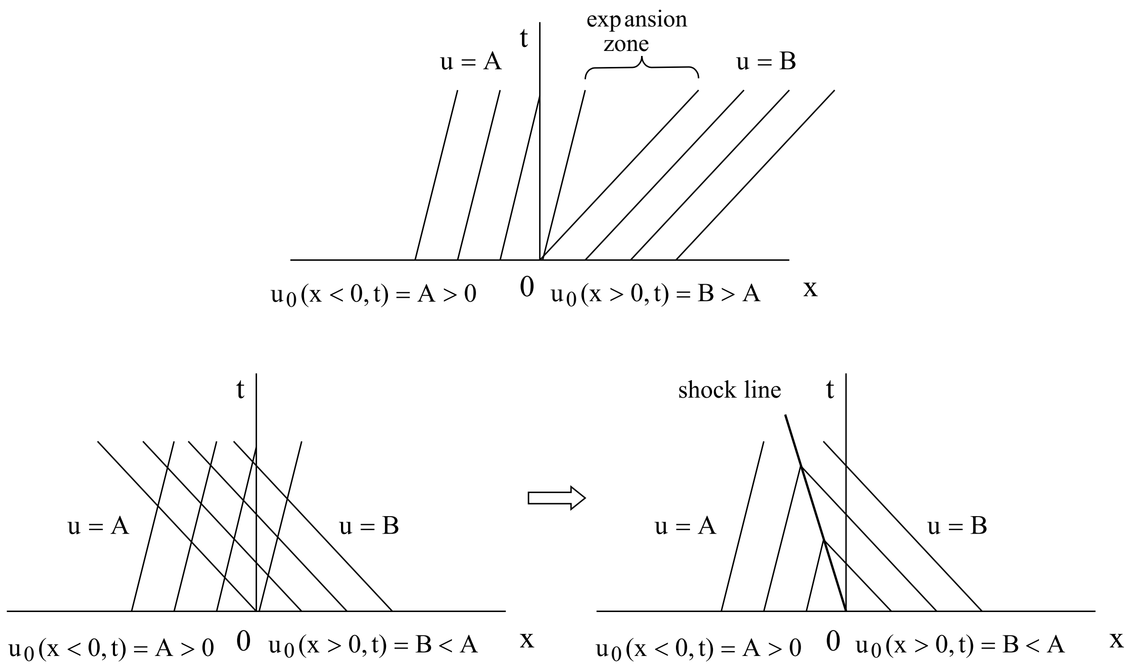


Figure IV.6 If the signal travels slower at the rear than at the front ($A < B$), the characteristic network is under-determined. Conversely, if the signal travels faster at the rear than in front ($A > B$), the characteristic network is over determined: the tentative network that displays intersecting characteristics, has to be modified to show a discontinuity line (curve).

Along a standard presentation, we would like the two relations,

$$
\begin{aligned}
0 &= \frac{\partial u}{\partial t} &+& \frac{\partial u}{\partial x}\, u \\
du &= \frac{\partial u}{\partial t}\, dt &+& \frac{\partial u}{\partial x}\, dx\,,
\end{aligned}
\tag{IV.2.6}
$$

to be identical. Therefore, we should have simultaneously $dx/dt = u$, and $du = 0$. In other words, the characteristic curves are

$$
\frac{dx}{dt} = u = \text{constant}\,. \tag{IV.2.7}
$$

The construction of the characteristic network starts from the x-axis, Fig. IV.6.

Clearly the properties of the network depends on the relative values of A and B:

- for $A < B$, the characteristic network is underdetermined. There is a fan in which no characteristic exists. The signal emanating from points $(x < 0, t = 0)$ travels at a speed A slower than the signal emanating from points $(x > 0, t = 0)$;
- for $A > B$, the characteristic network is overdetermined, i.e. the characteristics would tend to intersect. Indeed, the signal emanating from points $(x < 0, t = 0)$ travels at a speed A greater than the signal emanating from points $(x > 0, t = 0)$. However the characteristics can not cross because the solution, e.g. a mass density, would be multi-valued.

IV.2.2.2 The jump relation across a shock

We now return to the conservation law for the unknown $u = (x, t)$ where $q = q(u)$ is seen as the flux,

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \,. \tag{IV.2.8}$$

Let the symbol $[\![\cdot]\!]$ denote the jump across the shock,

$$[\![\cdot]\!] = (\cdot)_+ - (\cdot)_- \,, \tag{IV.2.9}$$

the symbol plus and minus indicating points right in front and right behind the shock. Of course the exact definition of the jump operator depends on what we call front and back, but the jump relation below does not.

The speed of propagation of the shock,

$$\frac{dX_s(t)}{dt} = \frac{[\![q]\!]}{[\![u]\!]} = \frac{q_+ - q_-}{u_+ - u_-} \,, \tag{IV.2.10}$$

depends on the jumps of the unknown $[\![u]\!]$ and flux $[\![q]\!]$ across the shock line.

The proof of this so-called **jump relation** begins by integration of the conservation law between two lagrangian positions $X_1 = X_1(t)$ and $X_2 = X_2(t)$,

$$\int_{X_1(t)}^{X_2(t)} \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} \; dx = 0 \,. \tag{IV.2.11}$$

This relation is further transformed using the standard formula that gives the derivative of an integral with variable and differentiable bounds,

$$\begin{aligned} &\frac{d}{dt} \int_{X_1(t)}^{X_2(t)} u(x,t) \; dx \\ &= \int_{X_1(t)}^{X_2(t)} \frac{\partial u(x,t)}{\partial t} \, dx + \frac{dX_2(t)}{dt} u(X_2(t), t) - \frac{dX_1(t)}{dt} u(X_1(t), t) \,. \end{aligned} \tag{IV.2.12}$$

Thus (IV.2.11) becomes

$$\frac{d}{dt} \int_{X_1}^{X_2} u(x,t) \, dx - \frac{dX_2}{dt} u(X_2, t) + \frac{dX_1}{dt} u(X_1, t) + q(X_2, t) - q(X_1, t) = 0 \,. \tag{IV.2.13}$$

Finally we account for the fact that the shock has an infinitesimal width, so that, X_s being a point on the shock line at time t, letting X_1 tend to X_{s-} and X_2 tend to X_{s+}, we get

$$-\frac{dX_s}{dt} u_+ + \frac{dX_s}{dt} u_- + q_+ - q_- = 0 \quad \square \,. \tag{IV.2.14}$$

Remark: the shock relation applied to the mass conservation

Conservation of mass corresponds to $u = \rho$ mass density and to $q = \rho\, v$ momentum. The jump relation can be transformed to the standard relation that involves the Lagrangian speed of propagation of the shock line,

$$[\![\rho\,(\frac{dX_s}{dt} - v)]\!] = 0\,. \tag{IV.2.15}$$

IV.2.2.3 The entropy condition

Exercises IV.4 and IV.5 present examples of under and over determined characteristic networks. Exercise IV.4 indicates how to construct a solution in absence of characteristics. The underlying construction is in agreement with the entropy condition,

$$\frac{dq_-}{du} > \frac{dX_s(t)}{dt} > \frac{dq_+}{du}\,. \tag{IV.2.16}$$

Remark: on the intrinsic form of the conservation law

Consider the two distinct conservation laws, written in integral (intrinsic) form,

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + \frac{1}{2}u^2(b,t) - \frac{1}{2}u^2(a,t) = 0\,, \tag{IV.2.17}$$

and

$$\frac{d}{dt}\int_a^b u^2(x,t)\,dx + \frac{2}{3}u^3(b,t) - \frac{2}{3}u^3(a,t) = 0\,, \tag{IV.2.18}$$

whose local forms (partial differential equations) are respectively,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,, \tag{IV.2.19}$$

and

$$2\,u\,\Big(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\Big) = 0\,. \tag{IV.2.20}$$

As a conclusion, two distinct conservation laws may have identical local form. An issue arises in presence of a shock: on the shock line, the relation to be accounted for is the jump relation, and no longer the local relation. Consequently, the original (intrinsic) flux corresponding to the physical problem to be solved should be known and referred to.

IV.3 Guidelines to solve PDEs via the method of characteristics

As already alluded for, the method of characteristics to solve PDEs is a bit tricky. The method is quite general. As a consequence, a number of decisions has to be taken. This concerns in particular the choice of the curvilinear system. Any inappropriate choice may be bound to failure. Some basic notions are listed below. They should be complemented by exercises.

We would like to find the solution to the quasi-linear partial differential equation for $u = u(x,y)$,

$$a(x,y,u)\,\frac{\partial u}{\partial x} + b(x,y,u)\,\frac{\partial u}{\partial y} = c(x,y,u)\ , \tag{IV.3.1}$$

where the functions a, b and c are sufficiently smooth, with the boundary data,

$$u = u_0(s), \quad \text{along } I_0 : \left\{ \begin{array}{l} x = F(s) \\ y = G(s) \end{array} \right. . \tag{IV.3.2}$$

I_0 should not be a characteristic: if it is differentiable, this implies,

$$\frac{F'(s)}{G'(s)} \neq \frac{a(F(s), G(s), u_0(s))}{b(F(s), G(s), u_0(s))}. \qquad \text{(IV.3.3)}$$

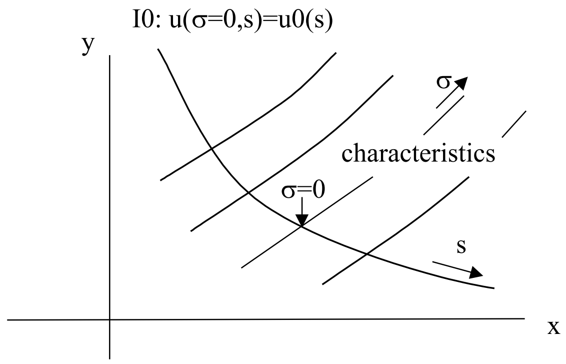


Figure IV.7 Data curve I_0, characteristic network and curvilinear coordinate σ associated with any characteristic and s associated with the curve I_0.

The method proceeds as follows. The curvilinear abscissa along the curve I_0 is s, Fig. IV.7. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

We would like the two relations,

$$\begin{aligned} c &= \frac{\partial u}{\partial x} a + \frac{\partial u}{\partial y} b \\ \frac{du}{d\sigma} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \sigma}, \end{aligned} \qquad \text{(IV.3.4)}$$

to be identical. Therefore,

$$\frac{\partial x}{\partial \sigma} = a, \quad \frac{\partial y}{\partial \sigma} = b, \quad \frac{du}{d\sigma} = c, \qquad \text{(IV.3.5)}$$

and, switching from the coordinates (x, y) to the coordinates (s, σ),

$$x(\sigma = 0, s) = F(s), \quad y(\sigma = 0, s) = G(s), \quad u(\sigma = 0, s) = u_0(s). \qquad \text{(IV.3.6)}$$

The solution is sought in the format,

$$x = x(\sigma, s), \quad y = y(\sigma, s), \quad u = u(\sigma, s). \qquad \text{(IV.3.7)}$$

The underlying idea is to fix s, so that (IV.3.4) becomes an ordinary differential equation (ODE). In other words, along each characteristic, (IV.3.4) is an ODE.

The system can be inverted into

$$\sigma = \sigma(x, y), \quad s = s(x, y), \qquad \text{(IV.3.8)}$$

if the determinant of the associated jacobian matrix does not vanish,

$$\frac{\partial(x, y)}{\partial(\sigma, s)} = \frac{\partial x}{\partial \sigma}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \sigma}\frac{\partial x}{\partial s} = a\, G'(s) - b\, F'(s) \neq 0. \qquad \text{(IV.3.9)}$$

Exercise IV.1: **Inhomogeneous waves over an infinite domain**.

Consider the initial value problem governing the axial displacement $u(x,t)$,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = h(x,t), \quad t>0,\ x \in]-\infty,\infty[;\\
&\text{(IC) initial conditions} \quad u(x,0) = f(x); \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,;\\
&\text{(BC) boundary conditions} \quad u(x \to \pm\infty, t) = 0\,,
\end{aligned} \tag{1}$$

in an infinite elastic bar,

$$-\infty \quad \cdots \overline{\overline{\qquad\qquad\qquad}} \cdots \quad +\infty \tag{2}$$

subject to prescribed initial displacement and velocity fields, $f = f(x)$ and $g = g(x)$ respectively. Here c is speed of elastic waves.

Show that the solution reads,

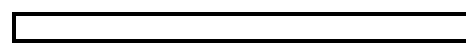
$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} h(y,\tau)\;dy\;d\tau\,. \tag{3}$$

As an alternative to the Fourier transform used in Exercise II.7, exploit the method of characteristics.

Exercise IV.2: **Reflection of waves at a fixed boundary. The method of images**.

Sect. IV.1.1 has considered the propagation of initial disturbances in an infinite bar. The idea was to avoid reflections of the signal that was bounded to complicate the initial exposition.

We turn here to the case of a semi-infinite bar,

0  ··· ∞ (1)

whose boundary $x = 0$ is fixed. The conditions (IV.1.5) modify to

$$
\begin{aligned}
&(\text{CL})_1 \quad u(x=0, t=0) = 0; \quad u(x \to +\infty, t=0) = 0;\\
&(\text{CL})_2 \quad \frac{\partial u}{\partial t}(x=0, t=0) = 0; \quad \frac{\partial u}{\partial t}(x \to +\infty, t=0) = 0\,.
\end{aligned} \tag{2}
$$

The method of images consist

- in thinking of a mirror bar over $]-\infty, 0[$;
- in complementing the initial data (IV.1.4) over the real bar by the data,

$$
f(x) = -f(-x); \quad g(x) = -g(-x); \quad x < 0\,. \tag{3}
$$

Show that the displacement and velocity fields (IV.1.12) for the infinite bar become,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\Big(\text{sgn(x − ct)}\, f(|x-ct|) + f(x+ct)\Big) + \frac{1}{2c}\int_{|x-ct|}^{x+ct} g(y)\, dy\\
\frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f'(|x-ct|) + f'(x+ct)\Big) + \frac{1}{2}\Big(\text{sgn(x − ct)}\, g(|x-ct|) + g(x+ct)\Big)\,,
\end{aligned} \tag{4}
$$

for the semi-infinite bar extending over $[0, +\infty[$.

Exercise IV.3: **A first order quasi-linear partial differential equation with boundary conditions**.

Find the solution to the partial differential equation for $u = u(x, y)$,

$$x \frac{\partial u}{\partial x} + y\,u \frac{\partial u}{\partial y} + x\,y = 0, \quad x > 0, \; y > 0\,, \tag{1}$$

with the boundary data,

$$u = 5, \quad \text{along } I_0 : x\,y = 1, \; x > 0\,. \tag{2}$$

Solution:

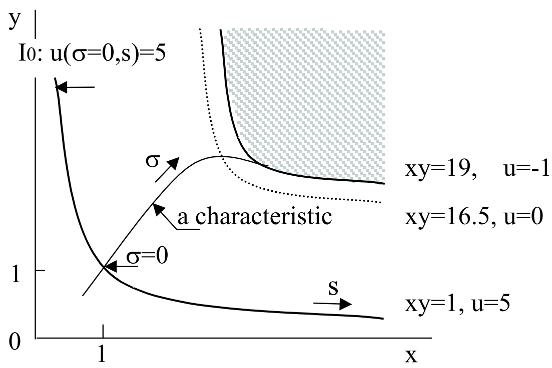


Figure IV.8 Curvilinear coordinates associated with the boundary value problem.

One can choose the curvilinear abscissa of the curve I_0 to be $s = x$, Fig. IV.8. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

Along the standard presentation, we would like the two relations,

$$\begin{aligned} -x\,y &= \frac{\partial u}{\partial x}\,x + \frac{\partial u}{\partial y}\,y\,u \\ \frac{du}{d\sigma} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \sigma}\,, \end{aligned} \tag{3}$$

to be identical. Therefore, the characteristic curves are defined by the relations $dy/dx = y\,u/x$, and

$$\frac{\partial x}{\partial \sigma} = x, \quad \frac{\partial y}{\partial \sigma} = y\,u, \quad \frac{\partial u}{\partial \sigma} = -x\,y\,, \tag{4}$$

and, switching from the coordinates (x, y) to the coordinates (s, σ),

$$x(\sigma = 0, s) = s, \quad y(\sigma = 0, s) = \frac{1}{s}, \quad u(\sigma = 0, s) = 5, \; s > 0\,. \tag{5}$$

To integrate (4), we note

$$\begin{aligned}
\frac{\partial(x\,y)}{\partial\sigma} &= \underbrace{\frac{\partial x}{\partial\sigma}}_{=x,\ (4)_1} \; y + x \; \underbrace{\frac{\partial y}{\partial\sigma}}_{=yu,\ (4)_2} \\
&= (1+u)\;\underbrace{x\,y}_{(4)_3} \\
&= -(1+u)\,\frac{\partial u}{\partial\sigma} \\
&= -\frac{\partial}{\partial\sigma}(u+\frac{u^2}{2})\,.
\end{aligned} \tag{6}$$

Therefore,

$$xy = -u - \frac{u^2}{2} + \phi(s)\,. \tag{7}$$

The function $\phi(s)$ is fixed by the boundary condition (2),

$$1 = -5 - \frac{5^2}{2} + \phi(s) \quad \Rightarrow \quad \phi(s) = \frac{37}{2}\,. \tag{8}$$

In summary, the solution of (7) which also satisfies the boundary condition along I_0, is

$$u(x,y) = -1 + \sqrt{38 - 2\,x\,y}, \quad x\,y < 19\,. \tag{9}$$

Exercise IV.4: **An IBVP with an expansion zone**.

1. Consider the first order partial differential equation for the unknown $u = u(x,t)$,

$$\frac{\partial u}{\partial t} + a(u)\,\frac{\partial u}{\partial x} = 0\,, \tag{1}$$

where a is a function of u. Show that the solutions of the form $u(x,t) = f(x/t)$ are the constants and the generalized inverses of a, that is, the functions such the composition of a and f is the identity function, $a \circ f = I$.

2. Solve the IBVP for $u = u(x,t)$,

$$\begin{array}{lll} \text{(FE)} & \dfrac{\partial u}{\partial t} + e^u\,\dfrac{\partial u}{\partial x} = 0, & x > 0,\ t > 0 \\ \text{(IC)} & u(x,0) = 2, & x > 0 \\ \text{(BC)} & u(0,t) = 1, & t > 0\,. \end{array} \tag{2}$$

Solution:

1. If $u(x,t) = f(x/t)$, then

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\,f', \quad \frac{\partial u}{\partial x} = \frac{1}{t}\,f', \tag{3}$$

and therefore,

$$\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = \frac{f'}{t}\Big(-\frac{x}{t} + a\Big), \tag{4}$$

whence, either f is constant or

$$\frac{x}{t} = a = a(u) = a\Big(f(\frac{x}{t})\Big) \quad \Rightarrow \quad a \circ f = I\,. \tag{5}$$

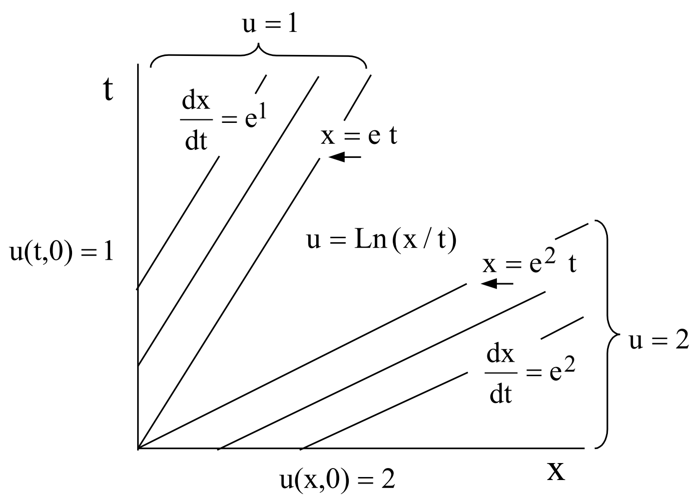


Figure IV.9 In the central fan where no characteristic exists, the solution is built heuristically. It connects continuously with the two zones where the solution carried out by the characteristics is constant.

2. Along the standard presentation, we would like the two relations,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} &+& \frac{\partial u}{\partial x}\, e^u \\ du &= \frac{\partial u}{\partial t}\, dt &+& \frac{\partial u}{\partial x}\, dx\,, \end{aligned} \tag{6}$$

to be identical. Therefore, we should have simultaneously $dx/dt = e^u$, and $du = 0$. In other words, the characteristic curves are

$$\frac{dx}{dt} = e^u = \text{constant}\,. \tag{7}$$

The construction of the characteristic network starts from the axes, Fig. IV.9.

There is no characteristic curve in a central fan. Still, we have a family of solutions via question 1. Since here the function a is the exponential, the inverse is the Logarithm. Continuity at the boundaries of the central fan is ensured simply by taking $u(x,t) = \mathrm{Ln}(x/t)$.

In summary,

$$u(x,t) = \begin{cases} 1, & 0 < x \le e\,t \\ \mathrm{Ln}(x/t), & e\,t \le x \le e^2\,t \\ 2, & e^2\,t \le x\,. \end{cases} \tag{8}$$

Note that we have not proved the uniqueness of the solution in the central fan. On the other hand, we can eliminate a jump from 1 to 0 along the putative shock line $X_s = \frac{1}{2}\,(1+0)\,t$, because this shock would not satisfy the entropy condition (IV.2.16).

Exercise IV.5: **An initial value problem (IVP) with a shock**.
Solve the initial value problem (IVP) for $u = u(x,t)$,

$$\begin{aligned} &(\text{FE}) \quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} &=& \; 0, \quad -\infty < x < \infty,\; t > 0 \\ &(\text{IC}) \qquad u(x,0) &=& \begin{cases} 1, & x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 \le x\,. \end{cases} \end{aligned} \tag{1}$$

Solution:
Along the standard presentation, we would like the two relations,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} &+& \frac{\partial u}{\partial x}\,u \\ du &= \frac{\partial u}{\partial t}\,dt &+& \frac{\partial u}{\partial x}\,dx\,, \end{aligned} \tag{2}$$

to be identical. Therefore, we should have simultaneously $dx/dt = u$, and $du = 0$. In other words, the characteristic curves are

$$\frac{dx}{dt} = u = \text{constant}\,. \tag{3}$$

The construction of the characteristic network starts from the x-axis, Fig. IV.10.

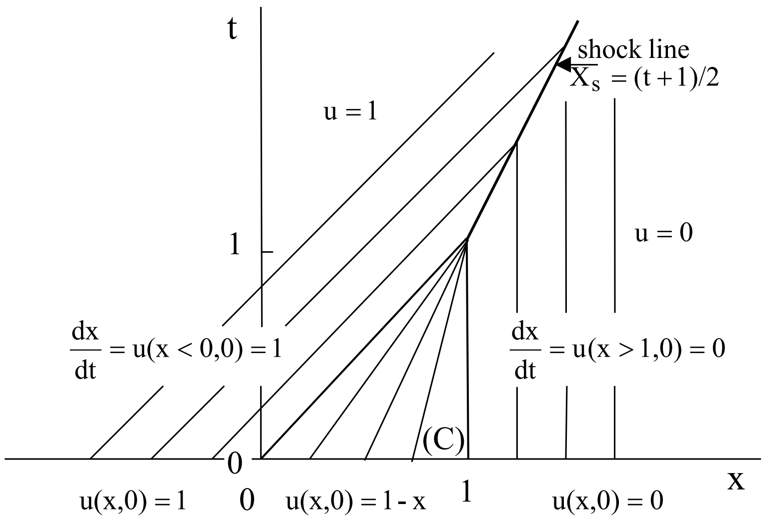


Figure IV.10 A shock develops to accommodate a faster information coming from behind. The shock line has equation $t = 2\,X_s - 1$, for $X_s > 1$. Elsewhere the solution is continuous.

The characteristics emanating from the x-axis for $x < 0$ carry the solution $u(x,t) = 1$, while the characteristics emanating from the x-axis for $x > 1$ carry the solution $u(x,t) = 0$.

However, we clearly have a problem, Fig. IV.10, because the above description implies that the characteristics cross each other, which is impossible. Consequently there is shock.

Let us first re-write the field equation as a conservation law,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{u^2}{2}\Big) = 0\,, \tag{4}$$

so as to identify the flux $q = q(u) = u^2/2$. The jump relation (IV.2.10) provides the speed of propagation of the shock line,

$$\frac{dX_s}{dt} = \frac{q_+ - q_-}{u_+ - u_-} = \frac{1}{2}\,(u_+ + u_-) = \frac{1}{2}\,, \tag{5}$$

the subscripts + and - denoting the two sides of the shock line. Therefore $X_s = t/2 +$ constant. The later constant is fixed by insisting that the point $(1, 1)$ belongs to the shock line. Therefore, the shock line is the semi-infinite segment,

$$X_s = \frac{t}{2} + \frac{1}{2}, \quad x \geq 1,\; t \geq 1\,. \tag{6}$$

To the left of the shock line, i.e. $x < (t + 1)/2$, the solution u is equal to 1, while it is equal to 0 to the right.

In the central fan (C), the slope of the characteristics dx/dt, which we know is constant, is equal to $u(x, 0) = 1 - x$. The tentative function,

$$u(x, t) = \frac{1-x}{1-t}, \quad x < 1,\; t < 1\,, \tag{7}$$

satisfies the field equation, has the proper slope at $t = 0$, and hence at any $t < 1$ since the slope is constant along characteristics, and fits continuously with the left and right characteristic networks.

Finally, note that the shock satisfies the entropy condition (IV.2.16).

Exercise IV.6: **A sudden surge in a river of Southern France**.

The height H and the horizontal velocity of water v in a long river, with a quasi horizontal bed, are governed by the equations of balance of mass and balance of horizontal momentum (here $g \sim 10$ m/s^2 is the gravitational acceleration):

$$\begin{aligned} &\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\,v) = 0, \\ &\frac{\partial}{\partial t}(Hv) + \frac{\partial}{\partial x}(H\,v^2 + \frac{g}{2}\,H^2) = 0\,. \end{aligned} \tag{1}$$

The horizontal velocity of water is $v_+ = 2\,\text{m/s}$ and the height is $H_+ = 1\,\text{m}$. A time $t = 0$, the height becomes suddenly equal to $H_- = 2\,\text{m}$ upstream $x \leq 0$, and it keeps that value at later times $t > 0$. Neglecting frictional resistance, bed slope, the local physical effects in the neighborhood of the shock, deduce the horizontal velocity v_- of water behind the shock, and the speed of displacement of the shock dX_s/dt.

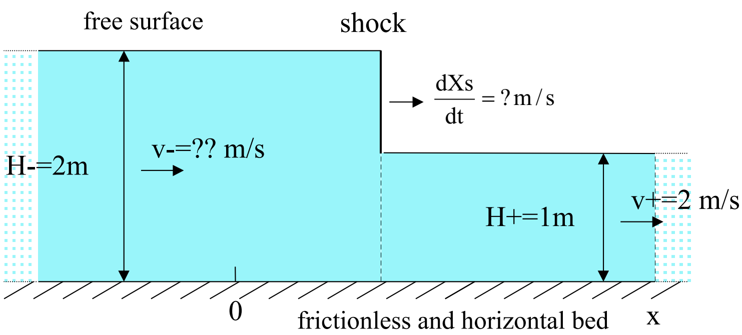


Figure IV.11 A shock develops to accommodate a faster information coming from upstream.

N.B. If the river bed is inclined downward with an angle $\theta > 0$, and if the coefficient of friction f is non zero, the rhs of the balance of momentum should be changed to $g\,H \sin\theta - f\,v^2$.

Solution:

The jump relation (IV.2.10) is applied to the two conservation equations,

$$\frac{dX_s}{dt} = \frac{(H\,v)_+ - (H\,v)_-}{H_+ - H_-} = \frac{(H\,v^2 + \frac{1}{2}g\,H^2)_+ - (H\,v^2 + \frac{1}{2}g\,H^2)_-}{(H\,v)_+ - (H\,v)_-}\,, \tag{2}$$

the subscripts + and - denoting the two sides of the shock line. Solving this equation for v_- yields,

$$\frac{v_- - v_+}{H_- - H_+} = \epsilon\,\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})}\,, \tag{3}$$

that is,

$$v_- = v_+ + \epsilon\,(H_- - H_+)\,\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})} \sim 4.74\,\text{m/s}\,, \tag{4}$$

from which follows the speed of propagation of the shock,

$$\frac{dX_s}{dt} = v_+ + \epsilon\,H_-\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})} \sim 7.48\,\text{m/s}\,. \tag{5}$$

In fact, there are two solutions to the problem as indicated by by $\epsilon = \pm 1$. The choice $\epsilon = +1$ is dictated by the entropy condition that implies that the upstream flow should be larger than the downstream flow.

Exercise IV.7: **Implicit solution to an initial value problem (IVP). Simple waves**.
The conservation law $\partial u/\partial t + \partial q(u)/\partial x = 0$ may also be written $\partial u/\partial t + a(u)\, \partial u/\partial x = 0$, with $a(u) = dq/du$. Let us assume $a(u) > 0$.
1. Show that the solution to the initial value problem (IVP) for $u = u(x,t)$,

$$\begin{array}{lrcl} \text{(FE)} & \dfrac{\partial u}{\partial t} + a(u)\,\dfrac{\partial u}{\partial x} & = & 0, \quad -\infty < x < \infty,\ t > 0 \\ \text{(IC)} & u(x,0) & = & u_0(x), \quad -\infty < x < \infty, \end{array} \tag{1}$$

is given implicitly under the format,

$$u = u_0(x - a(u)\,t)\,, \tag{2}$$

if

$$1 + u_0'(x - a(u)\,t)\,a'(u)\,t \neq 0\,. \tag{3}$$

Such a solution is a *forward simple wave*. Indeed, it travels at increasing x. Simple waves are waves that get distorted because their speed depends on the solution u.
2. Define similarly *backward* simple waves.

Solution:
1. The curve I_0 on which the data are given is the x-axis, so that one can choose the curvilinear abscissa of the curve I_0 to be $s = x$. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

Along the standard presentation, we would like the two relations,

$$\begin{array}{rclcl} 0 & = & \dfrac{\partial u}{\partial t} & + & \dfrac{\partial u}{\partial x}\,a(u) \\ \dfrac{du}{d\sigma} & = & \dfrac{\partial u}{\partial t}\,\dfrac{\partial t}{\partial \sigma} & + & \dfrac{\partial u}{\partial x}\,\dfrac{\partial x}{\partial \sigma}\,, \end{array} \tag{4}$$

to be identical. Therefore, the characteristic curves are defined by the relations $dx/d\sigma = a(u)$ and $du = 0$, that is, the solution u is constant along a characteristic,

$$\frac{\partial t}{\partial \sigma} = 1, \quad \frac{\partial x}{\partial \sigma} = a(u), \quad \frac{\partial u}{\partial \sigma} = 0\,, \tag{5}$$

and, switching from the coordinates (x,t) to the coordinates (s,σ),

$$t(\sigma = 0, s) = 0, \quad x(\sigma = 0, s) = s, \quad u(\sigma = 0, s) = u_0(s)\,. \tag{6}$$

The construction of the characteristic network starts from the x-axis, Fig. IV.12.

From $(5)_3$ and $(6)_3$ results

$$u(\sigma, s) = u(\sigma = 0, s) = u_0(s)\,. \tag{7}$$

Relations $(5)_1$ and $(6)_1$ imply,

$$t = \sigma + \phi(s) \overset{(6)_1}{=} \sigma\,. \tag{8}$$

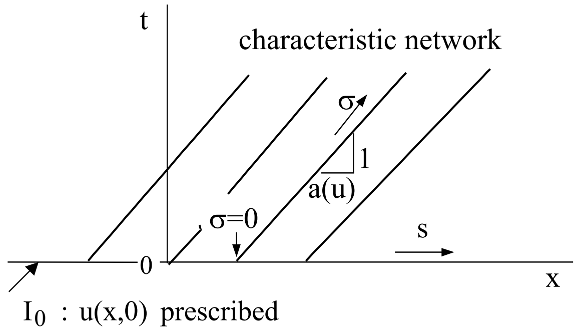


Figure IV.12 Curvilinear coordinates associated with an initial value problem.

In addition, relation $(5)_2$ can be integrated with help of $(6)_2$ and (7),

$$x = \sigma\, a(u_0) + \psi(s) \stackrel{(6)_2}{=} \sigma\, a(u_0) + s\,. \tag{9}$$

Finally, collecting these relations yields an implicit equation for u,

$$u \stackrel{(7)}{=} u_0(s) \stackrel{(9)}{=} u_0\Big(x - \sigma\, a(u)\Big) \stackrel{(8)}{=} u_0\Big(x - t\, a(u)\Big)\,. \tag{10}$$

Let us now try to solve this equation by differentiation,

$$du = \Big(dx - dt\, a(u) - t\, a'(u)\, du\Big)\, u_0'\,, \tag{11}$$

from which we can extract du,

$$(1 + t\, a'(u)\, u_0')\, du = (dx - dt\, a(u))\, u_0'\,, \tag{12}$$

only under the condition (3), which is in fact a particular form of the so-called theorem of implicit functions.

2. By deduction, backward simple waves are defined implicitly by the relation,

$$u = u_0(x + a(-u)\, t)\,, \tag{13}$$

and obey the PDE,

$$\frac{\partial u}{\partial t} - a(-u)\, \frac{\partial u}{\partial x} = 0\,. \tag{14}$$

Exercise IV.8: **Transient flow of a compressible fluid at constant pressure: a weakly coupled problem**.

Under adiabatic conditions, the velocity $u(x,t)$, mass density $\rho(x,t)$ and internal energy per unit volume $e(x,t)$ during the one-dimensional flow of a compressible fluid at constant pressure p are governed by the three coupled nonlinear partial differential equations, for $-\infty < x < \infty,\ t > 0$,

$$\begin{aligned} \text{momentum equation}: &\quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 \\ \text{mass conservation}: &\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ \text{energy equation}: &\quad \frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(ue) + p\,\frac{\partial u}{\partial x} = 0\,, \end{aligned} \tag{1}$$

subject to the initial data:

$$u(x,0) = u_0(x); \quad \rho(x,0) = \rho_0(x); \quad e(x,0) = e_0(x), \quad -\infty < x < \infty\,. \tag{2}$$

The function $u_0(x)$ is assumed to be differentiable and the functions $\rho_0(x)$ and $e_0(x)$ to be continuous.

1. Solve this system for the three unknowns velocity $u(x,t)$, density $\rho(x,t)$ and internal energy $e(x,t)$.

2. The above three equations have been worked out and organized so as to be brought into an easily solvable weakly coupled system. Show that this system of equations actually derives from the conservation laws of mass, momentum, and total (internal plus kinetic) energy, namely in turn,

$$\begin{aligned} \text{mass conservation}: &\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ \text{momentum balance}: &\quad \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho\, u^2) = 0 \\ \text{energy conservation}: &\quad \frac{\partial}{\partial t}(e + \tfrac{1}{2}\rho\, u^2) + \frac{\partial}{\partial x}\Big(u\,(e + \tfrac{1}{2}\rho\, u^2 + p)\Big) = 0\,. \end{aligned} \tag{3}$$

Solution:

Note that the order in which we have written the three equations matters. It is important to recognize that the first equation is independent and involves the sole unknown u, while the two other equations involve two unknowns. Therefore, the analysis begins by the uncoupled equation.

1.1 The first equation is a particular case of Exercise IV.7, with $a(u) = u$. From (8) and (9) of this Exercise, the characteristics are

$$x - u\,t = s\,, \tag{4}$$

and the solution is given implicitly by the equation,

$$u = u_0(s) = u_0(x - u\,t)\,. \tag{5}$$

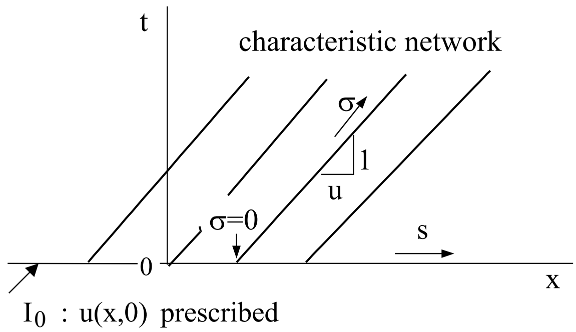


Figure IV.13 Curvilinear coordinates associated with the initial value problem.

1.2 The second equation may be re-written,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\, u = -\rho\, \frac{\partial u}{\partial x}\,. \tag{6}$$

The right hand side being known, this equation has the same characteristics defined by $dx/dt = u$ and $\sigma = t$, as the first one. Along a characteristic,

$$\frac{\partial \rho}{\partial t}\,\frac{\partial t}{\partial \sigma} + \frac{\partial \rho}{\partial x}\,\frac{\partial x}{\partial \sigma} = \frac{d\rho}{d\sigma}\,. \tag{7}$$

On comparing the last two equations,

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{d\sigma} = -\rho\, \frac{\partial u}{\partial x}\,. \tag{8}$$

Now, from (5),

$$\frac{\partial u}{\partial x} = u_0'(s)\,\Big(1 - t\,\frac{\partial u}{\partial x}\Big)\,, \tag{9}$$

and therefore,

$$\frac{\partial u}{\partial x} = \frac{u_0'(s)}{1 + t\,u_0'(s)}\,. \tag{10}$$

Insertion of this relation in (8),

$$\frac{1}{\rho}\,\frac{d\rho}{d\sigma} = -\frac{u_0'(s)}{1 + \sigma\,u_0'(s)}\,, \tag{11}$$

and integration with respect to σ, accounting for the fact that σ and s are independent variables, yields

$$\rho(\sigma, s) = \frac{\rho_0(s)}{1 + \sigma\,u_0'(s)}\,. \tag{12}$$

Return to the coordinates (x, t) uses the relations $t = \sigma$ and $s = x - u\,t$.

1.3 The third equation can be recast in terms of a new unknown $E(x, t) = e(x, t) + p$, namely

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial x}\, u = -E\, \frac{\partial u}{\partial x}\,, \tag{13}$$

with initial data $E_0(x) = E(x,0) = e(x,0) + p$. This equation is clearly identical in form to (6), and has therefore solution,

$$E(\sigma,s) = \frac{E_0(s)}{1+\sigma\,u_0'(s)} \quad \Rightarrow \quad e(\sigma,s) = \frac{e_0(s) - p\,\sigma\,u_0'(s)}{1+\sigma\,u_0'(s)}\,. \tag{14}$$

2. Just combine the three equations.

For those who want to know more.

1. As indicated above, the expressions of the energy equation assume adiabatic conditions. More generally, the energy equation is contributed by heat exchanges with the surroundings, heat sources and heat wells.
2. The set of equations (14) actually holds even for a space and time dependent pressure.

Exercise IV.9: **Traffic flow along a road segment devoid of entrances and exits**.

We have seen in Sect.IV.2.1 that the flux $q(x,t)$ in a conservation law for a density $u(x,t)$ is linked constitutively with this density, namely $q = q(u) = q(u(x,t))$. To illustrate the issue, let us consider the traffic of vehicles along a road segment devoid of entrances and exits. In absence of vehicles $u = 0$, the flux of course vanishes, $q(0) = 0$. On the other hand, one may admit that vehicles need some fluidity to move: at the maximum density u_m, bumper to bumper, the flux vanishes, $q(u_m) = 0$. Consequently the relation $q = q(u)$ can not be linear. The simplest possibility is perhaps,

$$\frac{q(u)}{q_m} = \begin{cases} 4\frac{u}{u_m}\Big(1 - \frac{u}{u_m}\Big), & 0 \leq \frac{u}{u_m} \leq 1 \\ 0, & 1 < \frac{u}{u_m} \end{cases} . \tag{1}$$

The speed of the traffic,

$$v = \frac{q}{u} = u_m \Big(1 - \frac{u}{u_m}\Big), \quad u_m \equiv 4\,\frac{q_m}{u_m}\,, \tag{2}$$

is therefore an affine function of the density.

1. Define the characteristics. Show Property (P): the slopes of the characteristics are constant and the density u is constant along a characteristic.

2. Consider the effects of a traffic light located at the position $x = 0$. The light has been red for some time. Behind the light, the density is maximum while there is no vehicle ahead. The light turned green at time $t = 0$. Thus the initial density is $u(x, t = 0) = u_m\, \mathcal{H}(-x)$. Obtain the density $u(x,t)$ at later times $t > 0$. Consider also the trajectory of a particular vehicle.

3. The light has been green for some time, and the density is uniform and smaller than u_m. At time $t = 0$, the light turns red, and the density becomes instantaneously maximum at $x = 0_-$. Describe the shock that propagates to the rear.

N.B. This exercise could be interpreted as well as describing a particle flow in a tube with opening and closing gates.

Exercise IV.10: **Transmission lines subject to initial data or boundary data**.

We return to the transmission line problem described in Exercise III.6. The equations governing the current $I(x,t)$ and potential $V(x,t)$ in a transmission line of axis x can be cast in the format of a linear system of two partial differential equations,

$$\begin{aligned} L\frac{\partial I}{\partial t} + \frac{\partial V}{\partial x} + R\,I &= 0 \\ C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial x} + G\,V &= 0 \end{aligned} \tag{1}$$

1. For an infinite transmission line,

$$-\infty \quad \cdots \Longrightarrow\!\!\!= \cdots \quad +\infty \tag{2}$$

solve the initial value problem, defined by the field equations (1) for $-\infty < x < \infty$, $t > 0$, subjected to the initial conditions,

$$I(x, t = 0) = I_0(x), \quad V(x, t = 0) = V_0(x), \quad -\infty < x < \infty\,. \tag{3}$$

Use the method of characteristics together with the normal form of the equations phrased in terms of alternative variables provided by equations (6) of Exercise III.6.

Obtain an integral solution in the general case, and an analytical solution for a distortionless line, $RC = LG$, for which the set of field equations uncouples. Highlight the parameters that characterize propagation and time decay.

2. Consider now a semi-infinite transmission line extending over $x > 0$,

$$0 \sqsubset\!\!= \cdots \quad \infty \tag{4}$$

Solve the boundary value problem defined by the field equations (1) for $0 < x < \infty$, $t > 0$, subjected to the boundary conditions,

$$I(x = 0, t) = I_0(t), \quad V(x = 0, t) = V_0(t), \quad t > 0\,. \tag{5}$$

Restrict the analysis to a distortionless line, $RC = LG$.