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Particles and fields in fluid turbulence

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The understanding of fluid turbulence has considerably progressed in recent years. The application of the methods of statistical mechanics to the description of the motion of fluid particles, i.e., to the Lagrangian dynamics, has led to a new quantitative theory of intermittency in turbulent transport. The first analytical description of anomalous scaling laws in turbulence has been obtained. The underlying physical mechanism reveals the role of statistical integrals of motion in nonequilibrium systems. For turbulent transport, the statistical conservation laws are hidden in the evolution of groups of fluid particles and arise from the competition between the expansion of a group and the change of its geometry. By breaking the scale-invariance symmetry, the statistically conserved quantities lead to the observed anomalous scaling of transported fields. Lagrangian methods also shed new light on some practical issues, such as mixing and turbulent magnetic dynamo.

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"Well," said Pooh, "we keep looking for Home and not finding it, so I thought that if we looked for this Pit, we'd be sure not to find it, which would be a Good Thing, because then we might find something that we weren't looking for, which might be just what we were looking for, really."

A. A. Milne, Tigger is unbounced

I. INTRODUCTION

Turbulence is the last great unsolved problem of classical physics which has evaded physical understanding and systematic description for many decades. Turbulence is a state of a physical system with many degrees of freedom strongly deviating from equilibrium. The first obstacle to its understanding stems from the large number of degrees of freedom actively involved in the problem. The scale of injection, where turbulence is excited, usually differs dramatically from the scale of damping, where dissipation takes place. Nonlinear interactions strongly couple the degrees of freedom by transferring excitations from the injection to the damping scale throughout the so-called inertial range of scales. The ensuing complicated and irregular dynamics calls for a statistical description. The main physical problem is to understand to what extent the statistics in the inertial interval is universal, i.e., independent of the conditions of excitation and dissipation. In such general formulation, the issue goes far beyond fluid mechanics, even though the main examples and experimental data are provided by turbulence in continuous media. From the standpoint of theoretical physics, turbulence is a nonequilibrium field-theoretical problem with many strongly interacting degrees of freedom. The second deeply rooted obstacle to its understanding is that far from equilibrium we do not possess any general guiding rule, like the Gibbs principle in equilibrium statistical physics. Indeed, to describe the single-time statistics of equilibrium systems, the only thing we need is the knowledge of dynamic integrals of motion. Then, our probability distribution in phase space is uniform over the surfaces of constant integrals of motion. Dynamically conserved quantities play an important role in turbulence description, too, as they flow throughout the inertial range in a cascadelike process. However, the conserved quantity alone does not allow one to describe the whole statistics but only a single correlation function which corresponds to its flux. The major problem is to obtain the rest of the statistics.

In every case, the starting point is to identify the dynamical integral of motion that cascades through the inertial interval. Let us consider the forced three-dimensional (3D) Navier-Stokes equation

$$\partial_t \mathbf{v}(\mathbf{r},t)+\mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t)-\nu\nabla^2\mathbf{v}(\mathbf{r},t) =-\nabla p(\mathbf{r},t)+f(\mathbf{r},t), \tag{1}$$

supplemented by the incompressibility condition $\nabla\cdot\mathbf{v}=0$. An example of injection mechanism is a random large-scale forcing $\mathbf{f}(\mathbf{r},t)$ with correlation length L. The relevant integral of motion, conserved in the absence of injection and dissipation, is kinetic energy $\int v^2 d\mathbf{r}/2$ and the quantity which cascades throughout the inertial interval is energy density in wave-number space. The energy flux-constancy relation was derived in Kolmogorov (1941) and it involves the third-order moment of the longitudinal velocity increments:

$$\langle\{[\mathbf{v}(\mathbf{r},t)-\mathbf{v}(\mathbf{0},t)]\cdot\mathbf{r}/r\}^3\rangle\equiv\langle(\Delta_r v)^3\rangle=-\frac{4}{5}\overline{\epsilon_v}r. \tag{2}$$

The separation r is supposed to lie in the inertial interval, ranging from the injection scale L down to the viscous dissipation scale. The major physical assumption made to derive the so-called 4/5 law is that the mean energy dissipation rate $\overline{\epsilon_v}=\nu\langle(\nabla\mathbf{v})^2\rangle$ has a nonzero limit as the viscosity ν tends to zero. This clearly points to the nonequilibrium flux nature of turbulence. The assumption of finite dissipation gives probably the first example of what is called "anomaly" in modern field-theoretical language: A symmetry of the inviscid equation (here, time-reversal invariance) is broken by the presence of the viscous term, even though the latter might have been expected to become negligible in the limit of vanishing viscosity. Note that the 4/5 law (2) implies that the third-order moment is universal, that is, it depends on the injection and the dissipation only via the mean energy injection rate, coinciding with $\overline{\epsilon_v}$ in the stationary state. To obtain the rest of the statistics, a natural first step made by Kolmogorov himself was to assume the statistics in the inertial range be scale invariant. The scale invariance amounts to assuming that the probability distribution function (PDF) of the rescaled velocity differences $r^{-h}\Delta_r v$ can be made r independent for an appropriate h. The nth order moment of the longitudinal velocity increments $\langle(\Delta_r v)^n\rangle$ (structure functions) would then depend on the separation as a power law r^{σ_n} with the "normal scaling" behavior $\sigma_n=hn$. The rescaling exponent may be determined by the flux law, e.g., $h=1/3$ for 3D Navier-Stokes turbulence. In the original Kolmogorov theory, the scale invariance was in fact following from the postulate of complete universality: the dependence on the injection and the dissipation is carried entirely by $\overline{\epsilon_v}$ not only for the third-order moment but for the whole statistics of the velocity increments. The velocity difference PDF could then involve only the dimensionless combination $(\overline{\epsilon_v}r)^{-1/3}\Delta_r v$ and would be scale invariant. There are cases, like weakly nonlinear wave turbulence (Zakharov *et al.*, 1992), where both scale invariance and complete universality are assured by the fact that the statistics in the inertial range is close to Gaussian. That does not hold for strongly nonlinear systems. Already in 1942, L. D. Landau pointed out that all the velocity structure functions (except the third one) are averages of nonlinear functions of the flux. They are therefore sensitive to its fluctuations, which depend on the specific injection mechanisms. Consequently, the velocity statistics in the inertial range may have nonuniversal features.

Experiments do not support scale invariance either. The structure functions are in fact found experimentally to have a power-law dependence on the separation r. However, the PDF of the velocity differences at various separations cannot be collapsed one onto another by simple rescaling and the scaling exponent σ_n of the structure functions is a nonlinear concave function of the order n. As the separation decreases in the inertial range, the PDF becomes more and more non-Gaussian, with a sharpening central peak and a tail that becomes longer and longer. In other words, the smaller the separations considered, the higher the probability of very weak and strong fluctuations. This manifests itself as a sequence of strong fluctuations alternating with quiescent periods, which is indeed observed in turbulence signals and is known as the phenomenon of intermittency. The violation of the dimensional predictions for the scaling laws is referred to as "anomalous scaling" for it reflects, again, a symmetry breaking. The Euler equation is scale invariant and the scales of injection and dissipation are supposed to be very large and small (formally, the limits to infinity and zero should be taken). However, the dynamics of turbulence is such that the limits are singular and scale invariance is broken. The presence of a finite injection scale L, irrespective of its large value, is felt throughout the inertial range precisely via the anomalies $\langle(\Delta_r v)^n\rangle \propto (\bar{\epsilon}_v r)^{n/3}(L/r)^{n/3-\sigma_n}$.

The non-Gaussianity of the statistics, the anomalous scaling, and the intermittency of the field occur as a rule rather than an exception in the context of fluid dynamics. The same phenomenology is observed in many other physical systems. An incomplete list includes compressible Navier-Stokes turbulence, Burgers' turbulence, and scalar and magnetic fields. Examples of scalar fields are provided by the temperature of a fluid, the humidity in the atmosphere, the concentration of chemical or biological species. The advection-diffusion equation governing the transport of a nonreacting scalar field by an incompressible velocity is

$$\partial_t \theta(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\cdot\nabla\theta(\mathbf{r},t) - \kappa\nabla^2\theta(\mathbf{r},t) = \varphi(\mathbf{r},t), \tag{3}$$

where φ describes the sources. For scalar dynamics, the space integral of any function of θ is conserved in the absence of sources and diffusion. In their presence, the corresponding relation for the flux of θ^2 was derived in Yaglom (1949):

$$\langle\{[\mathbf{v}(\mathbf{r},t)-\mathbf{v}(\mathbf{0},t)]\cdot\mathbf{r}/r\}[\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^2\rangle = -\frac{4}{3}\bar{\epsilon}r. \tag{4}$$

The major physical assumption is again that the mean scalar dissipation rate $\bar{\epsilon} = \kappa\langle(\nabla\theta)^2\rangle$ remains finite even in the limit where the molecular diffusivity κ vanishes. Consider the particular case when the advecting velocity $\mathbf{v}$ satisfies the 3D Navier-Stokes equation. Assuming again scale-invariance, the flux relations (2) and (4) would imply that the scaling exponent of both the velocity and the scalar field is 1/3. As it was expected for the velocity, the scalar structure functions $S_n(r) = \langle[\theta(\mathbf{r},t) - \theta(\mathbf{0},t)]^n\rangle$ would then depend on the separation as power laws r^{ζ_n} with $\zeta_n = n/3$. Experiments indicate that scale invariance is violated for a scalar field as well, that is $\zeta_n \neq n/3$. More importantly, the intermittency of the scalar is much stronger than that of the velocity, in particular, $n/3-\zeta_n$ is substantially larger than $n/3-\sigma_n$. It was a major intuition of R. H. Kraichnan to realize that the passive scalar could then be intermittent even in the absence of any intermittency of the advecting velocity.

The main ambition of the modern theory of turbulence is to explain the physical mechanisms of intermittency and anomalous scaling in different physical systems, and to understand what is really universal in the inertial-interval statistics. It is quite clear that strongly nonequilibrium systems generally do not enjoy the same degree of universality as those in equilibrium. In the absence of a unified approach to nonequilibrium situations, one tries to solve problems on a case-by-case basis, with the hope to learn if any universal guiding principle may be recognized. It is in solving the particular problems of passive scalar and magnetic fields that an important step in general understanding of turbulence has been recently made. The language most suitable for the description of the systems turned out to be the Lagrangian statistical formalism, i.e., the description of the motion of fluid particles. This line of analysis, pioneered by L. F. Richardson and G. I. Taylor in the 1920s and later developed by R. H. Kraichnan and others, has been particularly effective here. The results differ from case to case. Some fields are non-Gaussian but scale invariance is not broken, while others have turned out to be amenable to the first ever analytical description of anomalous scaling laws. The anomalous exponents have been found to be universal, but not the constants appearing in the prefactors of generic correlation functions. This has provided a quantitative clarification of Landau's previously mentioned remark and of the aspects of turbulence statistics that may still be expected to be universal. More importantly, the anomalous scaling has been traced to the existence of statistical integrals of motion. The mechanism is quite robust and relevant for transport by generic turbulent flows. The nature of those integrals of motion strongly differs from that of the dynamic conservation laws that determine equilibrium statistics. For any finite number of fluid particles, the conserved quantities are functions of the interparticle separations that are statistically preserved as the particles are transported by the random flow. For example, at scales where the velocity field is spatially smooth, the average distance R between two particles generally grows exponentially, while the ensemble average $\langle R^{-d}\rangle$ is asymptotically time independent in a statistically isotropic d-dimensional random flow. The integrals of motion change with the number of particles and generally depend nontrivially on the geometry of their configurations. In the connection between the advected fields and the particles, the order of the correlation functions is equal to the number of particles and the anomalous scaling issue may be recast as a problem in statistical geometry. The nonlinear behavior of the scaling exponents with the order is then due to the dependence of the integrals of motion on the number of particles. The existence of statistical conservation laws signals that the Lagrangian dynamics keeps trace of the particle initial configuration throughout the evolution. This memory is what makes the correlation functions at any small scale sensitive to the presence of a finite injection length L. We believe that, more generally, the notion of statistical integrals of motion is a key to understand the universal part of the steady-state statistics for systems far from equilibrium.

The aim of this review is a description of fluid turbulence from the Lagrangian viewpoint. Classical literature on Lagrangian dynamics mostly concentrated on turbulent diffusion and pair dispersion, i.e., the distance traveled by one particle or the separation between two particles as a function of time. By contrast, in that general picture that has emerged recently, the evolution of the multiparticle-configuration geometry takes center stage. The main body of the review will present these novel aspects of Lagrangian dynamics and their consequences for the advected fields. We shall adhere to the following plan. The knowledge accumulated on one- and two-particle dynamics has been extensively covered in literature (Pope, 1994; Majda and Kramer, 1999). The objective of the first three parts of Sec. II is to point out a few fundamental issues, with particular attention to the basic differences between the cases of spatially smooth and nonsmooth velocity fields. We then proceed to the multiparticle statistics and the analysis of hidden statistical conservation laws that cause the breakdown of scale invariance. Most of this analysis is carried out under the assumption of a prescribed statistics of the velocity field. In Sec. III we shall analyze passive scalar and vector fields transported by turbulent flow and what can be inferred about their statistics from the motion of fluid particles. In Sec. IV, we briefly discuss the Lagrangian dynamics in the Burgers and the Navier-Stokes equations. The statistics of the advecting velocity is not prescribed anymore, but it results from nonlinear dynamics. Conclusions focus on the impact of the results presented in this review on major directions of future research. Readers from other fields of physics interested mainly in the breakdown of scale invariance and statistical conservation laws may restrict themselves to Secs. II.C, II.E, III.C, and V.

II. PARTICLES IN FLUID TURBULENCE

As explained in the Introduction, understanding the properties of transported fields involves the analysis of the behavior of fluid particles. We have therefore decided to first present results on the time-dependent statistics of the Lagrangian trajectories $\mathbf{R}_n(t)$ and to devote the subsequent Sec. III to the description of transported fields. In the present section we sequentially increase the number of particles involved in the problem. We start from a single trajectory whose effective motion is a simple diffusion at times longer than the velocity correlation time in the Lagrangian frame (Sec. II.A). We then move to two particles. The separation law of two close trajectories depends on the scaling properties of the velocity field $\mathbf{v}(\mathbf{r},t)$. If the velocity is smooth, that is $|\mathbf{v}(\mathbf{R}_n)-\mathbf{v}(\mathbf{R}_m)|\propto|\mathbf{R}_n-\mathbf{R}_m|$, then the initial separation grows exponentially in time (Sec. II.B). The smooth case can be analyzed in much detail using the large deviation arguments presented in Sec. II.B.1. The reader mainly interested in applications to transported fields might wish to take the final results (21) and (27) for granted, skipping their derivation and the analysis of the few solvable cases where the large deviations may be calculated exactly. If the velocity is nonsmooth, that is, $|\mathbf{v}(\mathbf{R}_n)-\mathbf{v}(\mathbf{R}_m)|\propto|\mathbf{R}_n-\mathbf{R}_m|^{\alpha}$ with $\alpha<1$, then the separation distance between two trajectories grows as a power of time (Sec. II.C), as first observed by Richardson (1926). We discuss important implications of such a behavior on the nature of the Lagrangian dynamics. The difference between the incompressible flows, where the trajectories generally separate, and compressible ones, where they may cluster, is discussed in Sec. II.D. Finally, in the consideration of three or more trajectories, the new issue of geometry appears. Statistical conservation laws come to light in two-particle problem and then feature prominently in the consideration of multiparticle configurations. Geometry and statistical conservation laws are the main subject of Sec. II.E. Although we try to keep the discussion as general as possible, much of the insight into the trajectory dynamics is obtained by studying simple random ensembles of synthetic velocities where exact calculations are possible. The latter serve to illustrate the general features of the particle dynamics.

A. Single-particle diffusion

The Lagrangian trajectory $\mathbf{R}(t)$ of a fluid particle advected by a prescribed incompressible velocity field $\mathbf{v}(\mathbf{r},t)$ in d space dimensions and undergoing molecular diffusion with diffusivity κ is governed by the stochastic equation (Taylor, 1921), customarily written for differentials:

$$d\mathbf{R}=\mathbf{v}(\mathbf{R},t)\,dt+\sqrt{2\kappa}\,d\boldsymbol{\beta}(t). \tag{5}$$

Here, $\boldsymbol{\beta}(t)$ is the d-dimensional standard Brownian motion with zero average and covariance function $\langle\beta^i(t)\beta^j(t')\rangle=\delta^{ij}\min(t,t')$. The solution of Eq. (5) is fixed by prescribing the particle position at a fixed time, e.g., the initial position $\mathbf{R}(0)$.

The simplest instance of Eq. (5) is the Brownian motion, where the advection is absent. The probability density $\mathcal{P}(\Delta\mathbf{R};t)$ of the displacement $\Delta\mathbf{R}(t)=\mathbf{R}(t)-\mathbf{R}(0)$ satisfies the heat equation $(\partial_t-\kappa\boldsymbol{\nabla}^2)\mathcal{P}=0$ whose solution is the Gaussian distribution $\mathcal{P}(\Delta\mathbf{R};t)=(4\pi\kappa t)^{-d/2}\exp[-(\Delta\mathbf{R})^2/(4\kappa t)]$. The other limiting case is pure advection without noise. The properties of the displacement depend then on the specific trajectory under consideration. We shall always work in the frame of reference with no mean flow. We assume statistical homogeneity of the Eulerian velocities which implies that the Lagrangian velocity $\mathbf{V}(t)=\mathbf{v}[\mathbf{R}(t),t]$ is statistically independent of the initial position. If, additionally, the Eulerian velocity is statistically stationary, then so is the Lagrangian one.[1] The single-time expectations of the Lagrangian velocity coincide in particular with those of the Eulerian one, e.g., $\langle\mathbf{V}(t)\rangle=\langle\mathbf{v}\rangle=0$. The relation between the multitime statistics of the Eulerian and the Lagrangian velocities is, however, quite involved in the general case.

For $\kappa=0$, the mean-square displacement satisfies the differential equation:

$$\frac{d}{dt}\langle(\Delta\mathbf{R}(t))^2\rangle=2\int_0^t\langle\mathbf{V}(t)\cdot\mathbf{V}(s)\rangle ds$$
$$=2\int_0^t\langle\mathbf{V}(0)\cdot\mathbf{V}(s)\rangle ds, \tag{6}$$

where the second equality uses the stationarity of $\mathbf{V}(t)$. The behavior of the displacement is crucially dependent on the range of temporal correlations of the Lagrangian velocity. Let us define the Lagrangian correlation time as

$$\tau=\frac{\int_0^\infty\langle\mathbf{V}(0)\cdot\mathbf{V}(s)\rangle ds}{\langle\mathbf{V}^2\rangle}. \tag{7}$$

The value of τ provides a measure of the Lagrangian velocity memory. Divergence of τ is symptomatic of persistent correlations. As we shall discuss in the sequel, no general relation between the Eulerian and the Lagrangian correlation times can be established but for the case of short-correlated velocities. For times $t\ll\tau$, the two-point function in Eq. (6) is approximately equal to $\langle\mathbf{v}^2\rangle$ and the particle transport is ballistic: $\langle(\Delta\mathbf{R})^2\rangle\simeq\langle\mathbf{v}^2\rangle t^2$. When the Lagrangian correlation time is finite, a generic situation in a turbulent flow, an effective diffusive regime arises for $t\gg\tau$ with $\langle(\Delta\mathbf{R})^2\rangle=2\langle\mathbf{v}^2\rangle\tau t$ (Taylor, 1921). The particle displacements over time segments spaced by distances much larger than τ are indeed almost independent. At long times, the displacement $\Delta\mathbf{R}$ behaves then as a sum of many independent variables and falls into the class of stationary processes governed by the central limit theorem. In other words, the displacement for $t\gg\tau$ becomes a Brownian motion in d dimensions with

$$\langle\Delta R^i(t)\Delta R^j(t)\rangle\simeq 2D_e^{ij}t, \tag{8}$$

where

$$D_e^{ij}=\frac{1}{2}\int_0^\infty\langle V^i(0)V^j(s)+V^j(0)V^i(s)\rangle ds. \tag{9}$$

[1]This follows by averaging the expectations involving $\mathbf{V}(t+\tau)$ over the initial position $\mathbf{R}(0)$ (on which they do not depend) and by the change of variables $\mathbf{R}(0)\mapsto\mathbf{R}(\tau)$ under the velocity ensemble average. The argument requires the incompressibility of the velocity, see Sec. II.D.

The same arguments carry over to the case of a nonvanishing molecular diffusivity. The symmetric second-order tensor D_e^{ij} describes the effective diffusivity (also called eddy diffusivity). The trace of D_e^{ij} is equal to the long-time value $\langle\mathbf{v}^2\rangle\tau$ of the integral in Eq. (6), while its tensorial properties reflect the rotational symmetries of the velocity field. If it is isotropic, the tensor reduces to a diagonal form characterized by a single scalar value. The main problem of turbulent diffusion is to obtain the effective diffusivity tensor, given the velocity field $\mathbf{v}$ and the value of the diffusivity κ. Exhaustive reviews of the problem are available in the literature (Bensoussan *et al.*, 1978; Pope, 1994; Fannjiang and Papanicolaou, 1996; Majda and Kramer, 1999).

The other general issue in turbulent diffusion is about the conditions on the velocity field ensuring the Lagrangian correlation time τ be finite and an effective diffusion regime take place for large enough times. A sufficient condition (Kraichnan, 1970; Avellaneda and Majda, 1989; Avellaneda and Vergassola, 1995) is that the vector potential variance $\langle\mathbf{A}^2\rangle$ is finite, where the 3D incompressible velocity $\mathbf{v}=\nabla\times\mathbf{A}$. Similar conditions are valid for any space dimension. The condition $\kappa\neq 0$ is essential to the validity of the previous result, as shown by the counter example of Rayleigh-Bénard convective cells, see, e.g., Normand *et al.* (1977). In the absence of molecular noise, the particle circulates forever in the same convective cell, with no diffusion taking place at any time. This provides an example of subdiffusion: the integral in Eq. (6) goes to zero as $t\to\infty$ and the growth of the mean-square displacement is slower than linear. Note that any finite molecular diffusivity, however small, creates thin diffusive layers at the boundaries of the cells; particles can then jump from one cell to another and diffuse. Subdiffusion is particularly relevant for static 2D flows, where tools borrowed from percolation/statistical topography find most fruitful applications (Isichenko, 1992). Trapping effects required for subdiffusion are, generally speaking, favored by the compressibility of the velocity field, e.g., in random potentials (Bouchaud and Georges, 1990). Subdiffusive effects are expected to be overwhelmed by chaotic mixing in flows leading to Lagrangian chaos, i.e., to particle trajectories that are chaotic in the absence of molecular diffusion (Ottino, 1989; Bohr *et al.*, 1998). This is the generic situation for 3D and 2D time-dependent incompressible flows.

Physical situations having an infinite Lagrangian correlation time τ correspond to superdiffusive transport: divergences of the integral in Eq. (6) as $t\to\infty$ signal that the particle transport is faster than diffusive. A classical example of such behavior is the class of parallel flows presented by Matheron and de Marsily (1980). If the large-scale components of the velocity field are sufficiently strong to make the particle move in the same direction for arbitrarily long periods the resulting mean-square displacement grows more rapidly than t. Other simple examples of superdiffusive motion are Lévy-type models (Geisel *et al.*, 1985; Shlesinger *et al.*, 1987). A detailed review of superdiffusive processes in Hamiltonian systems and symplectic maps can be found in Shlesinger *et al.* (1993).

Having listed different subdiffusive and superdiffusive cases, from now on we shall be interested in random turbulent flows with finite Lagrangian correlation times, which are experimentally known to occur for sufficiently high Reynolds numbers (Pope, 1994). For the long-time description of the diffusion in such flows, it is useful to consider the extreme case of random homogeneous and stationary Eulerian velocities with a short correlation time. The formal way to get these processes is to change the time scale by taking the scaling limit $\lim_{\mu\to\infty}\mu^{1/2}\mathbf{v}(\mathbf{r},\mu t)$, i.e., considering the process as viewed in a sped-up film. We assume that the connected correlation functions[2] decay fast enough when time differences increase. The elementary consequences of those assumptions are the existence of the long-time asymptotic limit and the fact that it is governed by the central limit theorem. When $\mu\to\infty$, we recover a velocity field which is Gaussian and white in time, characterized by the two-point function

$$\langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t')\rangle = 2\,\delta(t-t')D^{ij}(\mathbf{r}-\mathbf{r}'). \tag{10}$$

The advection by such velocity fields was first considered by Kraichnan (1968) and it is common to call the Gaussian ensemble of velocities with two-point function (10) the Kraichnan ensemble. For the Kraichnan ensemble, the Lagrangian velocity $\mathbf{V}(t)$ has the same white-noise temporal statistics as the Eulerian one $\mathbf{v}(\mathbf{r},t)$ for fixed $\mathbf{r}$ and the displacement along a Lagrangian trajectory $\Delta\mathbf{R}(t)$ is a Brownian motion for all times. The eddy diffusivity tensor is $D_e^{ij}=D^{ij}(\mathbf{0})$, which is a special case of relation (9). In the presence of molecular diffusion, the overall diffusivity is the sum of the eddy contribution and the molecular value $\kappa\delta^{ij}$.

In realistic turbulent flows, the Lagrangian correlation time τ is comparable to the characteristic time scale of large eddies. Progress in numerical simulations (Yeung, 1997) and experimental technique (Voth *et al.*, 1998; La Porta *et al.*, 2001; Mordant *et al.*, 2001) has provided information on the single-particle statistics in the regime intermediate between ballistic and diffusive. Such behavior is captured by the the subtracted Lagrangian autocorrelation function $\langle\mathbf{V}(0)[\mathbf{V}(0)-\mathbf{V}(t)]\rangle$ or its second time derivative that is the autocorrelation function of the Lagrangian acceleration. This information has provided stringent tests on simple stochastic models (that eliminate velocity fields), often used in the past to describe the one-particle and two-particle statistics in turbulent flows (Pope, 1994). The Kraichnan ensemble that models stochastic velocity fields certainly misrepresents the single-particle statistics by suppressing the regime of times smaller than τ. It constitutes, however, as we shall see in the sequel, an important theoretical laboratory for studying the multiparticle statistics in fluid turbulence.

[2]The connected correlation functions, also called cumulants, are recursively defined by the relation $\langle v_1\cdots v_n\rangle = \Sigma_{\{\pi\alpha\}}\Pi_\alpha\langle\langle v_{\pi_\alpha(1)},\dots,v_{\pi_\alpha(n_\alpha)}\rangle\rangle$ with the sum over the partitions of $\{1,\dots,n\}$.

B. Two-particle dispersion in a smooth velocity

The separation $\mathbf{R}_{12}=\mathbf{R}_1-\mathbf{R}_2$ between two fluid particles with trajectories $\mathbf{R}_n(t)=\mathbf{R}(t;\mathbf{r}_n)$ passing at $t=0$ through the points $\mathbf{r}_n$ satisfies (in the absence of Brownian motion) the equation

$$\dot{\mathbf{R}}_{12}=\mathbf{v}(\mathbf{R}_1,t)-\mathbf{v}(\mathbf{R}_2,t). \tag{11}$$

We consider first an incompressible flow where the particles generally separate. In this subsection, we start from the smallest distances where the velocity field can be considered spatially smooth due to viscous effects. In next subsection (Sec. II.C), we treat the dispersion problem for larger distances (in the inertial interval of turbulence) where the velocity field has a nontrivial scaling. Finally, we describe a compressible flow and show how the separation among the particles is replaced by their clustering as the degree of compressibility grows.

1. General considerations

In smooth velocities, for separations $\mathbf{R}_{12}$ much smaller than the viscous scale of turbulence, i.e., in the so-called Batchelor regime (Batchelor, 1959), we may approximate $\mathbf{v}(\mathbf{R}_1,t)-\mathbf{v}(\mathbf{R}_2,t)\approx\sigma(t)\mathbf{R}_{12}(t)$ with the Lagrangian strain matrix $\sigma_{ij}(t)=\nabla_j v^i[\mathbf{R}_2(t),t]$. In this regime, the separation obeys the ordinary differential equation

$$\dot{\mathbf{R}}_{12}(t)=\sigma(t)\mathbf{R}_{12}(t), \tag{12}$$

leading to the linear propagation

$$\mathbf{R}_{12}(t)=W(t)\mathbf{R}_{12}(0), \tag{13}$$

where the evolution matrix is defined as $W^{ij}(t)=\partial R^i(\mathbf{r};t)/\partial r^j$ with $\mathbf{r}=\mathbf{r}_2$. We shall also use the notation $W(t;\mathbf{r})$ when we wish to keep track of the initial point or $W(t;\mathbf{r},s)$ if the initial time s is different from zero.

Equation (12), with the strain treated as given, may be explicitly solved for arbitrary $\sigma(t)$ only in the 1D case by expressing $W(t)$ as the exponential of the time-integrated strain:

$$\ln[R(t)/R(0)]=\ln W(t)=\int_0^t\sigma(s)ds\equiv X. \tag{14}$$

We have omitted subscripts replacing $\mathbf{R}_{12}$ by $\mathbf{R}$. When t is much larger than the correlation time τ of the strain, the variable X behaves as a sum of many independent equally distributed random numbers $X=\Sigma_1^N y_i$ with $N\propto t/\tau$. Its mean value $\langle X\rangle=N\langle y\rangle$ grows linearly in time. Its fluctuations $X-\langle X\rangle$ on the scale $\mathcal{O}(t^{1/2})$ are governed by the central limit theorem that states that $(X-\langle X\rangle)/N^{1/2}$ becomes for large N a Gaussian random variable with variance $\langle y^2\rangle-\langle y\rangle^2\equiv\Delta$. Finally, its fluctuations on the larger scale $\mathcal{O}(t)$ are governed by the large deviation theorem that states that the PDF of X has asymptotically the form

$$\mathcal{P}(X)\propto e^{-NH(X/N-\langle y\rangle)}. \tag{15}$$

This is an easy consequence of the exponential dependence on N of the generating function $\langle e^{zX}\rangle$ of the moments of X. Indeed, $\langle e^{zX}\rangle=e^{NS(z)}$, where we have denoted $\langle e^{zy}\rangle \equiv e^{S(z)}$ (assuming that the expectation exists for all complex z). The PDF $\mathcal{P}(X)$ is then given by the inverse Laplace transform $(1/2\pi i)\int e^{-zX+NS(z)}dz$ with the integral over any axis parallel to the imaginary one. For $X\propto N$, the integral is dominated by the saddle point z_0 such that $S'(z_0)=X/N$ and the large deviation relation (15) follows with $H=-S(z_0)+z_0S'(z_0)$. The function H of the variable $X/N-\langle y\rangle$ is called entropy function as it appears also in the thermodynamic limit in statistical physics (Ellis, 1985). A few important properties of H (also called rate or Cramér function) may be established independently of the distribution $\mathcal{P}(y)$. It is a convex function which takes its minimum at zero, i.e., for X taking its mean value $\langle X\rangle = NS'(0)$. The minimal value of H vanishes since $S(0)=0$. The entropy is quadratic around its minimum with $H''(0)=\Delta^{-1}$, where $\Delta = S''(0)$ is the variance of y. The possible non-Gaussianity of the y's leads to a nonquadratic behavior of H for (large) deviations of X/N from the mean of the order of $\Delta/S'''(0)$.

Coming back to the logarithm $\ln W(t)$ of the interparticle distance ratio in Eq. (14), its growth (or decay) rate $\lambda=\langle X\rangle/t$ is called the Lyapunov exponent. The moments $\langle [R(t)]^n\rangle$ behave exponentially as $\exp[\gamma(n)t]$ with $\gamma(n)$ a convex function of n vanishing at the origin. Even if $\lambda=\gamma'(0)<0$, high-order moments of R may grow exponentially in time; see, for instance, the behavior of the interparticle distance discussed in Sec. II.D. In this case, there must be one more zero n_1 of $\gamma(n)$ and a statistical integral of motion, $\langle R^{n_1}\rangle$, that does not depend on time at large times.

In the multidimensional case, the solution (13) for $\mathbf{R}(t)$ is determined by products of random matrices rather than just random numbers. The evolution matrix $W(t)$ may be written as

$$W(t)=\mathcal{T}\exp\left[\int_0^t \sigma(s)ds\right] = \sum_{n=0}^{\infty}\int_0^t \sigma(s_n)ds_n\cdots\int_0^{s_3}\sigma(s_2)ds_2\int_0^{s_2}\sigma(s_1)ds_1. \tag{16}$$

This time-ordered exponential form is, of course, not very useful for direct calculations except for the particular case of a short-correlated strain, see below. The main statistical properties of the separation vector $\mathbf{R}$ needed for most physical applications might still be established for quite arbitrary strains with finite temporal correlations. The basic idea goes back to Liapounoff (1907) and Furstenberg and Kesten (1960) and it found further development in the multiplicative ergodic theorem of Oseledec (1968). The modulus R of the separation vector may be expressed via the positive symmetric matrix W^TW. The main result states that in almost every realization of the strain, the matrix $(1/t)\ln W^TW$ stabilizes as $t\to\infty$. In particular, its eigenvectors tend to d fixed orthonormal eigenvectors $\mathbf{f}_i$. To understand that intuitively, consider some fluid volume, say a sphere, which evolves into an elongated ellipsoid at later times. As time increases, the ellipsoid is more and more elongated and it is less and less likely that the hierarchy of the ellipsoid axes will change. The limiting eigenvalues

$$\lambda_i=\lim_{t\to\infty} t^{-1}\ln|W\mathbf{f}_i| \tag{17}$$

define the so-called Lyapunov exponents. The major property of the Lyapunov exponents is that they are realization independent if the strain is ergodic. The usual convention is to arrange the exponents in nonincreasing order.

The relation (17) tells that two fluid particles separated initially by $\mathbf{R}(0)$ pointing into the direction $\mathbf{f}_i$ will separate (or converge) asymptotically as $\exp(\lambda_i t)$. The incompressibility constraints $\det(W)=1$ and $\Sigma\lambda_i=0$ imply that a positive Lyapunov exponent will exist whenever at least one of the exponents is nonzero. Consider indeed

$$E(n)=\lim_{t\to\infty} t^{-1}\ln\langle [R(t)/R(0)]^n\rangle, \tag{18}$$

whose slope at the origin gives the largest Lyapunov exponent λ_1. The function $E(n)$ obviously vanishes at the origin. Furthermore, $E(-d)=0$, i.e., incompressibility and isotropy make that $\langle R^{-d}\rangle$ is time-independent as $t\to\infty$ (Furstenberg, 1963; Zel'dovich *et al.*, 1984). Negative moments of orders $n<-1$ are indeed dominated by the contribution of directions $\mathbf{R}(0)$ almost aligned to the eigendirections $\mathbf{f}_2,\ldots,\mathbf{f}_d$. At $n<1-d$ the main contribution comes from a small subset of directions in a solid angle $\propto\exp(d\lambda_d t)$ around $\mathbf{f}_d$. It follows immediately that $\langle R^n\rangle\propto\exp[\lambda_d(d+n)t]$ and that $\langle R^{-d}\rangle$ is a statistical integral of motion. Since $E(n)$ is a convex function, it cannot have other zeros except $-d$ and 0 if it does not vanish identically between those values. It follows that the slope at the origin, and thus λ_1, is positive. The simplest way to appreciate intuitively the existence of a positive Lyapunov exponent is to consider, following Zel'dovich *et al.* (1984), the saddle-point 2D flow $v_x=\lambda x, v_y=-\lambda y$. A vector initially forming an angle ϕ with the x axis will be stretched after time T if $\cos\phi \geqslant [1+\exp(2\lambda T)]^{-1/2}$, i.e., the fraction of stretched directions is larger than 1/2.

A major consequence of the existence of a positive Lyapunov exponent for any random incompressible flow is an exponential growth of the interparticle distance $R(t)$. In a smooth flow, it is also possible to analyze the statistics of the set of vectors $\mathbf{R}(t)$ and to establish a multidimensional analog of Eq. (15) for the general case of a nondegenerate Lyapunov exponent spectrum. The final results will be the large deviation expressions (21) and (27) below. The idea is to reduce the d-dimensional problem to a set of d scalar problems excluding the angular degrees of freedom. We describe this procedure following Balkovsky and Fouxon (1999). Consider the matrix $I(t)=W(t)W^T(t)$, representing the tensor of inertia of a fluid element like the above-mentioned ellipsoid. The matrix is obtained by averaging $R^i(t)R^j(t)d/l^2$ over the initial vectors of length l. In contrast to W^TW that stabilizes at large times, the matrix I rotates in every realization. To account for that rotation, we represent the matrix as $O^T\Lambda O$ with the orthogonal O composed of the eigenvectors of I and the diagonal Λ having the eigenvalues $e^{2\rho_1},\ldots,e^{2\rho_d}$ arranged in nonincreasing order. The evolution equation $\partial_t I=\sigma I+I\sigma^T$ takes then the form

$$\partial_t\rho_i=\tilde\sigma_{ii},\quad \tilde\sigma=O\sigma O^T, \tag{19}$$

$$\partial_t O=\Omega O,\quad \Omega_{ij}=\frac{e^{2\rho_i}\tilde\sigma_{ji}+e^{2\rho_j}\tilde\sigma_{ij}}{e^{2\rho_i}-e^{2\rho_j}}, \tag{20}$$

with no summation over repeated indices. We assume isotropy so that at large times the SO(d) rotation matrix O is distributed uniformly over the sphere. Our task is to describe the statistics of the stretching and the contraction, governed by the eigenvalues ρ_i. We see from Eqs. (19) and (20) that the evolution of the eigenvalues is generally entangled to that of the angular degrees of freedom. As time increases, however, the eigenvalues will become widely separated ($\rho_1\gg\cdots\gg\rho_d$) for a majority of the realizations and $\Omega_{ij}\to\tilde\sigma_{ji}$ for $i<j$ (the upper triangular part of the matrix follows from antisymmetry). The dynamics of the angular degrees of freedom becomes then independent of the eigenvalues and the set of Eqs. (19) reduces to a scalar form. The solution $\rho_i=\int_0^t\tilde\sigma_{ii}(s)ds$ allows the application of the large deviation theory, giving the asymptotic PDF:

$$\begin{aligned}\mathcal{P}(\rho_1,\ldots,\rho_d;t)\propto&\exp[-tH(\rho_1/t-\lambda_1,\ldots,\rho_{d-1}/t-\lambda_{d-1})]\\ &\times\theta(\rho_1-\rho_2)\cdots\theta(\rho_{d-1}-\rho_d)\\ &\times\delta(\rho_1+\cdots+\rho_d).\end{aligned} \tag{21}$$

The Lyapunov exponents λ_i are related to the strain statistics as $\lambda_i=\langle\tilde\sigma_{ii}\rangle$ where the average is temporal. The expression (21) is not valid near the boundaries $\rho_i=\rho_{i+1}$ in a region of order unity, negligible with respect to $\lambda_i t$ at times $t\gg(\lambda_i-\lambda_{i+1})^{-1}$.

The entropy function H depends on the details of the strain statistics and has the same general properties as above: it is non-negative, convex, and it vanishes at zero. Near the minimum, $H(\mathbf{x})\approx\frac{1}{2}(C^{-1})_{ij}x_ix_j$ with the coefficients of the quadratic form given by the integrals of the connected correlation functions of $\tilde\sigma$ defined in Eq. (19):

$$C_{ij}=\int\langle\langle\tilde\sigma_{ii}(t),\tilde\sigma_{jj}(t')\rangle\rangle dt',\quad i,j=1,\ldots,d-1. \tag{22}$$

In the δ-correlated case, the entropy is everywhere quadratic. For a generic initial vector $\mathbf{r}$, the long-time asymptotics of $\ln(R/r)$ coincides with $\mathcal{P}(\rho_1)=\int\mathcal{P}(\rho_1,\ldots,\rho_d)d\rho_2\cdots d\rho_d$ which also takes the large-deviation form at large times, as follows from Eq. (21). The quadratic expansion of the entropy near its minimum corresponds to the lognormal distribution for the distance between two particles,

$$\mathcal{P}(r;R;t)\propto\exp\{-[\ln(R/r)-\bar\lambda t]^2/(2\Delta t)\}, \tag{23}$$

with $r=R(0)$, $\bar\lambda=\lambda_1$ and $\Delta=C_{11}$.

It is interesting to note that under the same assumption of nondegenerate Lyapunov spectrum one can analyze the eigenvectors $\mathbf{e}_i$ of the evolution matrix W (Goldhirsch *et al.*, 1987). Note the distinction between the eigenvectors $\mathbf{e}_i$ of W and $\mathbf{f}_i$ of W^TW. Let us order the eigenvectors $\mathbf{e}_i$ according to their eigenvalues. Those are real due to the assumed nondegeneracy and they behave asymptotically as $\exp(\lambda_1 t),\ldots,\exp(\lambda_d t)$. The $\mathbf{e}_d$ eigenvector converges exponentially to a fixed vector and any subspace spanned by $\{\mathbf{e}_{d-k},\ldots,\mathbf{e}_d\}$ for $0\le k\le d$ tends asymptotically to a fixed subspace for every realization. Note that the subspace is fixed in time but changes with the realization.

Molecular diffusion is incorporated into the above picture by replacing the differential Eq. (12) by its noisy version

$$d\mathbf{R}(t)=\sigma(t)\mathbf{R}(t)dt+2\sqrt{\kappa}d\beta(t). \tag{24}$$

The separation vector is subject to the independent noises of two particles, hence the factor 2 with respect to Eq. (5). The solution to the inhomogeneous linear stochastic Eq. (24) is easy to express via the matrix $W(t)$ in Eq. (16). The tensor of inertia of a fluid element $I^{ij}(t)=R^i(t)R^j(t)d/l^2$ is now averaged both over the initial vectors of length l and the noise, thus obtaining (Balkovsky and Fouxon, 1999):

$$\begin{aligned}I(t)=&W(t)W(t)^T+\frac{4\kappa d}{l^2}\int_0^t W(t)\\ &\times[W(s)^TW(s)]^{-1}W(t)^T ds.\end{aligned} \tag{25}$$

The matrix $I(t)$ evolves according to $\partial_t I=\sigma I+I\sigma^T+4\kappa d/l^2$ and the elimination of the angular degrees of freedom proceeds as previously. An additional diffusive term $2\kappa\exp(-2\rho_i)$ appears in Eq. (19) and the asymptotic solution becomes

$$\begin{aligned}\rho_i(t)=&\int_0^t\tilde\sigma_{ii}(s)ds\\ &+\frac{1}{2}\ln\left\{1+\frac{4\kappa}{l^2d}\int_0^t\exp\left[-2\int_0^s\tilde\sigma_{ii}(s')ds'\right]ds\right\}.\end{aligned} \tag{26}$$

The last term in Eq. (26) is essential for the directions corresponding to negative λ_i. The molecular noise will indeed start to affect the motion of the marked fluid volume when the respective dimension gets sufficiently small. If l is the initial size, the required condition $\rho_i\lesssim-\rho_i^*=-\ln(l^2|\lambda_i|/\kappa)$ is typically met for times $t\simeq\rho_i^*/|\lambda_i|$. For longer times, the respective ρ_i is prevented by diffusion to decrease much below $-\rho_i^*$, while the negative λ_i prevents it from increasing. As a result, the corresponding ρ_i becomes a stationary random process with a mean of the order $-\rho_i^*$. The relaxation times to the stationary distribution are determined by $\tilde\sigma$, which is diffusion independent, and they are thus much smaller than t. On the other hand, the components ρ_j corresponding to non-negative Lyapunov exponents are the integrals over the whole evolution time t. Their values at time t are not sensitive to the latest period of evolution lasting over the relaxation time for the contracting ρ_i. Fixing the values of ρ_j at times $t \gg \rho_i^*/|\lambda_i|$ will not affect the distribution of the contracting ρ_i and the whole PDF is factorized (Shraiman and Siggia, 1994; Chertkov *et al.*, 1997; Balkovsky and Fouxon, 1999). For Lagrangian dynamics in 3D developed Navier-Stokes turbulence there are, for instance, two positive and one negative Lyapunov exponents (Girimaji and Pope, 1990). For times $t \gg \rho_3^*/|\lambda_3|$ we have then

$$\mathcal{P} \propto \exp[-tH(\rho_1/t-\lambda_1, \rho_2/t-\lambda_2)]\mathcal{P}_{\rm st}(\rho_3), \tag{27}$$

with the same function H as in Eq. (21) since ρ_3 is independent of ρ_1 and ρ_2. Note that the account of the molecular noise violates the condition $\Sigma\rho_i = 0$ as fluid elements at scales smaller than $\sqrt{\kappa/|\lambda_3|}$ cannot be distinguished. To avoid misunderstanding, note that Eq. (27) does not mean that the fluid is getting compressible: the simple statement is that if one tries to follow any marked volume, the molecular diffusion makes this volume statistically growing.

Note that we have implicitly assumed l to be smaller than the viscous length $\eta = \sqrt{\nu/|\lambda_3|}$ but larger than the diffusion scale $\sqrt{\kappa/|\lambda_3|}$. Even though ν and κ are both due to molecular motion, their ratio widely varies depending on the type of material. The theory of this section is applicable for materials having large Schmidt (or Prandtl) numbers ν/κ.

The universal forms (21) and (27) for the two-particle dispersion are basically everything we need for physical applications. We will show in the next section that the highest Lyapunov exponent determines the small-scale statistics of a passively advected scalar in a smooth incompressible flow. For other problems, the whole spectrum of exponents and even the form of the entropy function are relevant.

2. Solvable cases

The Lyapunov spectrum and the entropy function can be derived exactly from the given statistics of σ for few limiting cases only. The case of a short-correlated strain allows for a complete solution. For a finite-correlated strain, one can express analytically λ and Δ for a $2d$ long-correlated strain and at large space dimensionality.

a. Short-correlated strain

Consider the case where the strain $\sigma(t)$ is a stationary white-in-time Gaussian process with zero mean and the two-point function

$$\langle \sigma_{ij}(t)\sigma_{kl}(t')\rangle = 2\,\delta(t-t')C_{ijkl}. \tag{28}$$

This case may be considered as the long-time scaling limit $\lim_{\mu\to\infty}\mu^{1/2}\sigma(\mu t)$ of a general strain along a Lagrangian trajectory, provided its temporal correlations decay fast enough. It may be also viewed as describing the strain in the Kraichnan ensemble of velocities decorrelated in time and smooth in space. In the latter case, the matrix $C_{ijkl} = -\nabla_j\nabla_l D^{ik}(\mathbf{0})$, where $D^{ij}(\mathbf{r})$ is the spatial part in the two-point velocity correlation (10). We assume $D^{ij}(\mathbf{r})$ to be smooth in $\mathbf{r}$ (or at least twice differentiable), a property assured by a fast decay of its Fourier transform $\hat{D}^{ij}(\mathbf{k})$. Incompressibility, isotropy, and parity invariance impose the form $D^{ij}(\mathbf{r}) = D_0\delta^{ij} - \frac{1}{2}d^{ij}(\mathbf{r})$ with

$$d^{ij}(\mathbf{r}) = D_1[(d+1)\,\delta^{ij}r^2 - 2r^ir^j] + o(r^2). \tag{29}$$

The corresponding expression for the two-point function of σ reads

$$C_{ijkl} = D_1[(d+1)\,\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}], \tag{30}$$

with the constant D_1 having the dimension of the inverse of time.

The solution of the stochastic differential Eq. (12) is given by Eq. (16) with the matrix $W(t)$ involving stochastic integrals over time. For a white-correlated strain, such integrals are not defined unambiguously but require a regularization that reflects finer details of the strain correlations wiped out in the scaling limit. An elementary discussion of this issue may be found in the Appendix. For an incompressible strain, however, the ambiguity in the integrals defining $W(t)$ disappears so that we do not need to care about such subtleties. The random evolution matrices $W(t)$ form a diffusion process on the group SL(d) of real matrices with unit determinant. Its generator is a second-order differential operator identified by Shraiman and Siggia (1995) as $M = D_1[dH^2 - (d+1)J^2]$, where H^2 and J^2 are the quadratic Casimir of SL(d) and its SO(d) subgroup. In other words, the PDF of $W(t)$ satisfies the evolution equation $(\partial_t - M)\mathcal{P}(W;t) = 0$. The matrix $W(t)$ may be viewed as a continuous product of independent random matrices. Such products in continuous or discrete versions have been extensively studied (Furstenberg and Kesten, 1960; Furstenberg, 1963; Le Page, 1982) and occur in many physical problems, e.g., in 1D localization (Lifshitz *et al.*, 1988; Crisanti *et al.*, 1993).

If we are interested in the statistics of stretching-contraction variables only, then $W(t)$ may be projected onto the diagonal matrix Λ with positive nonincreasing entries $e^{2\rho_1}, \ldots, e^{2\rho_d}$ by the decomposition $W = O\Lambda^{1/2}O'$, where the matrices O and O' belong to the group SO(d). As observed in Bernard *et al.* (1998) and Balkovsky and Fouxon (1999), the generator of the resulting diffusion of ρ_i is the d-dimensional integrable Calogero-Sutherland Hamiltonian. The ρ_i obey the stochastic Langevin equation

$$\partial_t\rho_i = D_1 d \sum_{j\neq i} \coth(\rho_i - \rho_j) + \eta_i, \tag{31}$$

where $\boldsymbol{\eta}$ is a white noise with two-point function $\langle \eta_i(t)\,\eta_j(t')\rangle = 2D_1(d\,\delta_{ij} - 1)\,\delta(t-t')$. At long times the separation between the ρ_i's becomes large and we may approximate $\coth(\rho_{ij})$ by ± 1. It is then easy to solve Eq. (31) and find the explicit expression of the PDF (21):

$$H(\mathbf{x}) = \frac{1}{4D_1 d}\sum_{i=1}^{d} x_i^2, \quad \lambda_i = D_1 d(d-2i+1). \tag{32}$$

Note the quadratic form of the entropy, implying that the distribution of $R(t)$ takes the lognormal form (23) with $\bar{\lambda}=\lambda_1$ and $\Delta=2D_1(d-1)$. The calculation of the long-time distribution of the leading stretching rate ρ_1 goes back to Kraichnan (1974). The whole set of d Lyapunov exponents was first computed in Le Jan (1985; see also Baxendale, 1986). Gamba and Kolokolov (1996) obtained the long-time asymptotics of ρ_i by a path integral calculation. The spectral decomposition of the Calogero-Sutherland Hamiltonian, see Olshanetsky *et al.* (1983), permits us to write explicitly the PDF of ρ_i for all times.

b. 2D slow strain

In 2D, one can reduce the vector Eq. (12) to a second-order scalar form. Let us indeed consider the case of a slow strain satisfying $\dot{\sigma}\ll\sigma^2$ and differentiate the equation $\dot{\mathbf{R}}=\sigma\mathbf{R}$ with respect to time. The term with $\dot{\sigma}$ is negligible with respect to σ^2 and a little miracle happens here: because of incompressibility, the matrix σ is traceless and σ^2 is proportional to the unit matrix in 2D. We thus come to a scalar equation for the wave function $\Psi=R_x+iR_y$:

$$\partial_t^2\Psi=(\sigma_{11}^2+\sigma_{12}\sigma_{21})\Psi. \tag{33}$$

This is the stationary Schrödinger equation for a particle in the random potential $U=S^2-\Omega^2$, where $S^2=\sigma_{11}^2+(\sigma_{12}+\sigma_{21})^2/4$ and $\Omega^2=(\sigma_{12}-\sigma_{21})^2/4$ is the vorticity. Time plays here the role of the coordinate. Our problem is thus equivalent to localization in the quasiclassical limit (Lifshitz *et al.*, 1988) and finding the behavior of Eq. (33) with given initial conditions is similar to the computation of a $1d$ sample resistivity, see, e.g., Abrikosov and Ryzhkin (1978) and Kolokolov (1993). Based on these results we can assert that the modulus $|\Psi|=R$ in random potentials grows exponentially in time, with the same exponent that controls the decay of the localized wave function.

The problem can be solved using semiclassical methods. The flow is partitioned in elliptic ($\Omega>S$) and hyperbolic ($S>\Omega$) regions (Weiss, 1991), corresponding to classical allowed ($U<0$) and forbidden ($U>0$) regions. The wave function Ψ is given by two oscillating exponentials or one decreasing and one increasing, respectively. Furthermore, the typical length of the regions is the correlation time τ, much larger than the inverse of the mean strain and vorticity S_{rms}^{-1} and Ω_{rms}^{-1}. It follows that the increasing exponentials in the forbidden regions are large and dominate the growth of $R(t)$. With exponential accuracy we have

$$\lambda(t)=\ln\left(\frac{R(t)}{R(0)}\right)=\frac{1}{t}\,\mathrm{Re}\int_0^t\sqrt{U(s)}\,ds, \tag{34}$$

where the real part restricts the integration to the hyperbolic regions. The parameters $\bar{\lambda}$ and Δ in the lognormal expression (23) are immediately read from Eq. (34):

$$\bar{\lambda}=\langle\mathrm{Re}\sqrt{U}\rangle,\quad \Delta=\int\langle\langle\mathrm{Re}\sqrt{U(0)},\mathrm{Re}\sqrt{U(t')}\rangle\rangle dt'. \tag{35}$$

Note that the vorticity gave no contribution in the δ-correlated case. For a finite correlation time, it suppresses the stretching by rotating fluid elements with respect to the axes of expansion. The real part in Eq. (35) is indeed filtering out the elliptic regions. Note that the Lyapunov exponent is given by a single-time average, while in the δ-correlated case it was expressed by the time integral of a correlation function. It follows that $\bar{\lambda}$ does not depend on the correlation time τ and it can be estimated as S_{rms} for $\Omega_{rms}\lesssim S_{rms}$. The corresponding estimate of the variance is $\Delta\sim\langle S^2\rangle\tau$. As the vorticity increases, the rotation takes over, the stretching is suppressed and $\bar{\lambda}$ reduces. One may show that the correlation time τ_s of the stretching rate is the minimum between $1/\Omega_{rms}$ and τ (Chertkov *et al.*, 1995a). For $\Omega_{rms}\tau\gg1$ we are back to a δ-correlated case and $\bar{\lambda}\sim\langle S^2\rangle/\Omega_{rms}$. All those estimates can be made systematic for a Gaussian strain (Chertkov *et al.*, 1995a).

c. Large space dimensionality

The key remark for this case is that scalar products like $R^i(t_1)R^i(t_2)$ are sums of a large number of random terms. The fluctuations of such sums are vanishing in the large-d limit and they obey closed equations that can be effectively studied for arbitrary strain statistics. This approach, developed in Falkovich *et al.* (1998), is inspired by the large-N methods in quantum field theory ('t Hooft, 1974) and statistical mechanics (Stanley, 1968). Here, we shall relate the behavior of the interparticle distance to the strain statistics and find explicitly $\bar{\lambda}$ and Δ in Eq. (23). The strain is taken Gaussian with zero mean and correlation function $\langle\sigma_{ij}(t)\sigma_{kl}(0)\rangle=(2D/\tau d)\delta_{ik}\delta_{jl}g(\zeta)$ where higher-order terms in $1/d$ are neglected. The integral of g is normalized to unity and $\zeta\equiv t/\tau$. At large d, the correlation function $F=\langle R^i(t_1)R^i(t_2)\rangle$ satisfies the equation

$$\begin{aligned}\frac{\partial^2}{\partial\zeta_1\,\partial\zeta_2}F(\zeta_1,\zeta_2)&=\tau^2\langle\sigma_{ij}(t_1)\sigma_{ik}(t_2)R^j(t_1)R^k(t_2)\rangle\\&=\beta g(\zeta_1-\zeta_2)F(\zeta_1,\zeta_2),\end{aligned} \tag{36}$$

with the initial condition $\partial_\zeta F(\zeta,0)=0$. The limit of large d is crucial for the factorization of the average leading to the second equality in Eq. (36). The dimensionless parameter $\beta=2D\tau$ measures whether the strain is long or short correlated. Since Eq. (36) is linear and the coefficient on the right-hand side depends only on the time difference, the solution may be written as a sum of harmonics, $F_\lambda(\zeta_1,\zeta_2)=\exp[\lambda\tau(\zeta_1+\zeta_2)]\Psi(\zeta_1-\zeta_2)$. Inserting it into Eq. (36), we get the Schrödinger equation for the even function $\Psi(t)$:

$$\partial_\zeta^2\Psi(\zeta)+[-(\lambda\tau)^2+\beta g(\zeta)]\Psi(\zeta)=0. \tag{37}$$

At large times, the dominant contribution comes from the largest exponent λ corresponding to the ground state in the potential $-\beta g(\zeta)$. Note that the "energy" is proportional to $-\lambda^2$ and $\overline{\lambda}$ is the Lyapunov exponent since $\langle R^2\rangle = F(\zeta,\zeta)$. From quantum mechanics textbooks it is known that the ground-state energy in deep and shallow potentials is proportional to their depth and its square, respectively. We conclude that $\overline{\lambda} \propto D$ for a fast strain (small β) and $\overline{\lambda} \propto \sqrt{D/\tau}$ in the slow case (large β). At large time differences, the potential term in Eq. (37) is negligible and $\overline{\lambda}$ determines both the growth of $\langle R^2\rangle$ and the decay of different-time-correlation function at $\zeta_1 + \zeta_2$ fixed. That also shows that the correlation function becomes independent of the larger of the times t_1 and t_2 when their difference exceeds τ.

For the fast strain case, one can put $g(\zeta) = \delta(\zeta)$ and the solution of Eq. (37) is amazingly simple: $F(\zeta_1,\zeta_2) = R^2(0)\exp[\beta \min(\zeta_1,\zeta_2)]$. The Lyapunov exponent $\overline{\lambda} = D$, in agreement with the result $\lambda_1 = D_1 d^2 + \mathcal{O}(d)$ obtained for the Kraichnan ensemble. For the slow case, the stretching rate is independent of τ at a given value of D/τ (determining the simultaneous correlation function of the strain). The analysis of the Schrödinger Eq. (37) with a deep potential also gives the correlation time τ_s of the stretching rate, which does not generally coincide with the strain correlation time τ (Falkovich *et al.*, 1998).

C. Two-particle dispersion in a nonsmooth incompressible flow

In this subsection we study the separation between two trajectories in the inertial range of scales $\eta \ll r \ll L$. The scales η and L stand in a 3D turbulent flow for the viscous and the injection scales (the latter is also called integral scale in that case). For a 2D inverse energy cascade flow, they would stand for the scales of injection and friction damping, respectively. We shall see that the behavior of the trajectories is quite different from that in smooth flows analyzed previously.

1. Richardson law

As discussed in the Introduction, velocity differences in the inertial interval exhibit an approximate scaling expressed by the power-law behavior of the structure functions $\langle (\Delta_r v)^n\rangle \propto r^{\sigma_n}$. Low-order exponents are close to the Kolmogorov prediction $\sigma_n = \alpha n$ with $\alpha = 1/3$. A linear dependence of σ_n on n would signal the scaling $\Delta_r v \propto r^\alpha$ with a sharp value of α. A nonlinear dependence of σ_n indicates the presence of a whole spectrum of exponents, depending on the space-time position in the flow (the so-called phenomenon of multiscaling). The 2D inverse and the 3D direct energy cascades in Navier-Stokes equation provide concrete examples of the two possible situations. Rewriting Eq. (11) for the fluid particle separation as $\dot{\mathbf{R}} = \Delta\mathbf{v}(\mathbf{R},t)$, we infer that $dR^2/dt = 2\mathbf{R}\cdot\Delta\mathbf{v} \propto R^{1+\alpha}$. If the value of α is fixed and smaller than unity, this is solved (ignoring the space-time dependence in the proportionality constant) by

$$R^{1-\alpha}(t) - R^{1-\alpha}(0) \propto t, \tag{38}$$

implying that the dependence on the initial separation is quickly wiped out and that R grows as $t^{1/(1-\alpha)}$. For the random process $\mathbf{R}(t)$, the relation (38) is, of course, of the mean-field type and should pertain to the long-time behavior of the averages

$$\langle R^\zeta(t)\rangle \propto t^{\zeta/(1-\alpha)}. \tag{39}$$

That implies their superdiffusive growth, faster than the diffusive one $\propto t^{\zeta/2}$. The scaling law (39) might be amplified to the rescaling property

$$\mathcal{P}(R;t) = \lambda \mathcal{P}(\lambda R; \lambda^{1-\alpha} t) \tag{40}$$

of the interparticle distance PDF. Possible deviations from a linear behavior in the order ζ of the exponents in Eq. (39) should be interpreted as a signal of multiscaling of the Lagrangian velocity $\Delta\mathbf{v}[\mathbf{R}(t),t] \equiv \Delta\mathbf{V}(t)$.

The power-law growth (39) for $\zeta = 2$ and $\alpha = 1/3$, i.e., $\langle R(t)^2\rangle \propto t^3$, is a direct consequence of the celebrated Richardson dispersion law (Richardson, 1926), the first quantitative phenomenological observation in developed turbulence. It states that

$$\frac{d}{dt}\langle R^2\rangle \propto \langle R^2\rangle^{2/3}. \tag{41}$$

The law (41) seems to be confirmed also by later experimental data, see Chap. 24 of Monin and Yaglom (1979) and Jullien *et al.* (1999), and by the numerical simulations (Zovari *et al.*, 1994; Elliott, Jr. and Majda, 1996; Boffetta *et al.*, 1998; Fung and Vassilicos, 1998). The more general property of self-similarity (40) (with $\alpha = 1/3$) has been observed in the inverse cascade of two-dimensional turbulence (Jullien *et al.*, 1999; Boffetta and Celani, 2000; Boffetta and Sokolov, 2000). It is likely that Eq. (41) is exact in that situation, while it may be only approximately correct in 3D, although the experimental data do not yet allow us to test it with sufficient confidence.

It is important to remark that, even assuming the validity of the Richardson law (41), it is impossible to establish general properties of the PDF $\mathcal{P}(R;t)$ such as those in Sec. II.B.1 for the single-particle PDF. The physical reason is easy to understand if one writes

$$\frac{d\langle R^2\rangle}{dt} = 2\tau_t \langle (\Delta\mathbf{V})^2\rangle, \tag{42}$$

similarly to Eqs. (6) and (7). Here

$$\tau_t = \int_0^t \langle \Delta\mathbf{V}(t)\cdot\Delta\mathbf{V}(s)\rangle ds / \langle (\Delta\mathbf{V})^2\rangle$$

is the correlation time of the Lagrangian velocity differences. If $d\langle R^2\rangle/dt$ is proportional to $\langle R^2\rangle^{2/3}$ and $\langle (\Delta\mathbf{V})^2\rangle$ behaves like $\langle R^2\rangle^{1/3}$ then τ_t grows as $\langle R^2\rangle^{1/3} \propto t$, i.e., the random process $\Delta\mathbf{V}(t)$ is correlated across its whole span. The absence of decorrelation explains why the central limit theorem and the large deviation theory cannot be applied. In general, there is no *a priori* reason to expect $\mathcal{P}(R;t)$ to be Gaussian with respect to a power of R either, although, as we shall see, this is what essentially happens in the Kraichnan ensemble.

2. Breakdown of the Lagrangian flow

It is instructive to contrast the exponential growth (18) of the distance between the trajectories within the viscous range with the power-law growth (39) in the inertial range. In the viscous regime the closer two trajectories are initially, the longer time is needed to reach a given separation. As a result, infinitesimally close trajectories never separate and trajectories in a fixed realization of the velocity field are continuously labeled by the initial conditions. Small deviations of the initial point are magnified exponentially, though. This sensitive dependence is usually considered as the defining feature of the chaotic behavior. Conversely, in the inertial interval the trajectories separate in a finite time independently of their initial distance $R(0)$, provided the latter was also in the inertial interval. The speed of this separation may depend on the detailed structure of the turbulent velocities, including their fine geometry (Fung and Vassilicos, 1998), but the very fact of the explosive separation is related to the scaling behavior $\Delta_r v \propto r^\alpha$ with $\alpha<1$. For high Reynolds numbers the viscous scale η is negligibly small, a fraction of a millimeter in the turbulent atmosphere. Setting it to zero (or equivalently the Reynolds number to infinity) is an appropriate abstraction if we want to concentrate on the behavior of the trajectories in the inertial range. In such a limit, the power-law separation between the trajectories extends down to arbitrarily small distances: infinitesimally close trajectories still separate in a finite time. This makes a marked difference in comparison to the smooth chaotic regime, clearly showing that developed turbulence and chaos are fundamentally different phenomena. As stressed in Bernard *et al.* (1998), the explosive separation of the trajectories results in a breakdown of the deterministic Lagrangian flow in the limit $\mathrm{Re}\to\infty$, see also Frisch *et al.* (1998) and Gawędzki (1998, 1999). The effect is dramatic since the trajectories cannot be labeled anymore by the initial conditions. Note that the sheer existence of the Lagrangian trajectories $\mathbf{R}(t;\mathbf{r})$ depending continuously on the initial position $\mathbf{r}$ would imply that $\lim_{\mathbf{r}_1\to\mathbf{r}_2}\langle|\mathbf{R}(t;\mathbf{r}_1)-\mathbf{R}(t;\mathbf{r}_2)|^\zeta\rangle=0$. That would contradict the persistence of a power-law separation of the Richardson type for infinitesimally close trajectories. Remark also that the breakdown of the deterministic Lagrangian flow does not violate the theorem about the uniqueness of solutions of the ordinary differential equation $\dot{\mathbf{R}}=\mathbf{v}(\mathbf{R},t)$. Indeed, the theorem requires the velocity to be Lipschitz in $\mathbf{r}$, i.e., that $\Delta_r v\leqslant\mathcal{O}(r)$. As first noticed by Onsager (1949), the velocities for $\mathrm{Re}=\infty$ are actually only Hölder continuous: $\Delta_r v\simeq\mathcal{O}(r^\alpha)$ with the exponent $\alpha<1$ (in Kolmogorov's phenomenology $\alpha=1/3$). The simple equation $\dot{x}=|x|^\alpha$ provides a classical example with two solutions $x=[(1-\alpha)t]^{1/(1-\alpha)}$ and $x=0$, both starting from zero, for the non-Lipschitz case $\alpha<1$. It is then natural to expect the existence of multiple Lagrangian trajectories starting or ending at the same point. Such a possibility was first noticed and exploited in a somewhat different context in the study of weak solutions of the Euler equations (Brenier, 1989; Shnirelman, 1999). Does then the Lagrangian description of the fluid break down completely at $\mathrm{Re}=\infty$?

Even though the deterministic Lagrangian description is inapplicable, the statistical description of the trajectories is still possible. As we have seen above, probabilistic questions like those about the averaged powers of the distance between initially close trajectories still have well-defined answers. We expect that for a typical velocity realization, one may maintain at $\mathrm{Re}=\infty$ a probabilistic description of the Lagrangian trajectories. In particular, objects such as the PDF $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ of the time t particle position $\mathbf{R}$, given its time s position $\mathbf{r}$, should continue to make sense. For a regular velocity with deterministic trajectories,

$$p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})=\boldsymbol{\delta}[\mathbf{R}-\mathbf{R}(t;\mathbf{r},s)], \tag{43}$$

where $\mathbf{R}(t;\mathbf{r},s)$ denotes the unique Lagrangian trajectory passing at time s through $\mathbf{r}$. In the presence of a small molecular diffusion, Eq. (5) for the Lagrangian trajectories has always a Markov process solution in each fixed velocity realization, irrespective of whether the latter be Lipschitz or Hölder continuous (Stroock and Varadhan, 1979). The resulting Markov process is characterized by the transition probabilities $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ satisfying the advection-diffusion equation[3]

$$[\partial_t-\nabla_{\mathbf{R}}\cdot\mathbf{v}(\mathbf{R},t)-\kappa\nabla_{\mathbf{R}}^2]p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})=0 \tag{44}$$

for $t>s$. The mathematical difference between smooth and rough velocities is that in the latter case the transition probabilities are weak rather than strong solutions. What happens if we turn off the molecular diffusion? If the velocity is Lipschitz in $\mathbf{r}$, then the Markov process describing the noisy trajectories concentrates on the deterministic Lagrangian trajectories and the transition probabilities converge to Eq. (43). It has been conjectured in Gawędzki (1999) that, for a generic $\mathrm{Re}=\infty$ turbulent flow, the Markov process describing the noisy trajectories still tends to a limit when $\kappa\to0$, but the limit stays diffused; see Fig. 1. In other words, the transition probability converges to a weak solution of the advection equation

$$[\partial_t-\nabla_{\mathbf{R}}\cdot\mathbf{v}(\mathbf{R},t)]p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})=0, \tag{45}$$

which does not concentrate on a single trajectory, as it was the case in Eq. (43). We shall then say that the limiting Markov process defines a stochastic Lagrangian flow. This way the roughness of the velocity would result in the stochasticity of the particle trajectories persisting even in the limit $\kappa\to0$. To avoid misunderstanding, let us stress again that, according to this claim, the Lagrangian trajectories behave stochastically already in a fixed realization of the velocity field and for negligible molecular diffusivities, i.e., the effect is not due to the molecular noise or to random fluctuations of the velocities. This spontaneous stochasticity of fluid particles seems to constitute an important aspect of developed turbulence. It is an unescapable consequence of the Richardson disper-

[3]For $\kappa>0$ and smooth velocities, the equation results from the Itô formula generalizing Eq. (A5) applied to Eq. (43) and averaged over the noise.

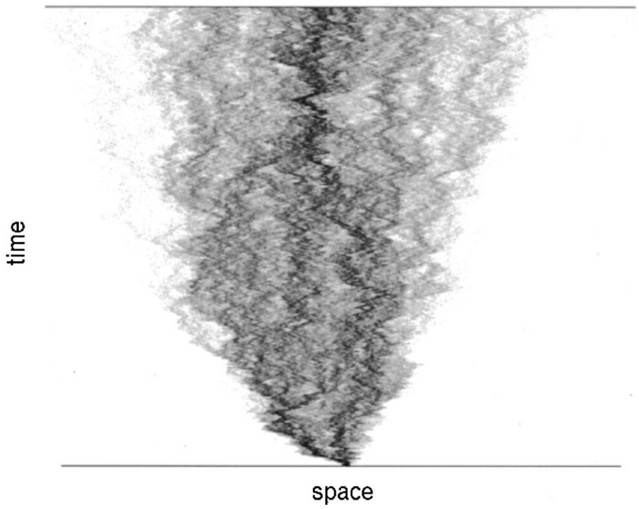

FIG. 1. An illustration of the breakdown of the Lagrangian flow in spatially nonsmooth flows: infinitesimally close particles reach a finite separation in a finite time. The consequence is the cloud observed in the figure. The particles evolve in a fixed realization of the velocity field and in the absence of any molecular noise.

sion law and of the Kolmogorov-like scaling of velocity differences in the limit $\mathrm{Re}\to\infty$ and it provides for a natural mechanism assuring the persistence of dissipation in the inviscid limit: $\lim_{\nu\to 0}\nu\langle|\nabla\mathbf{v}|^2\rangle\neq 0$.

3. The example of the Kraichnan ensemble

The general conjecture about the existence of stochastic Lagrangian flows for generic turbulent velocities, e.g., for weak solutions of the incompressible Euler equations locally dissipating energy, as discussed by Duchon and Robert (2000), has not been mathematically proven. The conjecture is known, however, to be true for the Kraichnan ensemble (10), as we are going to discuss in this subsection.

We should model the spatial part D^{ij} of the two-point function (10) so that it has proper scalings in the viscous and inertial intervals. This can be conveniently achieved by taking its Fourier transform

$$\hat{D}^{ij}(\mathbf{k})\propto\left(\delta^{ij}-\frac{k^ik^j}{k^2}\right)\frac{e^{-(\eta\mathbf{k})^2}}{(\mathbf{k}^2+L^{-2})^{(d+\xi)/2}},\tag{46}$$

with $0\leqslant\xi\leqslant 2$. In physical space,

$$D^{ij}(\mathbf{r})=D_0\delta^{ij}-\frac{1}{2}d^{ij}(\mathbf{r}),\tag{47}$$

where $d^{ij}(\mathbf{r})$ scales as r^ξ in the inertial interval $\eta\ll r\ll L$, as r^2 in the viscous range $r\ll\eta$ and tends to $2D_0\delta^{ij}$ at very large scales $r\gg L$. As we discussed in Sec. II.A, D_0 gives the single-particle effective diffusivity. Notice that $D_0=\mathcal{O}(L^\xi)$ indicating that turbulent diffusion is controlled by the velocity fluctuations at large scales of order L. On the other hand, $d_{ij}(\mathbf{r})$ describes the statistics of the velocity differences and it picks up contributions of all scales. In the limits $\eta\to 0$ and $L\to\infty$, it takes the scaling form

$$\lim_{\substack{\eta\to 0\\ L\to\infty}} d^{ij}(\mathbf{r})=D_1r^\xi\left((d-1+\xi)\delta^{ij}-\xi\frac{r^ir^j}{r^2}\right),\tag{48}$$

where the normalization constant D_1 has the dimensionality of $(\text{length}^{2-\xi})\times(\text{time}^{-1})$.

For $0<\xi<2$ and $\eta>0$, the typical velocities are smooth in space with the scaling behavior r^ξ visible only for scales much larger than the viscous cutoff η. When the cutoff is set to zero, however, the velocity becomes nonsmooth. The Kraichnan ensemble is then supported on velocities that are Hölder-continuous with the exponent $\xi/2-0$. That mimics the major property of turbulent velocities at the infinite Reynolds number. The limiting case $\xi=2$ describes the Batchelor regime of the Kraichnan model: the velocity gradients are constant and the velocity differences are linear in space. This is the regime that the analysis of Sec. II.B.2.a pertains to. In the other limiting case $\xi=0$, the typical velocities are very rough in space (distributional). For any ξ, the Kraichnan velocities have even rougher behavior in time. We may expect that the temporal roughness does not modify largely the qualitative picture of the trajectory behavior as it is the regularity of velocities in space, and not in time, that is crucial for the uniqueness of the trajectories (see, however, below).

For time-decorrelated velocities, both terms on the right-hand side of the Lagrangian Eq. (5) should be treated according to the rules of stochastic differential calculus. The choice of the regularization is irrelevant here even for compressible velocities, see the Appendix. The existence and the properties of solutions of such stochastic differential equations were extensively studied in the mathematical literature for velocities smooth in space, see, e.g., Kunita (1990). Those results apply to our case as long as $\eta>0$ both for positive or vanishing diffusivity. The advection-diffusion Eq. (44) for the transition probabilities also becomes a stochastic equation for white-in-time velocities. The choice of the convention, however, is important here even for incompressible velocities: the equation should be interpreted with the Stratonovich convention, see the Appendix. The equivalent Itô form contains an extra second-order term that amounts to the replacement of the molecular diffusivity by the effective diffusivity $(D_0+\kappa)$ in Eq. (44). The Itô form of the equation explicitly exhibits the contribution of the eddy diffusivity, hidden in the convention for the Stratonovich form. As pointed out by Le Jan and Raimond (1998, 1999), the regularizing effect of D_0 permits us to solve the equation by iteration also for the nonsmooth case giving rise to transition probabilities $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ defined for almost all velocities of the Kraichnan ensemble. Moreover, the vanishing diffusivity limit of the transition probabilities exist, defining a stochastic Lagrangian flow.

The velocity averages over the Kraichnan ensemble of the transition probabilities $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ are exactly calculable. We shall use a formal functional integral approach (Chertkov, 1997; Bernard *et al.*, 1998). In the phase space path integral representation of the solution of Eq. (44),

$$p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})=\int_{\substack{\mathbf{r}(s)=\mathbf{r}\\ \mathbf{r}(t)=\mathbf{R}}}\exp\left(\mp\int_s^t\{i\mathbf{p}(\tau)\cdot(\dot{\mathbf{r}}(\tau)-\mathbf{v}[\mathbf{r}(\tau),\tau)]+\kappa\mathbf{p}^2(\tau)\}d\tau\right)\mathcal{D}\mathbf{p}\,\mathcal{D}\mathbf{r}, \tag{49}$$

for $s\lessgtr t$, the Gaussian average over the velocities is easy to perform. It replaces the exponent in Eq. (49) by $\mp\int_s^t[i\mathbf{p}(\tau)\cdot\dot{\mathbf{r}}(\tau)+(D_0+\kappa)\mathbf{p}^2(\tau)]d\tau$ and results in the path-integral represention of the heat kernel of the Laplacian for which we shall use the operator notation $e^{|t-s|(D_0+\kappa)\nabla^2}(\mathbf{r};\mathbf{R})$. In other words, the average of Eq. (49) is the solution of the heat equation (with diffusivity $D_0+\kappa$) equal to $\boldsymbol{\delta}(\mathbf{R}-\mathbf{r})$ at time s. The above calculation confirms then the result discussed at the end of Sec. II.A about the all-time diffusive behavior of a single fluid particle in the Kraichnan ensemble.

In order to study the two-particle dispersion, one should examine the joint PDF of the equal-time values of two fluid particles averaged over the velocities

$$\langle p(\mathbf{r}_1,s;\mathbf{R}_1,t|\mathbf{v})p(\mathbf{r}_2,s;\mathbf{R}_2,t|\mathbf{v})\rangle\equiv\mathcal{P}_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{R}_1,\mathbf{R}_2;t-s). \tag{50}$$

The latter is given for the Kraichnan ensemble by the heat kernel $e^{|t-s|\mathcal{M}_2}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{R}_1,\mathbf{R}_2)$ of the elliptic second-order differential operator

$$\mathcal{M}_2=\sum_{n,n'=1}^{2}D^{ij}(\mathbf{r}_n-\mathbf{r}_{n'})\nabla_{r_n^i}\nabla_{r_{n'}^j}+\kappa\sum_{n=1}^{2}\boldsymbol{\nabla}^2_{\mathbf{r}_n}. \tag{51}$$

In other words, the PDF $\mathcal{P}_2$ satisfies the equation $(\partial_t-\mathcal{M}_2)\mathcal{P}_2=\delta(t-s)\,\delta(\mathbf{R}_1-\mathbf{r}_1)\,\delta(\mathbf{R}_2-\mathbf{r}_2)$, a result which goes back to the original work of Kraichnan (1968). Indeed, the Gaussian expectation (50) is again easily computable in view of the fact that the velocity enters through the exponential function in Eq. (49). The result is the path-integral expression

$$\int_{\substack{\mathbf{r}_n(s)=\mathbf{r}_n\\ \mathbf{r}_n(t)=\mathbf{R}_n}}\exp\left[\mp\int_s^t\left(\sum_{n=1}^{2}[i\mathbf{p}_n(\tau)\cdot\dot{\mathbf{r}}_n(\tau)+\kappa\mathbf{p}_n^2(\tau)]+\sum_{n,n'=1}^{2}D^{ij}[\mathbf{r}_n(\tau)-\mathbf{r}_{n'}(\tau)]p_{ni}(\tau)p_{n'j}(\tau)\right)d\tau\right]\prod_n\mathcal{D}\mathbf{p}_n\,\mathcal{D}\mathbf{r}_n \tag{52}$$

for the heat kernel of $\mathcal{M}_2$.

Let us concentrate on the relative separation $\mathbf{R}=\mathbf{R}_1-\mathbf{R}_2$ of two fluid particles at time t, given their separation $\mathbf{r}$ at time zero. The relevant PDF $\widetilde{\mathcal{P}}(\mathbf{r};\mathbf{R};t)$ is obtained by averaging over the simultaneous translations of the final (or initial) positions of the particles. Explicitly, it is given by the heat kernel of the operator $\widetilde{\mathcal{M}}=[d^{ij}(\mathbf{r})+2\kappa\delta^{ij}]\nabla_{r^i}\nabla_{r^j}$ equal to the restriction of $\mathcal{M}_2$ to the translationally invariant sector. Note that the eddy diffusivity D_0, dominated by the integral scale, drops out from $\widetilde{\mathcal{M}}$. The above result shows that the relative motion of two fluid particles is an effective diffusion with a distance-dependent diffusivity tensor scaling like r^{ξ} in the inertial range. This is a precise realization of the scenario for turbulent diffusion put up by Richardson (1926).

Similarly, the PDF of the distance R between two particles is given by the heat kernel $e^{|t|M}(r;R)$, where M is the restriction of $\mathcal{M}_2$ to the homogeneous and isotropic sector. Explicitly,

$$M=\frac{1}{r^{d-1}}\partial_r[(d-1)D_1r^{d-1+\xi}+2\kappa r^{d-1}]\partial_r \tag{53}$$

in the scaling regime and its heat kernel may be readily analyzed. In the Batchelor regime $\xi=2$ and for $\kappa\to0$, the heat kernel of M reproduces the lognormal distribution (23) with $\Delta=2D_1(d-1)$ and $\lambda=D_1d(d-1)$; see Sec. II.B.2.a.

The simple criterion allowing us to decide whether the Markov process stays diffused as $\kappa\to0$ is to control the limit $r\to0$ of the PDF $\mathcal{P}(r;R;t)$ (Bernard *et al.*, 1998). For smooth velocities, it follows from Eq. (23) that

$$\lim_{\substack{r\to0\\ \kappa\to0}}\mathcal{P}(r;R;t)=\delta(R). \tag{54}$$

In simple words, when the initial points converge, so do the end points of the process. Conversely, for $0\leqslant\xi<2$ we have

$$\lim_{\substack{r\to0\\ \kappa\to0}}\mathcal{P}(r;R;t)\propto\frac{R^{d-1}}{|t|^{d/(2-\xi)}}\exp\left[-\text{const}\times\frac{R^{2-\xi}}{|t|}\right], \tag{55}$$

in the scaling limit $\eta=0$, $L=\infty$. That confirms the diffused character of the limiting process describing the Lagrangian trajectories in fixed non-Lipschitz velocities: the end points of the process stay at finite distance even if the initial points converge. If we set the viscous cutoff to zero keeping L finite, the behavior (55) crosses over for $R\gg L$ to a simple diffusion with diffusivity $2D_0$: at such large distances the particle velocities are essentially independent and the single-particle behavior is recovered.

The stretched-exponential PDF (55) has the scaling form (40) for $\alpha=\xi-1$ and implies the power-law growth (39) of the averaged powers of the distance between trajectories. The PDF is Gaussian in the rough case $\xi=0$. Note that the Richardson law $\langle R^2(t)\rangle\propto t^3$ is reproduced for $\xi=4/3$ and not for $\xi=2/3$ (where the velocity has the spatial Hölder exponent 1/3). The reason is that the velocity temporal decorrelation cannot be ignored and the mean-field relation (38) should be replaced by $R^{1-\xi/2}(t)-R^{1-\xi/2}(0)\propto\beta(t)$ with the Brownian motion $\beta(t)$. Since $\beta(t)$ behaves as $t^{1/2}$, the replacement changes the power and indeed reproduces the large-time PDF (55) up to a geometric power-law prefactor. In general, the time dependence of the velocities plays a role in determining whether the breakdown of deterministic Lagrangian flow occurs or not. Indeed, the relation (42) implies that the scale-dependence of the correlation time τ_t of the Lagrangian velocity differences may change the time behavior of $\langle R^2\rangle$. In particular, $\langle R^2\rangle$ ceases to grow in time if $\tau_t\propto\langle R^2\rangle^{\beta}$ and $\langle(\Delta\mathbf{V})^2\rangle\propto\langle R^2\rangle^{\alpha}$ with $\beta\geqslant 1-\alpha$. It has been recently shown in Fannjiang *et al.* (2000) that the Lagrangian trajectories are deterministic in a Gaussian ensemble of velocities with Hölder continuity in space and such fast time decorrelation on short scales. The Kolmogorov values of the exponents $\alpha=\beta=1/3$ satisfy, however, $\beta<1-\alpha$.

Note the special case of the average $\langle R^{2-\xi-d}\rangle$ in the Kraichnan velocities. Since $Mr^{2-\xi-d}$ is a contact term $\propto\delta(r)$ for $\kappa=0$, one has $\partial_t\langle R^{2-\xi-d}\rangle\propto\mathcal{P}(r;0;t)$. The latter is zero in the smooth case so that $\langle R^{-d}\rangle$ is a true integral of motion. In the nonsmooth case, $\langle R^{2-\xi-d}\rangle \propto t^{1-d/(2-\xi)}$ and is not conserved due to a nonzero probability density to find two particles at the same place even when they started apart.

As stated, the result (55) holds when the molecular diffusivity is turned off in the velocity ensemble with no viscous cutoff, i.e., for vanishing Schmidt number Sc $=\nu/\kappa$, where ν is the viscosity defined in the Kraichnan model as $D_1\eta^{\xi}$. The same result holds also when ν and κ are turned off at the same time with $\mathrm{Sc}=\mathcal{O}(1)$, provided the initial distance r is taken to zero only afterwards (E and Vanden Eijnden, 2000b). This confirms that the explosive separation of close trajectories persists for finite Reynolds numbers as long as their initial distance is not too small, as anticipated by Bernard *et al.* (1998).

D. Two-particle dispersion in a compressible flow

Discussing the particle dispersion in incompressible fluids and exposing the different mechanisms of particle separation, we paid little attention to the detailed geometry of the flows, severely restricted by the incompressibility. The presence of compressibility allows for more flexible flow geometries with regions of ongoing compression effectively trapping particles for long times and counteracting their tendency to separate. To expose this effect and gauge its relative importance for smooth and nonsmooth flows, we start from the simplest case of a time-independent 1D flow $\dot{x}=v(x)$. In 1D, any velocity is potential: $v(x)=-\partial_x\phi(x)$, and the flow is the steepest descent in the landscape defined by the potential ϕ. The particles are trapped in the intervals where the velocity has a constant sign and they converge to the fixed points with lower value of ϕ at the ends of those intervals. In the regions where $\partial_x v$ is negative, nearby trajectories are compressed together. If the flow is smooth the trajectories take an infinite time to arrive at the fixed points (the particles might also escape to infinity in a finite time). Let us consider now a nonsmooth version of the velocity, e.g., a Brownian path with Hölder exponent 1/2. At variance with the smooth case, the solutions will take a finite time to reach the fixed points at the ends of the trapping intervals and will stick to them at subsequent times, as in the example of the equation $\dot{x}=|x-x_0|^{1/2}$. The nonsmoothness of the velocity clearly amplifies the trapping effects leading to the convergence of the trajectories. A time-dependence of the velocity changes somewhat the picture. The trapping regions, as defined for the static case, start wandering and they do not enslave the solutions which may cross their boundaries. Still, the regions of ongoing compression effectively trap the fluid particles for long time intervals. Whether the tendency of the particles to separate or the trapping effects win is a matter of detailed characteristics of the flow.

In higher dimensions, the behavior of potential flows is very similar to the 1D case, with trapping totally dominating in the time-independent case, its effects being magnified by the nonsmoothness of the velocity and blurred by the time dependence. The traps might of course have a more complicated geometry. Moreover, we might have both solenoidal and potential components in the velocity. The dominant tendency for the incompressible component is to separate the trajectories, as we discussed in the previous sections. On the other hand, the potential component enhances trapping in the compressed regions. The net result of the interplay between the two components depends on their relative strength, spatial smoothness and temporal rate of change.

Let us consider first a smooth compressible flow with a homogeneous and stationary ergodic statistics. Similarly to the incompressible case discussed in Sec. II.B.1, the stretching-contraction variables ρ_i, $i=1,\dots,d$, behave asymptotically as $t\lambda_i$ with the PDF of large deviations $x_i=\rho_i/t-\lambda_i$ determined by an entropy function $H(x_1,\dots,x_d)$. The asymptotic growth rate of the fluid volume is given by the sum of the Lyapunov exponents $s=\sum_{i=1}^{d}\lambda_i$. Note that density fluctuations do not grow in a statistically steady compressible flow because the pressure provides feedback from the density to the velocity field. That means that s vanishes even though the ρ_i variables fluctuate. However, to model the growth of density fluctuations in the intermediate regime, one can consider an idealized model with a steady velocity statistics having nonzero s. This quantity has the interpretation of the opposite of the entropy production rate, see Sec. III.A.4 below, and it is necessarily $\leqslant 0$ (Ruelle, 1997). Let us give here the argument due to Balkovsky *et al.* (1999) which goes as follows. In any statistically homogeneous flow, incompressible or compressible, the distribution of particle displacements is independent of their initial position and so is the distribution of the evolution matrix $W_{ij}(t;\mathbf{r})=\partial R^i(t;\mathbf{r})/\partial r^j$. Since the total volume V (assumed finite in this argument) is conserved, the average $\langle\det W\rangle$ is equal to unity for all times and initial positions although the determinant fluctuates in the compressible case. The average of $\det W = e^{\Sigma \rho_i}$ is dominated at long times by the saddle-point $\mathbf{x}^*$ giving the maximum of $\Sigma(\lambda_i + x_i) - H(\mathbf{x})$, which has to vanish to conform with the total volume conservation. Since $\Sigma x_i - H(\mathbf{x})$ is concave and vanishes at $x=0$, its maximum value has to be non-negative. We conclude that the sum of the Lyapunov exponents is nonpositive. The physics behind this result is transparent: there are more Lagrangian particles in the contracting regions, leading to negative average gradients in the Lagrangian frame. Indeed, the volume growth rate tends at large times to the Lagrangian average of the trace of the strain $\sigma = \boldsymbol{\nabla}\mathbf{v}$:

$$\frac{1}{t}\langle \ln\det[W(\mathbf{r};t)]\rangle \to \int \operatorname{tr}\sigma[\mathbf{R}(t;\mathbf{r}),t]\,\frac{d\mathbf{r}}{V} = \int \operatorname{tr}\sigma(\mathbf{R},t)\,\frac{d\mathbf{R}}{V\det[W(\mathbf{r};t)]}. \tag{56}$$

The Lagrangian average generally coincides with the Eulerian one $\int d\mathbf{r}\,\operatorname{tr}\sigma(t,\mathbf{r})/V$, only in the incompressible case (where it is zero). For compressible flow, the integrals in Eq. (56) vanish at the initial time (when we set the initial conditions for the Lagrangian trajectories so that the measure was uniform back then). The regions of ongoing compression with negative $\operatorname{tr}\sigma$ acquire higher weight in the average in Eq. (56) than the expanding ones. Negative values of $\operatorname{tr}\sigma$ suppress stretching and enhance trapping and that is the simple reason for the volume growth rate to be generally negative. Note that, were the trajectory $\mathbf{R}(\mathbf{r};t)$ defined by its final (rather than initial) position, the sign of the average strain trace would be positive. Let us stress again the essential difference between the Eulerian and the Lagrangian averages in the compressible case: an Eulerian average is uniform over space, while in a Lagrangian average every trajectory comes with its own weight determined by the local rate of volume change. For the corresponding effects on the single-particle transport, the interested reader is referred to Vergassola and Avellaneda (1997).

In the particular case of a short-correlated strain one can take t in Eq. (56) larger than the correlation time τ of the strain, yet small enough to allow for the expansion $\det[W(\mathbf{r};t)]^{-1} \approx 1 - \int_0^t \operatorname{tr}\sigma(\mathbf{r},t')\,dt'$ so that Eq. (56) becomes equal to $-\int_0^t \langle \operatorname{tr}\sigma(\mathbf{r},t)\operatorname{tr}\sigma(\mathbf{r},t')\rangle dt'$. More formally, let us introduce the compressible generalization of the Kraichnan ensemble for smooth velocities. Their (nonconstant part of the) pair correlation function is defined as

$$d^{ij}(\mathbf{r}) = D_1[(d+1-2\wp)\,\delta^{ij}r^2 + 2(\wp d-1)r^ir^j] + o(r^2), \tag{57}$$

compare to Eq. (29). The degree of compressibility $\wp \equiv \langle(\nabla_i v^i)^2\rangle/\langle(\nabla_i v^j)^2\rangle$ is between 0 and 1 for the isotropic case at hand, with the two extrema corresponding to the incompressible and the potential cases. The corresponding strain matrix $\sigma = \nabla\mathbf{v}$ has the Eulerian mean equal to zero and two-point function

$$\langle \sigma_{ij}(t)\sigma_{kl}(t')\rangle = 2\,\delta(t-t')D_1[(d+1-2\wp)\,\delta_{ik}\delta_{jl} + (\wp d-1)(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})]. \tag{58}$$

The volume growth rate $-\int_0^t \langle \sigma_{ii}(t)\sigma_{jj}(t')\rangle dt'$ is thus strictly negative, in agreement with the general discussion, and equal to $-\wp D_1 d(d-1)(d+2)$ if we set $\int_0^\infty \delta(t)\,dt = 1/2$. The same result is obtained more systematically by considering the Itô stochastic equation $dW = \sigma\,dt\,W$ for the evolution matrix and applying the Itô formula to $\ln\det(W)$; see the Appendix. One may identify the generator of the process $W(t)$ and proceed as for the incompressible case calculating the PDF $\mathcal{P}(\rho_1,\ldots,\rho_d;t)$. It takes again the large deviation form (21), with the entropy function and the Lyapunov exponents given by

$$H(\mathbf{x}) = \frac{1}{4D_1[d+\wp(d-2)]} \times \left[\sum_{i=1}^d x_i^2 + \frac{1-\wp d}{\wp(d-1)(d+2)}\left(\sum_{i=1}^d x_i\right)^2\right], \tag{59}$$

$$\lambda_i = D_1\{d(d-2i+1) - 2\wp[d+(d-2)i]\}, \tag{60}$$

to be compared to Eq. (32). Note how the form (59) of the entropy imposes the condition $\Sigma x_i = 0$ in the incompressible limit. The interparticle distance $R(t)$ has the lognormal distribution (23) with $\bar{\lambda} = \lambda_1 = D_1(d-1)(d-4\wp)$ and $\Delta = 2D_1(d-1)(1+2\wp)$. Explicitly, $t^{-1}\ln\langle R^n\rangle \propto n[n+d+2\wp(n-2)]$ (Chertkov, Falkovich, and Kolokolov, 1998). The quantity $R^{(4\wp-d)/(1+2\wp)}$ is thus statistically conserved. The highest Lyapunov exponent $\bar{\lambda}$ becomes negative when the degree of compressibility is larger than $d/4$ (Le Jan, 1985; Chertkov, Falkovich, and Kolokolov, 1998). Low-order moments of R, including its logarithm, would then decrease while high-order moments would grow with time.

It is instructive to decompose the strain into its "incompressible and compressible parts" $\sigma_{ij} - (1/d)\,\delta_{ij}\operatorname{tr}\sigma$ and $(1/d)\,\delta_{ij}\operatorname{tr}\sigma$. From the equality $\lambda_i = \langle\tilde{\sigma}_{ii}\rangle$, see Eq. (21), it follows that the Lyapunov exponents of the incompressible part (having $\bar{\lambda} > 0$) get uniformly shifted down by the Lagrangian average of $\operatorname{tr}\sigma/d$. In 1D, where the compressibility is maximal,[4] $\bar{\lambda} < 0$. The lowering of the Lyapunov exponents when $\wp$ grows clearly signals the increase of trapping. The regime with $\wp > d/4$, with all the Lyapunov exponents becoming negative, is the one where trapping effects dominate. The dramatic consequences for the scalar fields advected by such flow will be discussed in Sec. III.B.1.

As it was clear from the 1D example, we should expect even stronger effects of compressibility in nonsmooth velocity fields, with an increased tendency for

[4]One-dimensional results are recovered from our formulas by taking $\wp = 1$ and $D_1 \propto 1/(d-1)$.

the fluid particles to aggregate in finite time. This has, indeed, been shown to occur when the velocity has a short correlation time, i.e., for the nonsmooth version of the compressible Kraichnan ensemble (Gawędzki and Vergassola, 2000). The expression (46) for the Fourier transform of the two-point correlation function is easily modified. The functional dependence on k^2 remains the same, the solenoidal projector is simply multiplied by $1-\wp$ and one adds to it the compressible longitudinal component $\wp(d-1)k^ik^j/k^2$. This gives for the nonconstant part of the two-point function

$$d^{ij}(\mathbf{r})=D_1[(d-1+\xi-\wp\xi)\delta^{ij}r^{\xi}+\xi(\wp d-1)r^ir^jr^{\xi-2}]. \tag{61}$$

For $\xi=2$, Eq. (61) reproduces Eq. (57) without the $o(r^2)$ corrections. Most of the results discussed in Sec. II.C for the incompressible version of the model still go through, including the construction of the Markov process describing the noisy trajectories and the heat-kernel form of the joint PDF of two particle positions. The restriction M of $\mathcal{M}_2$ to the homogeneous and isotropic sector, whose heat kernel gives the PDF $\mathcal{P}(r;R;t)$ of the distance between two particles, now takes the form

$$M=\left[\frac{D_1(d-1)(1+\wp\xi)}{r^{d-1-\gamma}}\partial_r r^{d-1+\xi-\gamma}+\frac{2\kappa}{r^{d-1}}\partial_r r^{d-1}\right]\partial_r, \tag{62}$$

where $\gamma=\wp\xi(d+\xi)/(1+\wp\xi)$. As found in Gawędzki and Vergassola (2000), see also Le Jan and Raimond (1999) and E and Vanden Eijnden (2000b), depending on the smoothness exponent ξ and the degree of compressibility $\wp$, two different regimes arise in the limit $\kappa\to 0$.

For weak compressibility $\wp<\wp_c\equiv d/\xi^2$, the situation is very much the same as for the incompressible case and

$$\lim_{\substack{r\to 0\\ \kappa\to 0}}\mathcal{P}(r;R;t)\propto\frac{R^{d-\gamma-1}}{|t|^{(d-\gamma)/(2-\xi)}}\exp\left(-\text{const}\times\frac{R^{2-\xi}}{|t|}\right). \tag{63}$$

We still have an explosive separation of trajectories but, in comparison to the incompressible situation, the power prefactor $R^{-\gamma}$ with $\gamma\propto\wp$ suppresses large separations and enhances small ones. When $\wp$ crosses $\widetilde{\wp}_c=(d+\xi-2)/2\xi$, the particle-touching event $R=0$ becomes recurrent for the Markov process describing the distance between the two particles (Le Jan and Raimond, 1999). In other words, for $\widetilde{\wp}_c\leqslant\wp\leqslant\wp_c$ a pair of Lagrangian trajectories returns infinitely often to a near touch, a clear sign of increased trapping.

When $\wp$ crosses $\wp_c$, the singularity at $R=0$ of the right-hand side of Eq. (63) becomes nonintegrable and a different limit is realized. Indeed, in this regime,

$$\lim_{\kappa\to 0}\mathcal{P}(r;R;t)=\mathcal{P}^{reg}(r;R;t)+p(r;t)\,\delta(R), \tag{64}$$

with the regular part $\mathcal{P}^{reg}$ tending to zero and p approaching unity when $r\to 0$. This reproduces for $\eta=0$ the result (54), always holding when the viscous cutoff $\eta>0$ smooths the velocity realizations. In other words, even though the velocity is nonsmooth, the Lagrangian trajectories in a fixed velocity field are determined by their initial positions. Moreover, the contact term in Eq. (64) signals that trajectories starting at a finite distance r collapse to zero distance and stay together with a positive probability growing with time (to unity if the integral scale $L=\infty$). The strongly compressible regime $\wp>\wp_c$ is clearly dominated by trapping effects leading to the aggregation of fluid particles; see Fig. 2. The same results hold if we turn off the diffusivity κ and the viscosity ν at the same time, with the notable exception of the intermediate regime $\widetilde{\wp}_c\leqslant\wp<\wp_c$. In this interval, if the Schmidt number Sc diverges fast enough during the limiting process, the resulting PDF of the distance takes the form (64) rather than that of Eq. (55) arising for bounded Sc (E and Vanden Eijnden, 2000b). Sufficiently high Schmidt numbers thus lead to the particle aggregation in this case. Note that in the limit of smooth velocities $\xi\to 2$, the intermediate interval shrinks to the point $\wp=d/4$ where the highest Lyapunov exponent crosses zero.

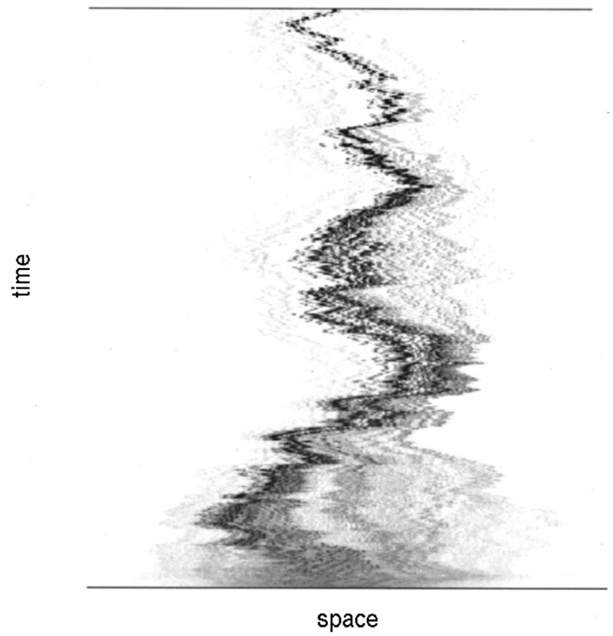

FIG. 2. The implosion of Lagrangian trajectories in a strongly compressible flow. Particles that are initially released uniformly across a sizable span of the interval are compressed and tend to produce a singular density field.

As was mentioned, the aggregation of fluid particles can take place only as a transient process. The back reaction of the density on the flow eventually stops the growth of the density fluctuations. The transient trapping should, however, play a role in the creation of the shocklet structures observed in high Mach number compressible flows (Zeman, 1990). There is another important physical situation that may be modeled by a smooth compressible random flow with a nonzero sum of the Lyapunov exponents. Let us consider a small inertial particle of density ρ and radius a in a fluid of density ρ_0. Its movement may be approximated by that of a Lagrangian particle in an effective velocity field provided that a^2/ν is much smaller than the velocity time scale in the Lagrangian frame. The inertial difference between the effective velocity $\mathbf{v}$ of the particle and the fluid velocity $\mathbf{u}(\mathbf{r},t)$ is proportional to the local acceleration: $\mathbf{v}=\mathbf{u}+(\beta-1)\tau_s\, d\mathbf{u}/dt$, where $\beta=3\rho/(\rho+2\rho_0)$ and $\tau_s=a^2/3\nu\beta$ is the Stokes time. Considering such particles distributed in the volume, one may define the velocity field $\mathbf{v}(\mathbf{r},t)$, whose divergence $\propto\boldsymbol{\nabla}[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}]$ does not vanish even if the fluid flow is incompressible. As discussed above, this leads to a negative volume growth rate and the clustering of the particles (Balkovsky *et al.*, 2001).

E. Multiparticle dynamics, statistical conservation laws, and breakdown of scale invariance

This subsection is a highlight of the review. We describe here the time-dependent statistics of multiparticle configurations with the emphasis on conservation laws of turbulent transport. As we have seen in the previous subsections, the two-particle statistics is characterized by a simple behavior of the single separation vector. In nonsmooth velocities, the length of the vector grows by a power law, while the initial separation is forgotten and there are no statistical integrals of motion. In contrast, the many-particle evolution exhibits nontrivial statistical conservation laws that involve geometry and are proportional to positive powers of the distances. The distance growth is balanced by the decrease of the shape fluctuations in those integrals. The existence of multiparticle conservation laws indicates the presence of a long-time memory and is a reflection of the coupling among the particles due to the simple fact that they all are in the same velocity field. The conserved quantities may be easily built for the limiting cases. For very irregular velocities, the fluid particles undergo independent Brownian motions and the interparticle distances grow as $\langle R_{nm}^2(t)\rangle=R_{nm}^2(0)+Dt$. Here, examples of statistical integrals of motion are $\langle R_{nm}^2-R_{pr}^2\rangle$ and $\langle 2(d+2)R_{nm}^2R_{pr}^2-d(R_{nm}^4+R_{pr}^4)\rangle$, and an infinity of similarly built harmonic polynomials where all the powers of t cancel out. Another example is the infinite-dimensional case, where the interparticle distances do not fluctuate. The two-particle law (38), $R_{nm}^{1-\alpha}(t)-R_{nm}^{1-\alpha}(0)\propto t$, implies then that the expectation of any function of $R_{nm}^{1-\alpha}-R_{pr}^{1-\alpha}$ does not change with time. A final example is provided by smooth velocities, where the particle separations at long times become aligned with the eigendirections of the largest Lyapunov exponent of the evolution matrix $W(t)$ defined in Eq. (16). All the interparticle distances R_{nm} will then grow exponentially and their ratios R_{nm}/R_{kl} do not change. Away from the degenerate limiting cases, the conserved quantities continue to exist, yet they cannot be constructed so easily and they depend on the number of particles and their configuration geometry. The very existence of conserved quantities is natural; what is generally nontrivial is their precise form and their scaling. The intricate statistical conservation laws of multiparticle dynamics were first discovered for the Kraichnan velocities. That came as a surprise since the Kraichnan velocity ensemble is Gaussian and time decorrelated, with no structure built in, except for the spatial scaling in the inertial range. The discovery has led to a new qualitative and quantitative understanding of intermittency, as we shall discuss in detail in Sec. III.C.1. Even more importantly, it has pointed to aspects of the multiparticle evolution that seem both present and relevant in generic turbulent flows. Note that those aspects are missed by simple stochastic processes commonly used in numerical Lagrangian models. There is, for example, a long tradition to take for each trajectory the time integral of a d-dimensional Brownian motion (whose variance is $\propto t^3$ as in the Richardson law) or an Ornstein-Uhlenbeck process. Such models, however, cannot capture correctly the subtle features of the N-particle dynamics such as the statistical conservation laws.

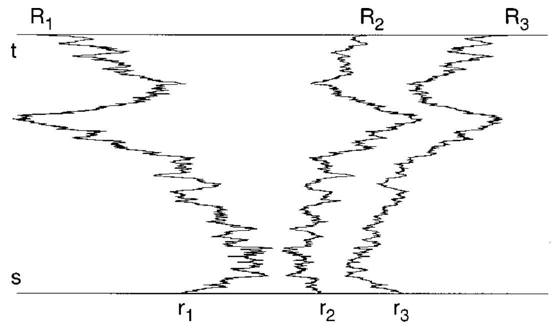

FIG. 3. An example of Lagrangian trajectories of three particles. The probability density of the positions $\underline{\mathbf{R}}$, conditional to the $\underline{\mathbf{r}}$'s, is described by the PDF $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ (in a fixed realization of the velocity). Its average over the statistics of the velocity field gives the Green functions $\mathcal{P}(\underline{\mathbf{r}};\underline{\mathbf{R}};t-s)$.

1. Absolute and relative evolution of particles

As for many-body problems in other branches of physics, e.g., in kinetic theory or in quantum mechanics, multiparticle dynamics brings about new aspects due to cooperative effects. In turbulence, such effects are mediated by the velocity fluctuations with long space correlations. Consider the joint PDF's of the equal-time positions $\underline{\mathbf{R}}=(\mathbf{R}_1,\ldots,\mathbf{R}_N)$ of N fluid trajectories

$$\left\langle\prod_{n=1}^{N}p(\mathbf{r}_n,s;\mathbf{R}_n,t|\mathbf{v})\right\rangle\equiv\mathcal{P}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t-s),\tag{65}$$

with the average over the velocity ensemble; see Fig. 3. More generally, one may study the different-time versions of Eq. (65). Such PDF's, called multiparticle Green functions, account for the overall statistics of the many-particle systems.

For statistically homogeneous velocities, it is convenient to separate the absolute motion of the particles from the relative one, as in other many-body problems with spatial homogeneity. For a single particle, there is nothing but the absolute motion which is diffusive at times longer than the Lagrangian correlation time (Sec. II.A). For N particles, we may define the absolute motion as that of the mean position $\overline{\mathbf{R}}=\Sigma_n \mathbf{R}_n/N$. When the particles separate beyond the velocity correlation length, they are essentially independent. The absolute motion is then diffusive with the diffusivity N times smaller than that of a single particle. The relative motion of N particles may be described by the versions of the joint PDF's (65) averaged over rigid translations:

$$\widetilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\int \mathcal{P}_N(\underline{\mathbf{r}};\underline{\mathbf{R}}+\boldsymbol{\rho};t)\,d\boldsymbol{\rho}, \tag{66}$$

where $\boldsymbol{\rho}=(\boldsymbol{\rho},\ldots,\boldsymbol{\rho})$. The PDF in Eq. (66) describes the distribution of the particle separations $\mathbf{R}_{nm}=\mathbf{R}_n-\mathbf{R}_m$ or of the relative positions $\underline{\widetilde{\mathbf{R}}}=(\mathbf{R}_1-\overline{\mathbf{R}},\ldots,\mathbf{R}_N-\overline{\mathbf{R}})$.

As for two particles, we expect that when $\kappa\to 0$ the multiparticle Green functions $\mathcal{P}_N$ tend to (possibly distributional) limits that we shall denote by the same symbol. The limiting PDF's are again expected to show a different short-distance behavior for smooth and nonsmooth velocities. For smooth velocities, the existence of deterministic trajectories leads for $\kappa=0$ to the collapse property

$$\lim_{\mathbf{r}_N\to\mathbf{r}_{N-1}} \mathcal{P}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\mathcal{P}_{N-1}(\underline{\mathbf{r}}';\underline{\mathbf{R}}';t)\,\delta(\mathbf{R}_{N-1}-\mathbf{R}_N), \tag{67}$$

where $\underline{\mathbf{R}}'=(\mathbf{R}_1,\ldots,\mathbf{R}_{N-1})$ and similarly for the relative PDF's. If all the distances among the particles are much smaller than the viscous cutoff, the velocity differences are approximated by linear expressions and

$$\widetilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\int \left\langle \prod_{n=1}^{N} \delta(\mathbf{R}_n+\boldsymbol{\rho}-W(t)\mathbf{r}_n)\right\rangle d\boldsymbol{\rho}. \tag{68}$$

The evolution matrix $W(t)$ was defined in Eq. (16) and the above PDF's clearly depend only on its statistics which has been discussed in Sec. II.B.

2. Multiparticle motion in Kraichnan velocities

The great advantage of the Kraichnan model is that the statistical Lagrangian integrals of motion can be found as zero modes of explicit evolution operators. Indeed, the crucial simplification lies in the Markov character of the Lagrangian trajectories due to the velocity time decorrelation. In other words, the processes $\underline{\mathbf{R}}(t)$ and $\underline{\widetilde{\mathbf{R}}}(t)$ are Markovian and the multiparticle Green functions $\mathcal{P}_N$ and $\widetilde{\mathcal{P}}_N$ give, for fixed N, their transition probabilities. The process $\underline{\mathbf{R}}(t)$ is characterized by its second-order differential generator $\mathcal{M}_N$, whose explicit form may be deduced by a straightforward generalization of the path-integral representation (52) to N particles. The PDF $\mathcal{P}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)=e^{|t-s|\mathcal{M}_N}(\underline{\mathbf{r}};\underline{\mathbf{R}})$ with

$$\mathcal{M}_N=\sum_{n,m=1}^{N} D^{ij}(\mathbf{r}_{nm})\nabla_{r_n^i}\nabla_{r_m^j}+\kappa\sum_{n=1}^{N}\nabla^2_{\mathbf{r}_n}. \tag{69}$$

For the relative process $\underline{\widetilde{\mathbf{R}}}(t)$, the operator $\mathcal{M}_N$ should be replaced by its translation-invariant version

$$\widetilde{\mathcal{M}}_N=-\sum_{n<m}[d^{ij}(\mathbf{r}_{nm})+2\kappa\delta^{ij}]\nabla_{r_n^i}\nabla_{r_m^j}, \tag{70}$$

with d^{ij} related to D^{ij} by Eq. (47). Note the multibody structure of the operators in Eqs. (69) and (70). The limiting PDF's obtained for $\kappa\to 0$ define the heat kernels of the $\kappa=0$ version of the operators that are singular elliptic and require some care in handling (Hakulinen, 2000).

As we have seen previously, the Kraichnan ensemble may be used to model both smooth and Hölder continuous velocities. In the first case, one keeps the viscous cutoff η in the two-point correlation (46) with the result that $d^{ij}(\mathbf{r})=\mathcal{O}(r^2)$ for $r\ll\eta$ as in Eq. (29), or one sets $\xi=2$ in Eq. (48). The latter is equivalent to the approximation (68) with $W(t)$ becoming a diffusion process on the group SL(d) of unimodular matrices, with an explicitly known generator, as discussed in Sec. II.B.2.a. The right-hand side of Eq. (68) may then be studied by using the representation theory (Shraiman and Siggia, 1995, 1996; Bernard *et al.*, 1998), see also Sec. II.E.5 below. It exhibits the collapse property (67).

From the form (70) of the generator of the process $\underline{\widetilde{\mathbf{R}}}(t)$ in the Kraichnan model, we infer that N fluid particles undergo an effective diffusion with the diffusivity depending on the interparticle distances. In the inertial interval and for a small molecular diffusivity κ, the effective diffusivity scales as the power ξ of the interparticle distances. Comparing to the standard diffusion with constant diffusivity, it is intuitively clear that the particles spend longer time together when they are close and separate faster when they become distant. Both tendencies may coexist and dominate the motion of different clusters of particles. It remains to find a more analytic and quantitative way to capture those behaviors. The effective short-distance attraction that slows down the separation of close particles is a robust collective phenomenon expected to be present also in time-correlated and non-Gaussian velocity fields. We believe that it is responsible for the intermittency of scalar fields transported by high Reynolds number flows, as it will be discussed in the second part of the review.

As for a single particle, the absolute motion of N particles is dominated by velocity fluctuations on scales of order L. In contrast, the relative motion within the inertial range is approximately independent of the velocity cutoffs and it is convenient to take directly the scaling limit $\eta=0$ and $L=\infty$. We shall also set the molecular diffusivity to zero. In these limits, $\widetilde{\mathcal{M}}_N$ has the dimension length$^{\xi-2}$, implying that time scales as length$^{2-\xi}$ and

$$\widetilde{\mathcal{P}}_N(\lambda\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\lambda^{-(N-1)d}\widetilde{\mathcal{P}}_N(\underline{\mathbf{r}};\lambda^{-1}\underline{\mathbf{R}};\lambda^{\xi-2}t). \tag{71}$$

The relative motion of N fluid particles may be tested by tracing the time evolution of the Lagrangian averages

$$\langle f(\underline{\mathbf{R}}(t))\rangle=\int f(\underline{\mathbf{R}})\widetilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)\,d\underline{\mathbf{R}}' \tag{72}$$

of translation-invariant functions f of the simultaneous particle positions. Think about the evolution of N fluid particles as that of a discrete cloud of marked points in physical space. There are two elements in the evolution of the cloud: the growth of its size and the change of its shape. We shall define the overall size of the cloud as $R=[(1/N)\Sigma_{n<m}R_{nm}^2]^{1/2}$ and its "shape" as $\hat{\underline{\mathbf{R}}}=\tilde{\underline{\mathbf{R}}}/R$. For example, three particles form a triangle in space (with labeled vertices) and the notion of shape that we are using includes the rotational degrees of freedom of the triangle. The growth of the size of the cloud might be studied by looking at the Lagrangian average of the positive powers R^ζ. More generally, let f be a scaling function of dimension ζ, i.e., such that $f(\lambda\underline{\mathbf{R}})=\lambda^\zeta f(\underline{\mathbf{R}})$. The change of variables $\underline{\mathbf{R}}\mapsto t^{1/(2-\xi)}\underline{\mathbf{R}}$, the relation (71), and the scaling property of f allow us to trade the Lagrangian PDF $\tilde{\mathcal{P}}_N$ in Eq. (72) for that at unit time $t^{\zeta/(2-\xi)}\tilde{\mathcal{P}}_N(t^{-1/(2-\xi)}\underline{\mathbf{r}};\underline{\mathbf{R}},1)$. As for two points, the limit of $\tilde{\mathcal{P}}_N$ when the initial points approach each other is nonsingular for nonsmooth velocities and we infer that

$$\langle f(\underline{\mathbf{R}}(t))\rangle=t^{\zeta/(2-\xi)}\int f(\underline{\mathbf{R}})\tilde{\mathcal{P}}_N(\underline{\mathbf{0}};\underline{\mathbf{R}};1)\,d\underline{\mathbf{R}}'+o(t^{\zeta/(2-\xi)}). \tag{73}$$

In particular, we obtain the N-particle generalization of the Richardson-type behavior (39): $\langle R(t)^\zeta\rangle\propto t^{\zeta/(2-\xi)}$. Hence in the Kraichnan model the size of the cloud of Lagrangian points grows superdiffusively $\propto t^{1/(2-\xi)}$. What about its shape?

3. Zero modes and slow modes

In order to test the evolution of the shape of the cloud one might compare the Lagrangian averages of different scaling functions. The relation (73) suggests that at large times they all scale dimensionally as $t^{\zeta/(2-\xi)}$. Actually, all do but those for which the integral in Eq. (73) vanishes. The latter scaling functions, whose evolution violates the dimensional prediction, may thus be better suited for testing the evolution of the shape of the cloud. Do such functions exist? Suppose that f is a scaling function of non-negative dimension ζ annihilated by $\tilde{\mathcal{M}}_N$, i.e., such that $\tilde{\mathcal{M}}_N f=0$. Its Lagrangian average, rather than obeying the dimensional law, does not change in time: $\langle f(\underline{\mathbf{R}}(t))\rangle=f(\underline{\mathbf{r}})$. Indeed, $\partial_t\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\tilde{\mathcal{M}}_N\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)$. Therefore the time-derivative of the right-hand side of Eq. (72) vanishes since it brings down the (Hermitian) operator $\tilde{\mathcal{M}}_N$ acting on f. Thus the zero modes of $\tilde{\mathcal{M}}_N$ are conserved in mean by the Lagrangian evolution. The importance of such conserved modes for the transport properties by δ-correlated velocities has been recognized independently by Shraiman and Siggia (1995), Chertkov *et al.* (1995b), and Gawędzki and Kupiainen (1995, 1996).

The above mechanism may be easily generalized (Bernard *et al.*, 1998). Suppose that f_k is a zero mode of the $(k+1)$th power of $\tilde{\mathcal{M}}_N$ (but not of a lower one), with scaling dimension $\zeta+(2-\xi)k$. Then, its Lagrangian average is a polynomial of degree k in time since its $(k+1)$th time derivative vanishes. Its temporal growth is slower than the dimensional prediction $t^{\zeta/(2-\xi)+k}$ if $\zeta>0$ so that the integral coefficient in Eq. (73) must vanish. We shall call such scaling functions slow modes. The slow modes may be organized into "towers" with the zero modes at the bottom.[5] One descends down the tower by applying the operator $\tilde{\mathcal{M}}_N$ which lowers the scaling dimension by $(2-\xi)$. The zero and the slow modes are natural candidates for probes of the shape evolution of the Lagrangian cloud. There is an important general feature of those modes due to the multibody structure of the operators: the zero modes of $\tilde{\mathcal{M}}_{N-1}$ are also zero modes of $\tilde{\mathcal{M}}_N$ and the same for the slow modes. Only those modes that depend nontrivially on all the positions of the N points may of course give new information on the N-particle evolution which cannot be read from the evolution of a smaller number of particles. We shall call such zero and slow modes irreducible.

To get convinced that zero and slow modes do exist, let us first consider the limiting case $\xi\to 0$ of very rough velocity fields. In this limit $d^{ij}(\mathbf{r})$ reduces to $D_1(d-1)\delta^{ij}$, see Eq. (48), and the operator $\tilde{\mathcal{M}}_N$ becomes proportional to $\underline{\nabla}^2$, the (Nd)-dimensional Laplacian restricted to the translation-invariant sector. The relative motion of the particles becomes a pure diffusion. If R denotes the size-of-the-cloud variable then

$$\underline{\nabla}^2=R^{-d_N+1}\partial_R R^{d_N-1}\partial_R+R^{-2}\hat{\underline{\nabla}}^2, \tag{74}$$

where $d_N\equiv(N-1)d$ and $\hat{\underline{\nabla}}^2$ is the angular Laplacian on the (d_N-1)-dimensional unit sphere of shapes $\hat{\underline{\mathbf{R}}}$. The spectrum of the latter may be analyzed using the properties of the rotation group. Its eigenvalues are $-j(j+d_N-2)$, where $j=0,1,\ldots$ is the angular momentum. The functions $f_{j,0}=R^j\phi_j(\hat{\underline{\mathbf{R}}})$, where ϕ_j is an angular momentum j eigenfunction, are zero modes of the Laplacian with scaling dimension j. The contributions coming from the radial and the angular parts in Eq. (74) indeed cancel out. The polynomials $f_{j,k}=R^{2k}f_{j,0}$ form the corresponding (infinite) tower of slow modes. All the scaling zero and slow modes of the Laplacian are of that form. The mechanism behind the special behavior of their Lagrangian averages is that mentioned at the beginning of the section. Beside the constant, the simplest zero mode has the form of the difference $\mathbf{R}_{12}^2-\mathbf{R}_{13}^2$. Both terms are slow modes in the tower of the constant zero mode and their Lagrangian averages grow linearly in time with the same leading coefficient. Their difference is thus constant. A similar mechanism stands behind the next example, the difference $2(d+2)\mathbf{R}_{12}^2\mathbf{R}_{34}^2-d(\mathbf{R}_{12}^4+\mathbf{R}_{34}^4)$, whose Lagrangian average is conserved due to the cancellation of linear and quadratic terms in time, and so on.

It was argued by Bernard *et al.* (1998) that the slow modes exist also for general ξ and show up in the asymptotic behavior of the multiparticle PDF's when the initial points get close:

[5] Note that $(\tilde{\mathcal{M}}_N)^k f_k$ is a zero mode of scaling dimension ζ.

$$\widetilde{\mathcal{P}}_N(\lambda\underline{\mathbf{r}};\underline{\mathbf{R}};t)=\sum_a\sum_{k=0}^{\infty}\lambda^{\zeta_a+(2-\xi)k}f_{a,k}(\underline{\mathbf{r}})g_{a,k}(\underline{\mathbf{R}},t),\tag{75}$$

for small λ. The first sum is over the zero modes $f_{a,0}\equiv f_a$ with scaling dimensions ζ_a, while higher k give the corresponding towers of slow modes. The functions in Eq. (75) may be normalized so that $f_{a,k-1}=\widetilde{\mathcal{M}}_N f_{a,k}$ and $g_{a,k+1}=\widetilde{\mathcal{M}}_N g_{a,k}=\partial_t g_{a,k}$. The leading term in the expansion comes from the constant zero mode $f_{0,0}=1$. The corresponding $g_{0,0}$ coincides with the PDF of N initially overlapping particles. The asymptotic expansion (75) is easy to establish for vanishing ξ and for $N=2$ with arbitrary ξ. In the general case, it has been obtained under some plausible, but yet unproven, regularity assumptions. Note that, due to Eq. (71), the expansion (75) describes also the asymptotics of the multiparticle PDF's when the final points get far apart and the times become large. The use of Eq. (75) allows us to extract the complete long-time asymptotics of the Lagrangian averages:

$$\langle f(\underline{\mathbf{R}}(t))\rangle=\sum_a\sum_{k=0}^{\infty}t^{(\zeta-\zeta_a)/(2-\xi)-k}f_{a,k}(\underline{\mathbf{r}})\times\int f(\underline{\mathbf{R}})g_{a,k}(\underline{\mathbf{R}},1)d\underline{\mathbf{R}}',\tag{76}$$

which is a detailed refinement of Eq. (73), corresponding just to the first term. Note that the pure polynomial-in-time behavior of the Lagrangian averages of the slow modes implies partial orthogonality relations between the slow modes and the g modes.

4. Shape evolution

The qualitative mechanism behind the preservation of the Lagrangian average of the zero modes is the compensation between its increase due to the size growth and its depletion due to the shape evolution. The size and the shape dynamics are mixed in the expansion (75) that, together with Eq. (71), describes the long-time long-distance relative evolution of the Lagrangian cloud. To get more insight into the cooperative behavior of the particles and the geometry of their configurations, it is useful to separate the shape evolution following Gat and Zeitak (1998); see also Arad and Procaccia (2000). The general idea is to trade time in the relative N-particle evolution $\underline{\widetilde{\mathbf{R}}}(t)$ for the size variable $R(t)$. This may be done as follows. Let us start with N particles in a configuration of size r and shape $\underline{\hat{\mathbf{r}}}$. Denote by $\underline{\hat{\mathbf{R}}}(R)$ the shape of the particle configuration the first time its size reaches $R\geqslant r$. Varying R, one obtains a description of the evolution of the shape of the particle cloud with its size (which may be discontinuous if the size does not grow at all moments of time). For scale invariant velocity fields, the PDF of the shapes $\underline{\hat{\mathbf{R}}}(R)$, i.e., of the first passages of $\underline{\widetilde{\mathbf{R}}}(t)$ through the sphere of size R, depends only on the ratio R/r. We shall denote it by $P_N(\underline{\hat{\mathbf{r}}};\underline{\hat{\mathbf{R}}};r/R)$. The shape evolution $\underline{\hat{\mathbf{R}}}(R)$ is still a Markov process: in order to compute the probability of the first passage through the sphere of size R, one may condition with respect to the first passage through a sphere of an intermediate size. As a result, the PDF's P_N obey a semigroup Chapman-Kolmogorov relation. As observed by Gat and Zeitak (1998), the eigenmode expansion of the (generally non-Hermitian) Markov semigroup $P_N(\lambda)$ involves the zero modes f_a:

$$P_N(\underline{\hat{\mathbf{r}}};\underline{\hat{\mathbf{R}}};\lambda)=\sum_a\lambda^{\zeta_a}f_a(\underline{\hat{\mathbf{r}}})h_a(\underline{\hat{\mathbf{R}}}).\tag{77}$$

The formal reason is as follows. The statistics of the first passage through a given surface may be obtained by imposing the Dirichlet boundary condition on the surface in the differential generator of the process. For the case at hand, if $G_N(\underline{\mathbf{r}},\underline{\mathbf{R}})$ denotes the kernel of the inverse of $-\widetilde{\mathcal{M}}_N$ and $G_N^{\rm Dir}$ refers to its version with the Dirichlet conditions at $R=1$, then

$$P_N(\underline{\hat{\mathbf{r}}};\underline{\hat{\mathbf{R}}};\lambda)=\frac{1}{2}\sum_{n,m=1}^{N}[\nabla_{R_n^i}R]\times[d^{ij}(\mathbf{R}_{nm})\nabla_{R_m^j}G_N^{\rm Dir}(\lambda\underline{\hat{\mathbf{r}}},\underline{\mathbf{R}})],\tag{78}$$

with the derivatives taken on the sphere of unit radius. The potential theory relation (78) expresses the simple fact that the probability of a first passage through a given surface is the normal component of the probability current (the expression in second parentheses on the right-hand side). On the other hand, by integrating the asymptotic expansion (75), one obtains the expansion

$$G_N(\lambda\underline{\mathbf{r}},\underline{\mathbf{R}})=\int_0^{\infty}\widetilde{\mathcal{P}}_N(\lambda\underline{\mathbf{r}};\underline{\mathbf{R}};t)dt=\sum_a\lambda^{\zeta_a}f_a(\underline{\mathbf{r}})\tilde{g}_a(\underline{\mathbf{R}}),\tag{79}$$

where $\tilde{g}_a=\int_0^{\infty}g_{a,0}\,dt$. Note that the slow modes do not appear since $g_{a,k+1}$ is the time derivative of $g_{a,k}$, that vanishes at the boundary of the integration interval for nonzero $\underline{\mathbf{R}}$. The Dirichlet boundary condition at $R=1$ should not affect the dependence on the asymptotically close positions $\lambda\underline{\mathbf{r}}$. An expansion analogous to Eq. (79) should then hold for $G_N^{\rm Dir}$ with the same zero modes and modified functions $\tilde{g}$. The potential theory formula Eq. (78) together with the Dirichlet version of Eq. (79) give Eq. (77).

Let us now consider the "shape only" version of the Lagrangian average (72). Substituting the expansion (77), we obtain

$$\langle f(\underline{\hat{\mathbf{R}}}(R))\rangle=\sum_a(R/r)^{-\zeta_a}f_a(\underline{\hat{\mathbf{r}}})\int f(\underline{\hat{\mathbf{R}}})h_a(\underline{\hat{\mathbf{R}}})d\underline{\hat{\mathbf{R}}}.\tag{80}$$

The interpretation of Eq. (80) is simple: when the size increases, the average of a generic function of the shape relaxes as a combination of negative powers of R to a constant, with the zero modes $f_a(\underline{\hat{\mathbf{r}}})$ giving the modes of relaxation (Gat and Zeitak, 1998). On the other hand, due to the orthogonality of the left and right eigenfunctions of the semigroup $P_N(\lambda)$, the shape averages of the zero modes decay: $\langle f_a(\underline{\hat{\mathbf{R}}}(R))\rangle=(R/r)^{-\zeta_a}f_a(\underline{\hat{\mathbf{r}}})$. Since the size grows as $t^{1/(2-\xi)}$, this quantitatively illustrates

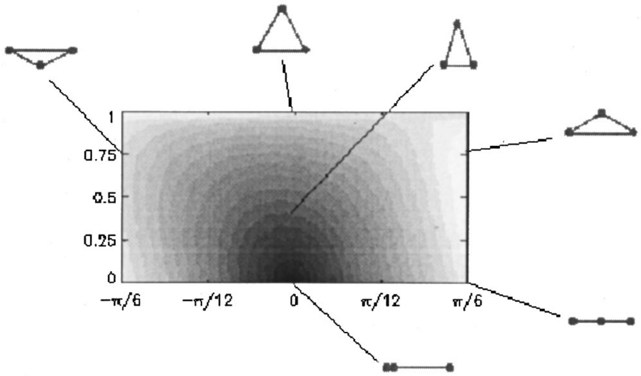

FIG. 4. The contour lines of a three-particle zero mode as a function of the shape of the triangle defined by the particles.

the decrease of the shape average responsible for the conservation in time of the Lagrangian average of the zero modes. A vivid and explicit example of the compensation is provided by the case of three particles. The contour lines of the relevant zero mode as a function of the shape of the triangle are shown in Fig. 4. The function tends to decrease for configurations where all the interparticle distances are comparable. It is then clear that the decrease in the shape average is simply due to particle evolving toward symmetrical configurations with aspect ratios of order unity (Pumir, 1998; Celani and Vergassola, 2001).

The relative motion of the particles in the limit $\xi\to 0$ becomes a pure diffusion. It is then easy to see that the zero modes of the Laplacian play indeed the role of relaxation modes of the $\xi=0$ shape evolution. Pure diffusion is the classical case of potential theory: $G_N(\underline{\mathbf{r}},\underline{\mathbf{R}}) \propto|\underline{\mathbf{r}}-\underline{\mathbf{R}}|^{-d_N+2}$ is the potential induced at $\underline{\mathbf{r}}$ by a unit charge placed at $\underline{\mathbf{R}}$ (the absolute value is taken in the sense of the size variable). The Dirichlet version $G_N^{\rm Dir}$ corresponds to the potential of a unit charge inside a grounded conducting sphere and it is obtained by the image charge method: $G_N^{\rm Dir}(\underline{\mathbf{r}},\underline{\mathbf{R}})=G_N(\underline{\mathbf{r}},\underline{\mathbf{R}}) -R^{-d_N+2}G_N(\underline{\mathbf{r}},\underline{\mathbf{R}}/R^2)$. The potential theory formula (78) gives then for the shape-to-shape transition probability $P_N(\hat{\underline{\mathbf{r}}};\hat{\underline{\mathbf{R}}};\lambda)\propto(1-\lambda^2)|\hat{\underline{\mathbf{R}}}-\lambda\hat{\underline{\mathbf{r}}}|^{-d_N}$, with the proportionality constant equal to the inverse volume of the sphere. On the other hand, calculating $G_N^{\rm Dir}$ in each angular momentum sector by Eq. (74), one easily describes the generator of the Markov semigroup $P_N(\lambda)$. It is the pseudodifferential Hermitian operator with the same eigenfunctions ϕ_j as the angular Laplacian $\hat{\underline{\nabla}}^2$ but with eigenvalues $-j$. The expansion (77) with $h_a=f_a$ follows (recall that the functions $R^j\phi_j$ form the zero modes of the Laplacian). The Markov process $\hat{\underline{\mathbf{R}}}(R)$ lives on distributional realizations (and not on continuous ones).

It is instructive to compare the shape dynamics of the Lagrangian cloud to the imaginary-time evolution of a quantum-mechanical many-particle system governed by the Hamiltonian $H_N=\Sigma_n(p_n^2/2m)+\Sigma_{n<m}V(\mathbf{r}_{nm})$. The (Hermitian) imaginary-time evolution operators e^{-tH_N} decompose in the translation-invariant sector as $\Sigma_a e^{-tE_{N,a}}|\psi_{N,a}\rangle\langle\psi_{N,a}|$. The ground-state energy is $E_{N,0}$ and the sum is replaced by an integral for the continuous part of the spectrum. An attractive potential between the particles may lead to the creation of bound states at the bottom of the spectrum of H_N. Breaking the system into subsystems of N_i particles by removing the potential coupling between them would then raise the ground-state energy: $E_{N,0}<\Sigma_i E_{N_i,0}$. A very similar phenomenon occurs in the stochastic shape dynamics. Consider indeed an even number of particles and denote by $\zeta_{N,0}$ the lowest scaling dimension of the irreducible zero mode invariant under translations, rotations, and reflections. For two particles there is no invariant irreducible zero mode and its role is played by the first slow mode $\propto r^{2-\xi}$ and $\zeta_{2,0}=2-\xi$. Suppose now that we break the system into subsystems with an even number N_i of particles by removing in $\widetilde{\mathcal{M}}_N$ the appropriate terms $d(\mathbf{r}_{nm})\nabla_{\mathbf{r}_n}\nabla_{\mathbf{r}_m}$ coupling the subsystems; see Eq. (70). For $N_i\geqslant 4$, the irreducible zero mode for the broken system factorizes into the product of such modes for the subsystems. If $N_i=2$ for some i, the factorization still holds modulo terms dependent on less variables. In any case, the scaling dimensions simply add up. The crucial observation, confirmed by perturbative and numerical analyses discussed below, is that the minimal dimension of the irreducible zero modes is raised: $\zeta_{N,0}<\Sigma_i\zeta_{N_i,0}$. In particular, $\zeta_{N,0}$ is smaller than $(N/2)(2-\xi)$. One even expects that $\zeta_{N,0}$ is a concave function of (even) N and that for $N\gg d$ its dependence on N saturates, see Secs. III.C.2, III.D.2, and III.F. By analogy with the multibody quantum mechanics, we may say that the irreducible zero modes are bound states of the shape evolution of the Lagrangian cloud. The effect is at the root of the intermittency of a passive scalar advected by nonsmooth Kraichnan velocities, as we shall see in Sec. III.C.1. It is a cooperative phenomenon exhibiting a short-distance attraction of close Lagrangian trajectories superposed on the overall repulsion of the trajectories. There are indications that similar bound states of the shape evolution persist in more realistic flows and that they are still responsible for the scalar intermittency; see Sec. III.D.2 and Celani and Vergassola (2001).

5. Perturbative schemes

The incompressible Kraichnan model has two parameters $d\in[2,\infty)$ and $\xi\in[0,2]$. It is then natural to ask if the problem is simplified at their limiting values and if perturbative methods might be used to get the zero modes near the limits. No significant simplification has been recognized for $d=2$ at arbitrary ξ. The other limits do allow for a perturbative treatment since the particle interaction is weak and the anomalous scaling disappears there. The perturbation theory is essentially regular for $\xi\to 0$ and $d\to\infty$. Conversely, the perturbation theory for $\xi\to 2$ is singular for two reasons. First, the advection by a smooth velocity field preserves the configurations with the particles aligned on a straight line. A small roughness of the velocity has little effect on the particle motion almost everywhere but for quasicollinear geometries. A separate treatment of those regions and a matching with the regular perturbation expansion for a general geometry is thus needed. Second, for almost smooth velocities, close particles separate very slowly and their collective behavior is masked by this effect which leads to an accumulation of zero modes with very close scaling dimensions. We shall start by the more regular cases of small ξ and large d. The scaling of the irreducible four-point zero mode with the lowest dimension was first calculated to the linear order in ξ by Gawędzki and Kupiainen (1995) by a version of degenerate Rayleigh-Schrödinger perturbation theory. In parallel, a similar calculation in the linear order in $1/d$ was performed by Chertkov *et al.* (1995b). Bernard *et al.* (1996) streamlined the small ξ analysis and generalized it to any even order, following a similar generalization by Chertkov and Falkovich (1996) for the $1/d$ expansion. We sketch here the main lines of those calculations.

As we discussed in Sec. II.E.3, the operator $\tilde{\mathcal{M}}_N$ is reduced to the Laplacian (74) for $\xi=0$. The zero modes of the Laplacian depend on the size of the particle configuration as R^j and on its shape as the eigenfunctions of $\hat{\underline{\nabla}}^2$ with angular momentum j. The zero modes invariant under d-dimensional translations, rotations, and reflections can be reexpressed as polynomials in $\mathbf{R}_{nm}^2$. For even N, the irreducible zero modes with the lowest scaling dimension have the form

$$f_{N,0}(\underline{\mathbf{R}})=\mathbf{R}_{12}^2\mathbf{R}_{34}^2\cdots\mathbf{R}_{(N-1)N}^2+[\cdots], \tag{81}$$

where $[\cdots]$ denotes a combination of terms that depend on the positions of $(N-1)$ or less particles. For four particles, the zero mode is $2(d+2)\mathbf{R}_{12}^2\mathbf{R}_{34}^2-d(\mathbf{R}_{12}^4+\mathbf{R}_{34}^4)$, our recurrent example. The terms $[\cdots]$ are not uniquely determined since any degree N zero mode for a smaller number of points might be added. Furthermore, permutations of the points in $f_{N,0}$ give other zero modes so that we may symmetrize the above expressions and look only at the permutation-invariant modes. The scaling dimension $\zeta_{N,0}$ of $f_{N,0}$ is clearly equal to N. This linear growth signals the absence of attractive effects between the particles diffusing with a constant diffusivity (no particle binding in the shape evolution). As we shall see in Sec. III.C.1, this leads to the disappearance of the intermittency in the advected scalar field, that becomes Gaussian in the limit $\xi\to 0$.

To the linear order in ξ, the operator $\tilde{\mathcal{M}}_N$ will differ from the Laplacian by a second-order differential operator $-\xi V$, involving logarithmic terms $\propto\ln(r_{nm})$. The zero mode and its scaling dimension are expanded as $f_0+\xi f_1$ and $N+\xi\zeta_1$, respectively. The lowest order term f_0 is given by the symmetrization of Eq. (81). As usual in such problems, the degeneracy hidden in $[\cdots]$ may be lifted by the perturbation that fixes f_0 for each zero mode, see below. At the first order in ξ, the equations that define the zero modes and their scaling dimension reduce to the relations

$$\underline{\nabla}^2 f_1=Vf_0,\quad (R\partial_R-N)f_1=\zeta_1 f_0. \tag{82}$$

Given an arbitrary zero mode f_0, one shows that the first equation admits a solution of the form $f_1=h+\Sigma_{n<m}h_{nm}\ln(r_{nm})$ with $O(d)$-invariant, degree N polynomials h_{nm} and h, the latter being determined up to zero modes of $\underline{\nabla}^2$. Note that the function $(R\partial_R-N)f_1=\Sigma_{n<m}h_{nm}$ is also annihilated by the Laplacian

$$\begin{aligned}\underline{\nabla}^2(R\partial_R-N)f_1&=(R\partial_R-N+2)\underline{\nabla}^2 f_1\\&=(R\partial_R-N+2)Vf_0\\&=([R\partial_R,V]+2V)f_0+V(R\partial_R-N)f_0\\&=0.\end{aligned} \tag{83}$$

The last equality follows from the scaling of f_0 and the fact that the commutators of $R\partial_R$ with $\tilde{\mathcal{M}}_N$ and V are $(\xi-2)\tilde{\mathcal{M}}_N$ and $-\underline{\nabla}^2-2V$, respectively. One obtains this way a linear map Γ on the space of the degree N zero modes of the Laplacian: $\Gamma f_0=(R\partial_R-N)f_1$. The second equation in Eq. (82) states that f_0 must be chosen as an eigenstate of the map Γ. Furthermore, the function should not belong to the subspace of unit codimension of the zero modes that do not depend on all the points. It is easy to see that such subspace is preserved by the map Γ. As the result, the eigenvalue ζ_1 is equal to the ratio between the coefficients of $\mathbf{R}_{12}^2\mathbf{R}_{34}^2\cdots\mathbf{R}_{(N-1)N}^2$ in Γf_0 and in f_0. The latter is easy to extract, see Bernard *et al.* (1996) for the details, and yields the result $\zeta_1=-N(N+d)/2(d+2)$ or, equivalently,

$$\zeta_{N,0}=\frac{N}{2}(2-\xi)-\frac{N(N-2)}{2(d+2)}\xi+\mathcal{O}(\xi^2), \tag{84}$$

giving the leading correction to the scaling dimension of the lowest irreducible zero mode. Note that to that order $\zeta_{N,0}$ is a concave function of N. Higher-order terms in ξ have been analyzed in Adzhemyan *et al.* (1998) (the second order) and in Adzhemyan *et al.* (2001) (the third order). The latter papers used a renormalization group resummation of the small ξ perturbative series for the correlation functions of the scalar gradients in conjunction with an operator product expansion; see Sec. III.C.1. The expression (84) may be easily generalized to the compressible Kraichnan ensemble of compressibility degree $\wp$. The correction ζ_1 for the tracer exponent picks up an additional factor $(1+2\wp)$ (Gawędzki and Vergassola, 2000). Higher-order corrections may be found in Antonov and Honkonen (2001). The behavior of the density correlation functions was analyzed in Adzhemyan and Antonov (1998), Gawędzki and Vergassola (2000), and Antonov and Honkonen (2001).

For large dimensionality d, it is convenient to use the variables $x_{nm}=R_{nm}^{2-\xi}$ as the independent coordinates[6] to make the d dependence in $\tilde{\mathcal{M}}_N$ explicit. Up to higher orders in $1/d$, the operator $\tilde{\mathcal{M}}_N\propto\mathcal{L}-(1/d)\,U$, where $\mathcal{L}=d^{-1}\Sigma_{n<m}[(d-1)\partial_{x_{nm}}+(2-\xi)x_{nm}\partial_{x_{nm}}^2]$ and U is a second-order d-independent differential operator mixing derivatives over different x_{nm}. We shall treat $\mathcal{L}$ as

[6]Their values are restricted only by the triangle inequalities between the interparticle distances.

the unperturbed operator and $-(1/d)\,U$ as a perturbation. The inclusion into $\mathcal{L}$ of the diagonal terms $\propto 1/d$ makes the unperturbed operator of the same (second) order in derivatives as the perturbation and renders the perturbative expansion less singular. The irreducible zero modes of $\mathcal{L}$ with the lowest dimension are given by an expression similar to Eq. (81):

$$f_{N,0}(\underline{\mathbf{R}})=x_{12}x_{34}\cdots x_{(N-1)N}+[\cdots],\tag{85}$$

and the permutations thereof. Their scaling dimension is $(N/2)(2-\xi)$. For $N=4$, one may, for example, take $f_{4,0}=x_{12}x_{34}-[d-1/2(2-\xi)]\,(x_{12}^2+x_{34}^2)$. As in the ξ expansion, in order to take into account the perturbation U, one has to solve the equations

$$\mathcal{L}f_1=Uf_0,\quad \left(\sum_{n<m}x_{nm}\partial_{x_{nm}}-\frac{N}{2}\right)f_1=\frac{\zeta_1}{2-\xi}f_0.\tag{86}$$

One checks again that $\Gamma f_0\equiv(\Sigma_{n<m}x_{nm}\partial_{x_{nm}}-N/2)f_1$ is annihilated by $\mathcal{L}$. In order to calculate ζ_1, it remains to find the coefficient of $x_{12}\cdots x_{(N-1)N}$ in Γf_0. In its dependence on the x's, the function Uf_0 scales with power $(N/2-1)$. One finds f_1 by applying the inverse of the operator $\mathcal{L}$ to it. Γf_0 is then obtained by gathering the coefficients of the logarithmic terms in f_1; see Chertkov *et al.* (1995b) for the details. When $d\to\infty$, the operator $\mathcal{L}$ reduces to the first order one $\mathcal{L}'\equiv\Sigma_{n<m}\partial_{x_{nm}}$. This signals that the particle evolution becomes deterministic at $d=\infty$, with all x_{nm} growing linearly in time. If one is interested only in the $1/d$ correction to the scaling exponent and not in the zero mode, then it is possible to use directly the more natural (but more singular) decomposition $\widetilde{\mathcal{M}}_N\propto\mathcal{L}'-(1/d)\,U'$. The leading zero modes of $\mathcal{L}'$ also have the form (85). Noting that $\mathcal{L}'$ is a translation operator, the zero mode $\Gamma' f_0$ may be obtained as the coefficient of the logarithmically divergent term in $\int_0^\infty U' f_0(x_{nm}-t)\,dt$; see Chertkov and Falkovich (1996). In both approaches, the final result is

$$\zeta_{N,0}=\frac{N}{2}(2-\xi)-\frac{N(N-2)}{2d}\,\xi+\mathcal{O}\left(\frac{1}{d^2}\right),\tag{87}$$

which is consistent with the small ξ expression (84).

The nonisotropic zero modes, as well as those for odd N, may be studied similarly. The zero modes of fixed scaling dimension form a representation of the rotation group SO(d) which may be decomposed into irreducible components. In particular, one may consider the components corresponding to the symmetric tensor products of the defining representation of SO(d), labeled by the angular momentum j (the multiplicity of the tensor product). For two particles, no other representations of SO(d) appear. The two-point operator $\widetilde{\mathcal{M}}_2$ becomes in each angular momentum sector an explicit second-order differential operator in the radial variable. It is then straightforward to extract the scaling dimensions of its zero modes:

$$\zeta_{2,0}^j=-\frac{d-2+\xi}{2}+\frac{1}{2}\sqrt{(d-2+\xi)^2+\frac{4(d-1+\xi)j(j+d-2)}{d-1}}.\tag{88}$$

Note that $\zeta_{2,0}^1=1$ in any d corresponding to linear zero modes. For the three-point operator, the lowest scaling dimensions are $\zeta_{3,0}^0=4-[2(d-2)/(d-1)]\,\xi+\mathcal{O}(\xi^2)$ (Gat *et al.*, 1997; Gat, L'vov, and Procaccia, 1997) and $\zeta_{3,0}^1=3-(d+4)/(d+2)\,\xi+\mathcal{O}(\xi^2)$ (Pumir, 1996, 1997) or $\zeta_{3,0}^1=3-\xi-2\xi/d+\mathcal{O}(1/d^2)$ (Gutman and Balkovsky, 1996). For even N and j, the generalization of Eq. (84) takes the form (Antonov, 1999, 2000; Arad, L'vov, *et al.*, 2000; Wiese, 2000)

$$\zeta_{N,0}^j=\frac{N}{2}(2-\xi)-\left(\frac{N(N-2)}{2(d+2)}-\frac{j(j+d-2)(d+1)}{2(d+2)(d-1)}\right)\xi+\mathcal{O}(\xi^2).\tag{89}$$

The effective expansion parameter in the small ξ or large d approach turns out to be $N\xi/[(2-\xi)d]$ so that neither of them is applicable to the region of the almost smooth velocity fields. This region requires a different perturbative technique exploiting the numerous symmetries exhibited by the multiparticle evolution in the limiting case $\xi=2$. Those symmetries were first noticed and employed to derive an exact solution for the zero modes by Shraiman and Siggia (1996). The expression of the multiparticle operators at $\xi=2$ reads

$$\widetilde{\mathcal{M}}_N=D_1[dH^2-(d+1)J^2],\tag{90}$$

with $H^2=\Sigma_{ij}H_{ij}H_{ji}$ and $J^2=-\Sigma_{i<j}J_{ij}^2$ denoting the Casimir operators of the group SL(d) and of its SO(d) subgroup acting on the index $i=1,\dots,d$ of the particle positions r_n^i. The corresponding generators are given by $H_{ij}=\Sigma_n[-r_n^i\nabla_{r_n^j}+(1/d)\,\delta_{ij}(r_n^k\nabla_{r_n^k})]$ and $J_{ij}=H_{ij}-H_{ji}$. The relation (90), that may be easily checked directly, is consistent with the expression (68) for the heat kernel of $\widetilde{\mathcal{M}}_N$. As mentioned in Sec. II.B.1, the right-hand side of Eq. (90) is indeed the generator of the diffusion process $W(t)$ on the group SL(d). In their analysis, Shraiman and Siggia (1995) employed an alternative expression for the multiparticle operators, exhibiting yet another symmetry of the smooth case:

$$\widetilde{\mathcal{M}}_N=D_1\left(dG^2-(d+1)J^2+\frac{d-N+1}{N-1}\Lambda(\Lambda+d_N)\right),\tag{91}$$

where $G^2=\Sigma_{n,m=1}^{N-1}G_{nm}G_{mn}$ is the quadratic Casimir of SL$(N-1)$ acting on the index $n=1,\dots,N-1$ of the difference variables r_{nN}^i and $\Lambda=\Sigma_{i,n}r_n^i\nabla_{r_n^i}$ is the generator of the overall dilations. For three points, one may then decompose the scaling translationally invariant functions into the eigenfunctions of G^2, L^2 and other generators commuting with the latter and with Λ. The zero modes of $\widetilde{\mathcal{M}}_3$ at $\xi=2$ have the lowest scaling dimension equal to unity and vanishing in the angular momentum sectors $j=1$ and $j=0$; see Pumir *et al.* (1997) and Balkovsky, Falkovich, and Lebedev (1997). They have infinite multiplicity since their space carries an infinite-dimensional representation of $SL(2)$. Similar zero modes exist for any higher scaling dimension. In 2D, for example, the scaling dimension of a three-particle zero mode may be raised multiplying it by a power of $\det(r^{\,i}_{n3})$. The continuous spectrum of the dimensions and the infinite degeneracy of the zero modes in the smooth case is one source of the difficulties. Another related difficulty is that for $\xi=2$ the principal symbol of $\tilde{\mathcal{M}}_3$ looses positive-definiteness not only when two of the points coincide, but also when all the three points become collinear. That leads to the domination for quasicollinear geometries of the perturbative terms in $\tilde{\mathcal{M}}_3$ over the unperturbed ones. The problem requires a boundary layer approach developed first by Pumir *et al.* (1997) for the $j=1$ sector with the conclusion that the minimal scaling dimension of the zero mode behaves as $1+o(2-\xi)$. Similar techniques led Balkovsky, Falkovich, and Lebedev (1997) to argue for a $\mathcal{O}(\sqrt{2-\xi})$ behavior of the minimal scaling dimension of the isotropic zero modes. The three-particle zero-mode equation was solved numerically for the whole range of values of ξ by Pumir (1997) for $j=1$ and $d=2,3$ and by Gat (1997) and Gat, L'vov, and Procaccia (1997) in the isotropic case for $d=2,3,4$. Their results are compatible with the perturbative analysis around $\xi=0$ and $\xi=2$, with a smooth interpolation for the intermediate values (no crossing between different branches of the zero modes). Analytical nonperturbative calculations of the zero modes were performed for the passive scalar shell models, where the degrees of freedom are discrete. We refer the interested reader to the original works (Benzi *et al.*, 1997; Andersen and Muratore-Ginanneschi, 1999) and to Bohr *et al.* (1998) for an introduction to shell models.

III. PASSIVE FIELDS

The results on the statistics of Lagrangian trajectories presented in Sec. II will be used here to analyze the properties of passively advected scalar and vector fields. The qualification "passive" means that we disregard the back reaction of the advected fields on the advecting velocity. We shall treat both a scalar per unit mass (a tracer field), satisfying the equation

$$\partial_t\theta+\mathbf{v}\cdot\boldsymbol{\nabla}\theta=\kappa\boldsymbol{\nabla}^2\theta, \tag{92}$$

and the density per unit volume, whose evolution is governed by

$$\partial_t n+\boldsymbol{\nabla}\cdot(n\mathbf{v})=\kappa\boldsymbol{\nabla}^2 n. \tag{93}$$

For incompressible flows, the two equations are obviously coinciding. Examples of passive vector fields are provided by the gradient of a tracer $\boldsymbol{\omega}=\nabla\theta$, obeying

$$\partial_t\boldsymbol{\omega}+\boldsymbol{\nabla}(\mathbf{v}\cdot\boldsymbol{\omega})=\kappa\nabla^2\boldsymbol{\omega}, \tag{94}$$

and the divergenceless magnetic field evolving in incompressible flow according to

$$\partial_t\mathbf{B}+\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{B}-\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{v}=\kappa\boldsymbol{\nabla}^2\mathbf{B}. \tag{95}$$

Two broad situations will be distinguished: forced and unforced. The evolution equations for the latter are Eqs. (92)–(95). They will be analyzed in Sec. III.A. For the former, a pumping mechanism such as a forcing term is present and steady states might be established. The rest of the section treats steady cascades of passive fields under the action of a permanent pumping. As we shall see below, the advection equations may be easily solved in terms of the Lagrangian flow, hence the relation between the behavior of the advected fields and of the fluid particles. In particular, the multipoint statistics of the advected fields will appear to be closely linked to the collective behavior of the separating Lagrangian particles. An introduction to the passive advection problem may be found in Shraiman and Siggia (2000).

A. Unforced evolution of scalar and vector fields

The physical situation of interest is that the initial passive field or its distribution is prescribed and the problem is to determine the field distribution at a later time t. The simplest question to address is which fields have their amplitudes decaying in time and which growing, assuming the velocity field to be statistically steady. A tracer field always decays because of dissipative effects, with the law of decay depending on the velocity properties. The fluctuations of a passive density may grow in a compressible flow, with this growth saturated by diffusion after some time. The fluctuations of both $\boldsymbol{\omega}$ and $\mathbf{B}$ may grow exponentially as long as diffusion is unimportant. After diffusion comes into play, their destinies are different: $\boldsymbol{\omega}$ decays, while the magnetic field continues to grow. This growth is known as dynamo process and it continues until saturated by the back reaction of the magnetic field on the velocity. Another important issue here is the presence or absence of a dynamic self-similarity: for example, is it possible to present the time-dependent PDF $\mathcal{P}(\theta;t)$ as a function of a single argument? In other words, does the form of the PDF remain invariant in time apart from a rescaling of the field? We shall show that for large times the scalar PDF tends to a self-similar limit when the advecting velocity is nonsmooth, while self-similarity is broken in smooth velocities.

1. Backward and forward in time Lagrangian description

If the advecting velocities are smooth and if the diffusive terms are negligible,[7] the advection equations may be easily solved in terms of the Lagrangian flow. To calculate the value of a passively advected field at a given time one has to trace the field evolution backwards

[7]Recall that the Schmidt number ν/κ, also called Prandtl number when considering temperature or magnetic fields, is assumed to be large.

along the Lagrangian trajectories. This is to be contrasted with the description of the particles in the previous section, which was developed in terms of the forward evolution. The tracer θ stays constant along the Lagrangian trajectories:

$$\theta(\mathbf{r},t)=\theta(\mathbf{R}(0;\mathbf{r},t),0), \tag{96}$$

where $\mathbf{R}(\cdot;\mathbf{r},t)$ denotes the Lagrangian trajectory passing at time t through the point $\mathbf{r}$. The density n changes along the trajectory as the inverse of the volume contraction factor. Let us consider the matrix $\widetilde{W}(t;\mathbf{r})=W(t;\mathbf{R}(0;\mathbf{r},t))$, where $W(t;\mathbf{r})$ is given by Eq. (13) and describes the forward evolution of small separations of the Lagrangian trajectories starting at time zero near $\mathbf{r}$. The volume contraction factor is $\det(\widetilde{W}(t;\mathbf{r}))$ and

$$n(\mathbf{r},t)=[\det(\widetilde{W}(t;\mathbf{r}))]^{-1}n(\mathbf{R}(0;\mathbf{r},t),0). \tag{97}$$

Note that the matrix $\widetilde{W}(t;\mathbf{r})$ is the inverse of the backward-in-time evolution matrix $W'(t;\mathbf{r})$ with the matrix elements $\partial R^i(0;\mathbf{r},t)/\partial r^j$. This is indeed implied by the identity $\mathbf{R}(t;\mathbf{R}(0;\mathbf{r},t),0)=\mathbf{r}$ and the chain rule for differentiation. The solution of the evolution equation for the gradient of the tracer is obtained by differentiating Eq. (96):

$$\boldsymbol{\omega}(\mathbf{r},t)=[\widetilde{W}(t;\mathbf{r})^{-1}]^T\boldsymbol{\omega}(\mathbf{R}(0;\mathbf{r},t),0). \tag{98}$$

Finally, the magnetic field satisfies

$$\mathbf{B}(\mathbf{r},t)=\widetilde{W}(t;\mathbf{r})\mathbf{B}(\mathbf{R}(0;\mathbf{r},t),0). \tag{99}$$

The relations (96)–(99) give, in the absence of forcing and diffusion, the solutions of the initial value problem for the advection equations in a given realization of a smooth velocity.

For nonzero κ, the solutions of the scalar equations are given essentially by the same expressions. However, $\mathbf{R}(\cdot;\mathbf{r},t)$ denotes now a noisy Lagrangian trajectory satisfying the stochastic Eq. (5) and passing through $\mathbf{r}$ at time t and the right-hand sides of Eqs. (96)–(99) should be averaged over the noise using the Itô formula of stochastic calculus discussed in the Appendix. These solutions may be rewritten using the transition PDF's $p(\mathbf{r},s;\mathbf{R},t|\mathbf{v})$ introduced in Sec. II.C; see Eq. (43) and describing the probability density to find the noisy particle at time t at position $\mathbf{R}$, given its time s position $\mathbf{r}$. One has

$$\theta(\mathbf{r},t)=\int p(\mathbf{r},t;\mathbf{R},0|\mathbf{v})\,\theta(\mathbf{R},0)\,d\mathbf{R},$$

$$n(\mathbf{r},t)=\int p(\mathbf{R},0;\mathbf{r},t|\mathbf{v})\,n(\mathbf{R},0)\,d\mathbf{R}. \tag{100}$$

The two PDF's appearing in these formulas, one backward and the other forward in time, coincide for incompressible velocities but they are generally unequal for the compressible cases. For nonsmooth velocities, those PDF's continue to make sense and we shall use Eq. (100) to define the solutions of the scalar advection equations in that case. As for the vector fields, their properties depend both on the noisy Lagrangian trajectory end points and on the matrices $\widetilde{W}(t;\mathbf{r})$, that are well defined only in smooth velocities. The formal procedure for nonsmooth velocities is to first impose a viscous cutoff, smoothing the velocity behavior at small scales, and then removing it. When this is done, some properties of the field remain well defined and may be analyzed (see, for instance, Sec. III.C.3).

In random velocity fields, the advected quantities become random fields whose statistics may be probed by considering the equal-time correlation functions. In particular, those of a tracer evolve according to

$$C_N(\underline{\mathbf{r}},t)\equiv\langle\theta(\mathbf{r}_1,t)\cdots\theta(\mathbf{r}_N,t)\rangle$$
$$=\int\mathcal{P}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};-t)\,\theta(\mathbf{R}_1,0)\cdots\theta(\mathbf{R}_N,0)\,d\underline{\mathbf{R}}. \tag{101}$$

Here, as in Sec. II.C, the Green functions $\mathcal{P}_N$ are the joint PDF's of the equal-time positions of N fluid particles; see Eq. (65). For the correlators of the density n, the backward propagator in Eq. (101) should be replaced by its forward version, in agreement with Eq. (100). If the initial data are random and independent of the velocities, they may be easily averaged over. For a Gaussian initial distribution with zero mean,

$$C_{2n}(\underline{\mathbf{r}},t)=\int\mathcal{P}_{2n}(\underline{\mathbf{r}};\underline{\mathbf{R}};-t)$$
$$\times[C_2(\mathbf{R}_{12},0)\cdots C_2(\mathbf{R}_{(2n-1)2n},0)+\cdots]\,d\underline{\mathbf{R}}, \tag{102}$$

where, according to the Wick rule, the dots stand for the other pairings of the $2n$ points.

Let us now briefly discuss the compressible case, where the statistics of the matrices $\widetilde{W}$ and W generally do not coincide. As we have already discussed in Sec. II.D, every trajectory then comes with its own weight determined by the local rate of volume change and exhibited by the Lagrangian average of a function $f(\widetilde{W})$:

$$\int f(\widetilde{W}(t;\mathbf{r}))\frac{d\mathbf{r}}{V}=\int f(W(t;\mathbf{R}))\det(W(t;\mathbf{R}))\frac{d\mathbf{R}}{V}. \tag{103}$$

The relation in Eq. (103) simply follows from the definition of $\widetilde{W}$. The volume change factor $\det(W)=\exp(\Sigma\rho_i)$, with the same notation as in Sec. II.B. Recall that only the average of the determinant is generally equal to unity for compressible flow. The averages of the SO(d)-invariant functions of $\widetilde{W}$ are described for large times by the large-deviation function $\widetilde{H}=H-\Sigma\rho_i/t$, with the last term coming from the volume factor. The corresponding Lyapunov exponents $\widetilde{\lambda}_i$ are determined by the extremum of $\widetilde{H}$ (Balkovsky, Falkovich, and Fouxon, 1999). The exponents generally depend on the form of the entropy function H and cannot be expressed via the Lyapunov exponents λ_i only. Since the matrix W' of the backward evolution is the inverse of $\widetilde{W}$, the backward Lyapunov exponents are given by $-\widetilde{\lambda}_{d-i+1}$

and not by the naive guess $-\lambda_{d-i+1}$. In particular, the interparticle distance diverges backward in time with the exponent $-\tilde{\lambda}_d$. The same way as we have shown in Sec. II.D that $\Sigma\lambda_i\leqslant 0$ in a compressible flow, one shows that $\Sigma\tilde{\lambda}_i\geqslant 0$ (implying $\tilde{\lambda}_1\geqslant 0$). For the forward Lagrangian evolution we thus have an average compression of volumes, whereas passive fields rather feel an average expansion. Indeed, as we go away from the moment where we imposed a uniform Lagrangian measure, the rate of change of the volume is becoming negative in a fluctuating compressible flow.

The forward and backward in time Lyapunov exponents coincide if the statistics of the velocities is time reversible, i.e., if $\mathbf{v}(\mathbf{r},t)$ and $-\mathbf{v}(\mathbf{r},-t)$ are identically distributed. More generally, the entire distributions of the forward and the backward in time stretching rates coincide in that case:

$$H(\rho_1/t-\lambda_1,\dots,\rho_d/t-\lambda_d)$$
$$=H(-\rho_d/t-\lambda_1,\dots,-\rho_1/t-\lambda_d)+\sum_i \rho_i/t. \tag{104}$$

This is an example of the time-reversibility symmetry of the large deviation entropy function that was described by Gallavotti and Cohen (1995). The symmetry holds also for a δ-correlated strain, although the above finite-volume argument (103) does not apply directly to this case. Recall that in that instance the entropy function (59) describes the large deviations of the stretching rates of the matrix $W(t)$ given by the Itô version of Eq. (16). For the inverse evolution, the strain $\sigma(s)$ should be replaced by $\sigma'(s)=-\sigma(t-s)$ and the matrix $W'(t)$ has the same distribution as $W(t)$. The matrix $\tilde{W}(t)=W'(t)^{-1}$ is then given by Eq. (16) with the anti-Itô regularization and the relation between the conventions [see Eq. (A4)] implies $\tilde{W}(t)=W(t)e^{-2\Sigma\lambda_i t/d}$. Realistic turbulent flows are irreversible because of the dissipation so that the symmetry (104), that was confirmed in an experimental situation (Ciliberto and Laroche, 1998), may be at most approximate.

2. Quasi-Lagrangian description of the advection

Important insights into the advection mechanisms are obtained by eliminating global sweeping effects and describing the advected fields in a frame whose origin moves with the fluid. This picture of the hydrodynamic evolution, known under the name of quasi-Lagrangian description, is intermediate between the static Eulerian and the dynamic Lagrangian ones (Monin, 1959; Belinicher and L'vov, 1987). Specifically, quasi-Lagrangian fields are defined as

$$\tilde{\psi}(\mathbf{r},t)=\psi[\mathbf{r}+\mathbf{R}(t;\mathbf{r}_0,0),t], \tag{105}$$

where ψ stands for any Eulerian field, scalar or vector, introduced previously and $\mathbf{R}(t;\mathbf{r}_0,0)$ is the Lagrangian trajectory passing through $\mathbf{r}_0$ at time zero. The quasi-Lagrangian fields satisfy the same evolution equations as the Eulerian ones except for the replacement of the advective term by $[\tilde{\mathbf{v}}(\mathbf{r},t)-\tilde{\mathbf{v}}(\mathbf{0},t)]\cdot\boldsymbol{\nabla}$. If the incompressible velocity and the initial values of the advected field are statistically homogeneous, the equal-time statistics of the quasi-Lagrangian and the Eulerian fields coincide. The equal-time statistics is indeed independent of the initial point $\mathbf{r}_0$. The equality of the equal-time Eulerian and quasi-Lagrangian distributions follows then by averaging first over $\mathbf{r}_0$ and then over the velocity and by changing the variables $\mathbf{r}_0\mapsto\mathbf{R}(t;\mathbf{r}_0,0)$. The equality does not hold if the initial values of the advected fields are nonhomogeneous.

It will be especially convenient to use the quasi-Lagrangian picture for distances r much smaller than the viscous scale, i.e., in the Batchelor regime (Falkovich and Lebedev, 1994). The variations in the velocity gradients may then be ignored so that $\tilde{\mathbf{v}}(\mathbf{r},0)-\tilde{\mathbf{v}}(\mathbf{0},t)\approx\sigma(t)\mathbf{r}$. In this case, the velocity field enters into the advection equations only through the time-dependent strain matrix. For the tracer, one obtains then the evolution equation

$$\partial_t\tilde{\theta}+(\sigma\mathbf{r})\cdot\boldsymbol{\nabla}\tilde{\theta}=\kappa\boldsymbol{\nabla}^2\tilde{\theta}. \tag{106}$$

This may be solved as before using now the noisy Lagrangian trajectories for a velocity linear in the spatial variables:

$$\tilde{\theta}(\mathbf{r},t)=\overline{\theta\left(W(t)^{-1}\mathbf{r}-\sqrt{2\kappa}\int_0^t W(s)^{-1}d\boldsymbol{\beta}(s),0\right)}, \tag{107}$$

where $W(t)$ is the evolution matrix of Eq. (16). The overbar denotes the average over the noise which is easily performed for incompressible velocity fields using the Fourier representation:

$$\tilde{\theta}(\mathbf{r},t)=\int \hat{\theta}(W(t)^T\mathbf{k},0)\exp[i\mathbf{k}\cdot\mathbf{r}-\mathbf{k}\cdot Q(t)\mathbf{k}]\frac{d\mathbf{k}}{(2\pi)^d} \tag{108}$$

with

$$Q(t)=\kappa\int_0^t W(t)[W(s)^TW(s)]^{-1}W(t)^T ds. \tag{109}$$

3. Decay of tracer fluctuations

For practical applications, e.g., in the diffusion of pollution, the most relevant quantity is the average $\langle\theta(\mathbf{r},t)\rangle$. It follows from Eq. (101) that the average concentration is related to the single-particle propagation discussed in Sec. II.A. For times longer than the Lagrangian correlation time, the particle diffuses and $\langle\theta\rangle$ obeys the effective heat equation

$$\partial_t\langle\theta(\mathbf{r},t)\rangle=D_e^{ij}\nabla_i\nabla_j\langle\theta(\mathbf{r},t)\rangle, \tag{110}$$

with the effective diffusivity D_e^{ij} given by Eq. (9). The decay of higher-order moments and multipoint correlation functions involves multiparticle propagation and it is sensitive to the degree of smoothness of the velocity field.

The simplest decay problem is that of a uniform scalar spot of size l released in the fluid. Another relevant situation is that where a homogeneous statistics with correlations decaying on the scale l is initially prescribed. The corresponding decay problems are discussed hereafter for the two cases of smooth and nonsmooth incompressible flow.

a. Smooth velocity

Let us consider an initial scalar configuration given in the form of a single spot of size l. Its average spatial distribution at later times is given by the solution of Eq. (110) with the appropriate initial condition. On the other hand, the decay of the scalar in the spot as it is carried with the flow corresponds to the evolution of $\tilde{\theta}$, defined by Eq. (105). We assume the Schmidt/Prandtl number ν/κ to be large so that the viscous scale η is much larger than the diffusive scale r_d. We shall be considering the 3D situation, where the diffusive scale $r_d=\sqrt{\kappa/|\lambda_3|}$ and λ_3 is the most negative Lyapunov exponent defined in Sec. II.B.1. As shown there, for times $t \ll t_d=|\lambda_3|^{-1}\ln(l/r_d)$, the diffusion is unimportant and the values of the scalar field inside the spot do not change. At later times, the width in the direction of the negative Lyapunov exponent λ_3 is frozen at r_d, while the spot keeps growing exponentially in the other two directions.[8] The freezing of the contracting direction at r_d thus results in an exponential growth of the volume $\propto\exp(\rho_1+\rho_2)$. Hence the scalar moment of order α measured at locations inside the spot will decrease as the average of $\exp[-\alpha(\rho_1+\rho_2)]$. The resulting decay laws $\exp(-\gamma_\alpha t)$ may be calculated using the PDF (27) of the stretching variables ρ_i. More formally, the scalar moments inside the spot are captured by the quasi-Lagrangian single-point statistics. Following Balkovsky and Fouxon (1999), let us take in Eq. (108) a Gaussian initial configuration $\hat{\theta}(\mathbf{k},0)=\exp[-\frac{1}{12}(l\mathbf{k})^2]$. As a result,

$$\tilde{\theta}(\mathbf{0},t)=\int \exp\left[-\frac{1}{12}l^2\mathbf{k}\cdot I(t)\mathbf{k}\right]\frac{d\mathbf{k}}{(2\pi)^3} \propto \det I(t)^{-1/2}=e^{-\Sigma\rho_i}, \tag{111}$$

where $I(t)$ is the mean tensor of inertia introduced in Sec. II.B.1; see Eq. (25). Using the PDF (27), one obtains then

$$\langle\tilde{\theta}^\alpha(t)\rangle\propto\int \exp[-\alpha(\rho_1+\rho_2) -tH(\rho_1/t-\lambda_1,\rho_2/t-\lambda_2)]d\rho_1\,d\rho_2. \tag{112}$$

At large times, the integral is determined by the saddle point. At small α, it lies within the parabolic domain of H and the decay rate γ_α increases quadratically with the order α. At large enough orders, the integral is dominated by the rare realizations where the volume of the spot does not grow in time and the growth rates become independent of the order (Shraiman and Siggia, 1994; Balkovsky and Fouxon, 1999; Son, 1999). That conclusion is confirmed experimentally (Groisman and Steinberg, 2000).

An alternative way to describe the decay of $\langle\tilde{\theta}^N(t)\rangle$ is to take N fluid particles that come to the given point $\mathbf{r}$ at time t and to track them back to the initial time. The realizations contributing to the moments are those for which all the particles were initially inside the original spot of size l, see Eq. (107). Looking backward in time, we see that the molecular noise splits the particles by small separations of order $\mathcal{O}(r_d)$ during a time interval of the order $t_d=r_d^2/\kappa\approx|\lambda_3^{-1}|$ near t. After that, the advection takes over. The realizations contributing to the moments are those with the interparticle separations almost orthogonal to the (backward) expanding direction ρ_3 of $\tilde{W}^{-1}$. More exactly, they should form an angle $\lesssim(l/r_d)e^{\rho_3}$ with the plane orthogonal to the expanding direction. Such separations occupy a solid angle fraction of the same order. Since we now track particles moving due to the advection (the molecular noise is accounted for by the finite splitting) then $\Sigma\rho_i=0$ and Eq. (112) follows.

The same simple arguments lead to the result for the case of a random initial condition with zero mean. Let us first consider the case when it is Gaussian with correlation length l. It follows from Eq. (102) that the realizations contributing to $\langle\tilde{\theta}^{2n}\rangle=\langle\theta^{2n}\rangle$ are those where n independent pairs of particles are separated by distances smaller than l at time $t=0$. The moments are therefore given by

$$\langle\theta^{2n}(t)\rangle\propto\int \exp[-n(\rho_1+\rho_2) -tH(\rho_1/t-\lambda_1,\rho_2/t-\lambda_2)]d\rho_1\,d\rho_2. \tag{113}$$

Note that the result is in fact independent of the scalar field initial statistics. Indeed, for a non-Gaussian field we should average over the Lagrangian trajectories the initial correlation function $C_{2n}(\underline{\mathbf{R}}(0),0)$ that involves a nonconnected and a connected part. The latter is assumed to be integrable with respect to the $2n-1$ separation vectors (and thus to depend on them). Each dependence brings an $\exp[-\rho_1-\rho_2]$ factor and the connected part will thus give a subleading contribution with respect to the nonconnected one. The above results were first obtained by Balkovsky and Fouxon (1999) using different arguments (see Secs. III.4 and III.5). Remark the square root of the volume factor appearing in Eq. (113) as distinct from Eq. (112). In the language of spots, this is explained by the mutual cancellations of the tracer from different spots and the ensuing law of large numbers. Indeed, different blobs of size l with initially uncorrelated values of the scalar will overlap at time t and the rms value of θ will be proportional to the square root of the number of spots $\propto\exp(\rho_1+\rho_2)$. The same qualitative conclusions drawn previously about the decay rates γ_α may be obtained from Eq. (113). In particu-

[8] As in Sec. II.B, we consider the case of two non-negative Lyapunov exponents; the arguments are easily modified for two nonpositive exponents.

lar, Balkovsky and Fouxon (1999) performed the explicit calculation for the short-correlated case (28). The result is

$$\gamma_\alpha = -\lim_{t\to\infty}\frac{1}{t}\ln\langle|\theta|^\alpha\rangle \propto \alpha\left(1-\frac{\alpha}{8}\right), \tag{114}$$

for $\alpha<4$ and $\gamma_\alpha=$const for $\alpha>4$. An important remark is that the PDF of the decaying scalar is not self-similar in a smooth velocity field. The PDF is indeed becoming more and more intermittent with time, as signaled by the growth of the kurtosis $\langle|\theta|^\alpha\rangle/\langle\theta^2\rangle^{\alpha/2}$ for $\alpha>2$. The previous arguments may be easily generalized to the case of compressible flow.

b. Nonsmooth velocity

For the decay in incompressible nonsmooth flow, we shall specifically consider the case of a time reversible Kraichnan velocity field. The comments on the general case are reserved to the end of the section. The simplest objects to investigate are the single-point moments $\langle\theta^{2n}(t)\rangle$ and we are interested in their long-time behavior $t\gg l^{2-\xi}/D_1$. Here, l is the correlation length of the random initial field and D_1 enters the velocity two-point function as in Eq. (48). Using Eq. (101) and the scaling property (71) of the Green function we obtain

$$\langle\theta^{2n}(t)\rangle = \int \mathcal{P}_{2n}(\underline{\mathbf{0}};\underline{\mathbf{R}};-1)\,C_{2n}(t^{1/(2-\xi)}\underline{\mathbf{R}},0)\,d\underline{\mathbf{R}}. \tag{115}$$

There are two universality classes for this problem, corresponding to either nonzero or vanishing value of the so-called Corrsin integral $J_0=\int C_2(\mathbf{r},t)\,d\mathbf{r}$. Note that the integral is generally preserved in time by the passive scalar dynamics.

We concentrate here on the case $J_0\neq 0$ and refer the interested reader to the original paper by Chaves *et al.* (2001) for more details. For $J_0\neq 0$, the function $t^{d/(2-\xi)}C_2(t^{1/(2-\xi)}\mathbf{r},0)$ tends to $J_0\delta(\mathbf{r})$ in the long-time limit and Eq. (115) is reduced to

$$\langle\theta^{2n}(t)\rangle \approx (2n-1)!!\frac{J_0^n}{t^{nd/(2-\xi)}} \times\int \mathcal{P}_{2n}(\underline{\mathbf{0}};\mathbf{R}_1,\mathbf{R}_1,\dots,\mathbf{R}_n,\mathbf{R}_n;-1)\,d\underline{\mathbf{R}}, \tag{116}$$

for a Gaussian initial condition. A few remarks are in order. First, the previous formula shows that the behavior in time is self-similar. In other words, the single point PDF $\mathcal{P}(t,\theta)$ takes the form $t^{d/2(2-\xi)}Q(t^{d/2(2-\xi)}\theta)$. That means that the PDF of $\theta/\sqrt{\epsilon}$ is asymptotically time independent as was hypothesized by Sinai and Yakhot (1989), with $\overline{\epsilon}(t)=\kappa\langle(\nabla\theta)^2\rangle$ being time-dependent (decreasing) dissipation rate. This should be contrasted with the lack of self-similarity found previously for the smooth case. Second, the result is asymptotically independent of the initial statistics (of course, within the universality class $J_0\neq 0$). As in the previous subsection, this follows from the fact that the connected non-Gaussian part of C_{2n} depends on more than n separation vectors. Its contribution is therefore decaying faster than $t^{-nd/(2-\xi)}$. Third, it follows from Eq. (116) that the long-time PDF, although universal, is generally non-Gaussian. Its Gaussianity would indeed imply the factorization of the probability for the $2n$ particles to collapse in pairs at unit time. Due to the correlations existing among the particle trajectories, this is generally not the case, except for $\xi=0$ where the particles are independent. The degree of non-Gaussianity is thus expected to increase with ξ, as confirmed by the numerical simulations presented in (Chaves *et al.*, 2001).

Other statistical quantities of interest are the structure functions $S_{2n}(\mathbf{r},t)=\langle[\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^{2n}\rangle$ related to the correlation functions by

$$S_{2n}(\mathbf{r},t)=\int_0^1\cdots\int_0^1 \partial_{\mu_1}\cdots\partial_{\mu_{2n}} C_{2n}(\mu_1\mathbf{r},\dots,\mu_{2n}\mathbf{r},t)\prod d\mu_i \equiv \Delta(\mathbf{r})C_{2n}(\cdot). \tag{117}$$

To analyze their long-time behavior, we proceed similarly as in Eq. (115) and use the asymptotic expansion (75) to obtain

$$\begin{aligned} S_{2n}(\mathbf{r},t) &= \int \Delta(t^{-1/(2-\xi)}\mathbf{r})\mathcal{P}_{2n}(\cdot;\mathbf{R};-1)\,C_{2n}(t^{1/(2-\xi)}\underline{\mathbf{R}},0)\,d\underline{\mathbf{R}} \\ &\approx \frac{\Delta(\mathbf{r})f_{2n,0}(\cdot)}{t^{\zeta_{2n,0}/(2-\xi)}}\int g_{2n,0}(\underline{\mathbf{R}},-1)\,C_{2n}(t^{1/(2-\xi)}\underline{\mathbf{R}},0)\,d\underline{\mathbf{R}} \\ &\propto \left(\frac{r}{l(t)}\right)^{\zeta_{2n}}\langle\theta^{2n}\rangle(t). \end{aligned} \tag{118}$$

Here, $f_{2n,0}$ is the irreducible zero mode in Eq. (75) with the lowest dimension and the scalar integral scale $l(t)\propto t^{1(2-\xi)}$. As we shall see in Sec. III.C.1, the quantities $\zeta_{2n}=\zeta_{2n,0}$ give also the scaling exponents of the structure function in the stationary state established in the forced case.

Let us conclude this subsection by briefly discussing the scalar decay for velocity fields having finite correlation times. The key ingredient for the self-similarity of the scalar PDF is the rescaling (71) of the propagator. Such property is generally expected to hold (at least for large enough times) for self-similar velocity fields regardless of their correlation times. This has been confirmed by the numerical simulations in Chaves *et al.* (2001). For an intermittent velocity field the presence of various scaling exponents makes it unlikely that the propagator possesses a rescaling property like Eq. (71). The self-similarity in time of the scalar distribution might then be broken.

4. Growth of density fluctuations in compressible flow

The evolution of a passive density field $n(\mathbf{r},t)$ is governed by Eq. (93). In smooth velocities and in the absence of diffusion, its solution is read from Eq. (97), where we shall take the initial field on the right-hand side to be uniform. This gives $n(\mathbf{r},t)=[\det(\widetilde{W}(t;\mathbf{r}))]^{-1}$.

Performing the velocity average and recalling the long-time asymptotics of the $\tilde{W}$ statistics, we obtain

$$\langle n^{\alpha}(t)\rangle \propto \int \exp\Big[(1-\alpha)\times\sum_i \rho_i - tH(\rho_1/t-\lambda_1,\ldots,\rho_d/t-\lambda_d)\Big]\prod d\rho_i . \tag{119}$$

The moments at long times may be calculated by the saddle-point method and they are generally behaving as $\propto\exp(\gamma_\alpha t)$. The growth rate function γ_α is convex, due to Hölder inequality, and vanishes both at the origin and for $\alpha=1$ (by the total mass conservation). This leads to the conclusion that γ_α is negative for $0<\alpha<1$ and is otherwise positive: low-order moments decay, whereas high-order and negative moments grow. For a Kraichnan velocity field, the large deviations function H is given by Eq. (59) and the density field becomes lognormal with $\gamma_\alpha\propto\alpha(\alpha-1)$ (Klyatskin and Gurarie, 1999). Note that the asymptotic rate $\langle \ln n\rangle/t$ is given by the derivative at the origin of γ_α and it is equal to $-\Sigma\tilde{\lambda}_i\leqslant 0$. The density is thus decaying in almost any realization if the sum of the Lyapunov exponents is nonzero. Since the mean density is conserved, it has to grow in some (smaller and smaller) regions, which implies the growth of high moments. The amplification of negative moments is due to the growth of low-density regions. The positive quantity $-\Sigma\lambda_i$ has the interpretation of the mean (Gibbs) entropy production rate per unit volume. Indeed, if we define the Gibbs entropy $S(n)$ as $-\int(\ln n)n\,d\mathbf{r}$ then, by Eq. (103), $S(n)=\int \ln\det(W(t;\mathbf{r}))d\mathbf{r}$. Since $\ln\det(W)=\Sigma\rho_i$, the entropy transferred to the environment per unit time and unit volume is $-\Sigma\rho_i/t$ and it is asymptotically equal to $-\Sigma\lambda_i>0$; see Ruelle (1997).

The behavior of the density moments discussed above is the effect of a linear but random hyperbolic stretching and contracting evolution (13) of the trajectory separations. In a finite volume, the linear evolution is eventually superposed with nonlinear bending and folding effects. In order to capture the combined impact of the linear and the nonlinear dynamics at long times, one may observe at fixed time t the density produced from an initially uniform distribution imposed at much earlier times t_0. When $t_0\to-\infty$ and if $\lambda_1>0$, the density approaches weakly, i.e., in integrals against test functions, a realization-dependent fractal density $n_*(\mathbf{r},t)$ in almost all the realizations of the velocity. The resulting density field is the so-called SRB (Sinai-Ruelle-Bowen) measure, see, e.g., Kiffer (1988). The fractal dimension of the SRB measures may be read from the values of the Lyapunov exponents (Frederikson *et al.*, 1983). For the Kraichnan ensemble of smooth velocities, the SRB measures were first discussed by Le Jan (1985). In 2D, they have a fractal dimension equal to $1+(1-2\wp)/(1+2\wp)$ if $0<\wp<\frac{1}{2}$. In 3D, the dimension is $2+(1-3\wp)/(1+2\wp)$ if $0<\wp\leqslant\frac{1}{3}$ and $1+(3-4\wp)/5\wp$ if $\frac{1}{3}\leqslant\wp<\frac{3}{4}$, where $\wp$ is the compressibility degree.

The above considerations show that, as long as one can neglect diffusion, the passive density fluctuations grow in a random compressible flow. One particular case of the above phenomena is the clustering of inertial particles in an incompressible turbulent flow; see Balkovsky *et al.* (2001) where the theory for a general flow and the account of the diffusion effects that eventually stops the density growth were presented.

5. Gradients of the scalar in a smooth velocity

For the passive scalar gradients $\boldsymbol{\omega}=\boldsymbol{\nabla}\theta$ in an unforced incompressible situation, the equation to be solved is Eq. (94). The initial distribution is assumed statistically homogeneous with a finite correlation length. As discussed previously, one may treat diffusion either by adding a Brownian motion to the backward Lagrangian trajectories or by using the Fourier transform method (108). For pedagogical reasons, we choose here the latter and solve Eq. (94) by simply taking the gradient of the scalar expression (108). The long-time limit is independent of the initial scalar statistics (Balkovsky and Fouxon, 1999) and it is convenient to take it Gaussian with the two-point function $\propto\exp[-(-d/2)(r/l)^2]$. The averaging over the initial statistics for the generating function $\mathcal{Z}(\mathbf{y})=\langle\exp[i\mathbf{y}\cdot\boldsymbol{\omega}]\rangle$ reduces then to Gaussian integrals involving the matrix $I(t)$ determined by Eq. (25). The inverse Fourier transform is given by another Gaussian integral over $\mathbf{y}$ and one finally obtains for the PDF of $\boldsymbol{\omega}$

$$\mathcal{P}(\boldsymbol{\omega})\propto\langle(\det I)^{d/4+1/2}\exp[-\mathrm{const}\times\sqrt{\det I}(\boldsymbol{\omega},I\boldsymbol{\omega})]\rangle. \tag{120}$$

As may be seen from Eq. (25), during the initial period $t<t_d=|\lambda_d^{-1}|\ln(l/r_d)$, the diffusion is unimportant, the contribution of the matrix Q to I is negligible, the determinant of the latter is unity, and ω^2 grows as the trace of I^{-1}. In other words, the statistics of $\ln\omega$ and of $-\rho_d$ coincide in the absence of diffusion. The statistics of the gradients can therefore be immediately taken over from Sec. II.B. The growth rate $(2t)^{-1}\langle\ln\boldsymbol{\omega}^2\rangle$ approaches $|\lambda_d|$ while the gradient PDF depends on the entropy function. For the Kraichnan model (28), the PDF is lognormal with the average $D_1 d(d-1)t$ and the variance $2D_1(d-1)t$ read directly from Eq. (32). This result was obtained by Kraichnan (1974) using the fact that, without diffusion, ω satisfies the same equation as the distance between two particles, whose PDF is given by Eq. (23).

As time increases, the wavenumbers (evolving as $\dot{\mathbf{k}}=-\sigma^T\mathbf{k}$) reach the diffusive scale r_d^{-1} and the diffusive effects start to modify the PDF, propagating to lower and lower values of ω. High moments first and then lower ones will start to decrease. The law of decay at $t\gg t_d$ can be deduced from Eq. (120). Considering this expression in the eigenbasis of the matrix I, we observe that the dominant component of $\boldsymbol{\omega}$ coincides with the largest eigendirection of the I^{-1} matrix, i.e., the one along the ρ_d axis. Recalling from Sec. II.B that the distribution of ρ_d is stationary, we infer that $\langle|\omega|^{\alpha}(t)\rangle$

$\propto\langle(\det I)^{-\alpha/4}\rangle$. The comparison with Eq. (113) shows that the decay laws for the scalar and its gradients coincide (Balkovsky and Fouxon, 1999; Son, 1999). This is qualitatively understood by estimating $\omega\sim\theta/l_{\min}$, where $l_{\min}$ is the smallest size of the spot. Noting that θ and $l_{\min}$ are independent and that $l_{\min}\approx e^{\rho_d}l$ at large times has a stationary statistics concentrated around r_d, it is quite clear that the decrease of ω is due to the decrease of θ.

6. Magnetic dynamo

In this subsection, we consider the generation of inhomogeneous magnetic fluctuations below the viscous scale of incompressible turbulence. The question is relevant for astrophysical applications as the magnetic fields of stars and galaxies are thought to have their origin in the turbulent dynamo action (Moffatt, 1978; Parker, 1979; Zel'dovich *et al.*, 1983; Childress and Gilbert, 1995). In this problem, the magnetic field can be treated as passive. Furthermore, the viscosity-to-diffusivity ratio is often large enough for a sizable interval of scales between the viscous and the diffusive cutoffs to be present. That is the region of scales with the fastest growth rates of the magnetic fluctuations. Their dynamics, modeled by the passive advection of magnetic field by a large-scale (smooth) velocity field, will be described here.

The dynamo process is caused by the stretching of fluid elements already extensively discussed above and the major new point to be noted is the role of the diffusion. In a perfect conductor, when the diffusion is absent, the magnetic field satisfies the same equation as the infinitesimal separation between two fluid particles (12): $d\mathbf{B}/dt=\sigma\mathbf{B}$. Any chaotic flow would then produce dynamo, with the growth rate

$$\overline{\gamma}=\lim_{t\to\infty}(2t)^{-1}\langle\ln B^2\rangle, \tag{121}$$

equal to the highest Lyapunov exponent λ_1. Recall that the gradients of a scalar grow with the growth rate $-\lambda_3$ during the diffusionless stage. In fact, any real fluid has a nonzero diffusivity and, even though it can be very small, its effects may be dramatic. The longstanding problem solved by Chertkov, Falkovich, *et al.* (1999) was whether the presence of a small, yet finite, diffusivity could stop the dynamo growth process at large times (as it is the case for the gradients of a scalar).

Our starting point is Eq. (99), expressing the magnetic field in terms of the stretching matrix $\widetilde{W}$ and the backward Lagrangian trajectory. In incompressible flow, matrices $\widetilde{W}$ and W are identically distributed and we do not distinguish them here. For example, the second-order correlation function is given by

$$\mathcal{C}_2^{ij}(\mathbf{r}_{12},t)\equiv\langle B^i(\mathbf{r}_1,t)B^j(\mathbf{r}_2,t)\rangle$$
$$=\langle W^{il}W^{jm}\mathcal{C}_2^{lm}(\mathbf{R}_{12}(0;\mathbf{r}_{12},t),0)\rangle, \tag{122}$$

with the average taken over the velocity and the molecular noise. For the sake of simplicity, we assume that the initial statistics of $\mathbf{B}$ is homogeneous, Gaussian, of zero mean and of correlation length l. We concentrate on the behavior at scales $r_{12}\ll l$. For times less than $t_d=|\lambda_3|^{-1}\ln(l/r_d)$, the Lagrangian separations $R_{12}\ll l$ and the magnetic field is stretched by the W matrix as in a perfect conductor; see Eq. (122). For longer times, the separation R_{12} can reach l, irrespective of its original value. This is the long-time asymptotic regime of interest, where the destinies of the scalar gradients and the magnetic field are different.

It follows from Eq. (122) that the correlations are due to those realizations where $R_{12}(0)\lesssim l$. As in Sec. III.A.3, the initial separation $\mathbf{r}_{12}$ should then be quasiorthogonal to the expanding direction ρ_3 of W^{-1} and the fraction of solid angle occupied by those realizations is $\propto(l/r_{12})e^{\rho_3}$. Together with the $e^{2\rho_1}$ factor coming from the perfect conductor amplification, we thus obtain for the trace of the correlation function

$$\mathrm{tr}\mathcal{C}_2(\mathbf{r},t)\propto\int\mathcal{P}(\rho_1,\rho_2,t)e^{\rho_1-\rho_2}\,d\rho_1\,d\rho_2, \tag{123}$$

with $\mathcal{P}(\rho_1,\rho_2;t)$ as in Eq. (21). The integration is constrained by $-\ln(l/r_{12})\lesssim\rho_2$, required for the separation along ρ_2 to remain smaller than l. Note that the gradients of a scalar field are stretched by the same W^{-1} matrix that governs the growth of the Lagrangian separations. It is therefore impossible to increase the stretching factor of the gradient and keep the particle separation within the correlation length l at the same time. That is why diffusion eventually kills all the gradients while the component B^i that points into the direction of stretching survives and grows with ∇B^i perpendicular to it. This simple picture also explains the absence of dynamo in 2D incompressible flow, where the stretching in one direction necessarily means the contraction in the other one.

Let us now consider the single-point moments $\langle B^{2n}\rangle(t)$. The $2n$ particles, all at the same point at time t, are split by the molecular diffusion by small separations of length $\mathcal{O}(r_d)$ in a time of the order $t_d=r_d^2/\kappa$ near t. For the subsequent advection not to stretch the separations beyond l, the "diffusive" separations at time $t-t_d$ should be quasiorthogonal to the expanding direction ρ_3. More exactly, they should form an angle $\lesssim(l/r_d)e^{\rho_3}$ with the plane orthogonal to the expanding direction. Together with the pure conductor stretching factor, we are thus left with a contribution $\propto\exp[n(2\rho_1+\rho_3)]$. Two possible classes of Lagrangian trajectories should now be distinguished, depending on whether the angle formed by the "diffusive" separations with the ρ_2 direction is arbitrary or constrained to be small (see also Molchanov *et al.*, 1985). For the former, the contribution is simply given by the average of the expression $\exp[n(\rho_1-\rho_2)]$ derived previously, with the constraint $-\ln l/r_d\lesssim\rho_2$ ensuring the control of the particle separation along ρ_2. For the latter, the contribution is proportional to the average of $\exp[n(\rho_1+2\rho_2)]$. Indeed, the condition of quasiorthogonality to the ρ_2 direction contributes a $n\rho_2$ term in the exponent and the remaining $2n\rho_2$ term is coming from the solenoidality condition $\boldsymbol{\nabla}\cdot\mathbf{B}=0$. The magnetic-field correlation is in fact proportional to the solenoidal projector and the component stretched by the W matrix, see Eq. (122), is $\mathcal{C}_2^{11} \propto [1 - (\boldsymbol{\nabla}^2)^{-1}\nabla_1^2]\mathcal{C}(r)$. The realizations having the particle separations precisely aligned with the ρ_1 direction will therefore not contribute. For separations almost aligned to ρ_1, one may show that the square of the angle with respect to ρ_1 appears in $\mathcal{C}_2^{11}$, thus giving the additional small factor $(l/r_d e^{\rho_2})^2$.

Which one of the two previous classes of Lagrangian trajectories dominates the moments depends on the specific form of the entropy function. For the growth rate (121), the situation is simpler as the average is dominated by the region around $\rho_i = \lambda_i$. The average of the logarithm is indeed obtained by taking the limit $n \to 0$ in $\langle B^{2n} - 1 \rangle / 2n$ and the saddle point at large times sits at the minimum of the entropy function. The two previous classes of Lagrangian trajectories dominate for positive and negative λ_2, respectively. Using the identity $\Sigma\lambda_i = 0$, we finally obtain

$$\overline{\gamma} = \min\{(\lambda_1 - \lambda_2)/2, (\lambda_2 - \lambda_3)/2\}. \tag{124}$$

The validity of this formula is restricted, first, by the condition that $l \exp(\lambda_1 t)$ is still less than the viscous scale. The stretching in the third direction also imposes a restriction: the finiteness of the maximal possible size l_0 of the initial fluctuations gives the constraint $l_0 \exp(\lambda_3 t) > r_d$. At larger time, $\langle \ln |\mathbf{B}|^2 \rangle$ decays.

The most important conclusion coming from Eq. (124) is that the growth rate is always non-negative for a chaotic incompressible flow. Note that the growth rate vanishes if two of the Lyapunov exponents coincide, corresponding to the absence of dynamo for axially symmetric cases. For time-reversible and two-dimensional flows, the intermediate Lyapunov exponent vanishes and $\overline{\gamma} = \lambda_1/2$. Note that 3D magnetic field does grow in a 2D flow; when, however, both the flow and the field are two dimensional, one finds $\overline{\gamma} = -\lambda_1/2$. For isotropic Navier-Stokes turbulence, numerical data suggest $\lambda_2 \approx \lambda_1/4$ (Girimaji and Pope, 1990) and the long-time growth rate is then $\overline{\gamma} \approx 3\lambda_1/8$.

The moments of positive order all grow in a random incompressible flow with a nonzero Lyapunov exponent. Indeed, the curve $E_n = \ln\langle B^{2n}\rangle / 2t$ is a convex function of n (due to Hölder inequality) and it vanishes at the origin, where its derivative coincides by definition with the non-negative growth rate. Even when $\overline{\gamma} = 0$, the growth rates for $n > 0$ are positive if the entropy function has a finite width. For $n = 1$ this was stated in Gruzinov *et al.* (1996). As discussed previously, the behavior of the growth rate curve E_n is nonuniversal and it depends on the specific form of the entropy function. For the Kraichnan case, we can use the result (32) for the entropy function and the calculation is elementary. The dominant contribution is coming from the average of $\exp[n(\rho_1 - \rho_2)]$ and the ρ_2 integration is dominated by the lower bound $-\ln l/r_d$. The answer $E_2 = 3\lambda_1$ was first obtained by Kazantsev (1968). The general result is $E_n = \lambda_1 n(n+4)/4$, to be compared with the perfect conductor result $\lambda_1 n(2n+3)/2$. The difference between them formally means that the two limits of large times and small diffusivity do not commute (what is called "dissipative anomaly," see the next section). Multipoint correlation functions were calculated by Chertkov, Falkovich, *et al.* (1999). They reflect the prevailing strip structure of the magnetic field. An initially spherical blob evolves indeed into a strip structure, with the diffusive effects neutralizing one of the two directions that are contracted in a perfect conductor. The strips induce strong angular dependences and anomalous scalings similar to those described in Sec. III.B.3 below.

7. Coil-stretch transition for polymer molecules in a random flow

At equilibrium, a polymer molecule coils up into a spongy ball whose typical radius is kept at R_0 by thermal noise. Being placed in a flow, such molecule is deformed into an elongated ellipsoid which can be characterized by its end-to-end extension $\mathbf{R}$. As long as the elongation is much smaller than the total length of the molecule, the entropy is quadratic in R so that the molecule is brought back to its equilibrium shape by a damping linear in $\mathbf{R}$. The equation for the elongation is as follows (Hinch, 1977):

$$\partial_t \mathbf{R} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{R} = \mathbf{R} \cdot \boldsymbol{\nabla} \mathbf{v} - \frac{1}{\tau} \mathbf{R} + \boldsymbol{\eta}. \tag{125}$$

The left-hand side describes advection of the molecule as a whole, the first term on the right-hand side is responsible for stretching, τ is the relaxation time and $\boldsymbol{\eta}$ is the thermal noise with $\langle \eta^i(t) \eta^j(0) \rangle = \delta^{ij} \delta(t) R_0^2/\tau$. Since the size of the molecules is always much smaller than the viscous length then $\nabla \mathbf{v} = \sigma$ and one can solve Eq. (125) using the evolution matrix W introduced in Sec. II.B.1. At long enough times (when the initial condition is forgotten) the statistics of the elongation is given by $\mathbf{R} = \int_0^\infty ds\, W(s) \boldsymbol{\eta}(s) \exp[-t/\tau]$. We are interested in the tail of the PDF $\mathcal{P}(R)$ at $R \gg R_0$. The events contributing to it are related to the realizations with a long-time history of stretching where the variable ρ_1 (corresponding to the largest Lyapunov exponent λ_1) is large. The tail of the PDF is estimated analyzing the behavior of $R/R_0 = \int_0^\infty \exp[\rho_1(s) - s/\tau] ds$. The realizations dominating the tail are those where $\rho_1(s) - s/\tau$ takes a sharp maximum at some time s_* before relaxing to its typical negative values. The probability of those events is read from the large deviation expression (21): $\ln \mathcal{P} \sim -s_* H[s_*^{-1} \rho_1(s_*) - \lambda_1]$, where H is the entropy function. With logarithmic accuracy one can then replace $\rho_1(s_*) = \ln(R/R_0) + s_*/\tau$ and what is left is just to find the maximum with respect to s_*. The extremum value $X_* \equiv (s_*)^{-1} \ln(R/R_0)$ is fixed by the saddle-point condition that $H - X_* H'$ should vanish at $X_* + \tau^{-1} - \lambda_1$. The final answer for the PDF is as follows:

$$\mathcal{P}(R) \propto R_0^\alpha R^{-1-\alpha} \quad \text{with} \quad \alpha = H'(X_* + \tau^{-1} - \lambda_1). \tag{126}$$

The convexity of the entropy function ensures that α is positive if $\lambda_1 < 1/\tau$.

In accordance with Eq. (126), the exponent α decreases when λ_1 increases and it tends to zero as $\lambda_1 \to 1/\tau$. In this region, the entropy function is quadratic and the exponent is expressed via the average value of ρ_1 and its dispersion only: $\alpha=2(1-\lambda_1\tau)/\tau\Delta$. The integral of the PDF diverges at large R as α tends to zero. The transition at $\lambda_1\to 1/\tau$ is called the coil-stretch transition as the majority of the polymer molecules got stretched. This stretching can be stopped by nonlinear elastic effects or by the back reaction of the polymers onto the flow. The understanding of the coil-stretch transition goes back to the works by Lumley (1972, 1973). The power-law tail (126) has been derived by Balkovsky *et al.* (2000). The influence of nonlinear effects on the statistics of the elongation was examined by Chertkov (2000).

B. Cascades of a passive scalar

This section describes forced turbulence of a passive tracer statistically stationary in time and homogeneous in space. We consider the advection-diffusion equation

$$\partial_t\theta+(\mathbf{v}\cdot\boldsymbol{\nabla})\theta\kappa\boldsymbol{\nabla}^2\theta+\varphi \tag{127}$$

with the pumping φ assumed stationary, homogeneous, isotropic, Gaussian, of zero mean, and with covariance

$$\langle\varphi(\mathbf{r},t)\varphi(\mathbf{0},0)\rangle\delta(t)\Phi(r/l). \tag{128}$$

The function Φ is taken constant for $r/l\ll 1$ and decaying rapidly for large ratios. The following considerations are valid for a pumping finite correlated in time provided its correlation time in the Lagrangian frame is much smaller than the stretching time from a given scale to the pumping correlation scale l. Note that in most physical situations the sources do not move with the fluid so that the Lagrangian correlation time of the pumping is the minimum between its Eulerian correlation time and l/V, where V is the typical fluid velocity. Most of the general features of the advection are however independent of the details of the pumping mechanism and its Gaussianity and δ correlation are not a very serious restriction, as it will be shown in Sec. III.C.1.

Equation (127) implies for incompressible velocities the balance relation for the "scalar energy" density $e=\theta^2/2$:

$$\partial_t e+\boldsymbol{\nabla}\cdot\mathbf{j}=-\epsilon+\phi, \tag{129}$$

where $\epsilon=\kappa(\boldsymbol{\nabla}\theta)^2$ is the rate of dissipation, $\phi=\varphi\theta$ is that of energy injection, and $\mathbf{j}=\frac{1}{2}\theta^2\mathbf{v}-\kappa\theta\boldsymbol{\nabla}\theta$ is the flux density. In a steady state, the injection must be balanced by the diffusive dissipation, while the stretching and the contraction by the velocity provide for a steady cascade of the scalar from the pumping scale l to the diffusion scale r_d (where diffusion is comparable to advection).

The advection-diffusion dynamics induces the Hopf equations of evolution for the equal-time correlation functions. For a white-in-time pumping, one obtains

$$\partial_t\langle\theta_1\cdots\theta_N\rangle+\sum_{n=1}^{N}\langle\theta_1\cdots\mathbf{v}_n\cdot\boldsymbol{\nabla}_n\theta_n\cdots\theta_N\rangle$$
$$=\kappa\sum_{n=1}^{N}\langle\theta_1\cdots\boldsymbol{\nabla}_n^2\theta_n\cdots\theta_N\rangle+\sum_{n,m}\langle\theta_1\underset{\hat n\ \hat m}{\ldots\ldots}\theta_N\rangle\Phi_{nm} \tag{130}$$

in the shorthand notation $\theta_n\equiv\theta(\mathbf{r}_n,t)$, $\Phi_{nm}\equiv\Phi(\mathbf{r}_{nm}/l)$, etc. These equations are clearly not closed since the left-hand side involves the mixed correlators of the advected fields and the velocities. An exception is provided by the case of the Kraichnan ensemble of velocities where the mixed correlators may be expressed in terms of those containing only the advected fields, see Sec. III.C.1 below. The stationary version of the two-point Hopf equation may be written in the form

$$\langle(\mathbf{v}_1\cdot\nabla_1+\mathbf{v}_2\cdot\nabla_2)\theta_1\theta_2\rangle+2\kappa\langle\boldsymbol{\nabla}_1\theta_1\cdot\boldsymbol{\nabla}_2\theta_2\rangle=\Phi_{12}. \tag{131}$$

The relative strength of the two terms on the left-hand side depends on the distance. For velocities scaling as $\Delta_r v\propto r^\alpha$, the ratio of advection and diffusion terms $\mathrm{Pe}(r)=\Delta_r v r/\kappa$ may be estimated as $r^{\alpha+1}/\kappa$. In particular, $\mathrm{Pe}\equiv\mathrm{Pe}(l)$ is called the Péclet number, and the diffusion scale r_d is defined by the relation $\mathrm{Pe}(r_d)=1$.

In the "diffusive interval" $r_{12}\ll r_d$, the diffusion term dominates in the left-hand side of Eq. (131). Taking the limit of vanishing separations, we infer that the mean dissipation rate is equal to the mean injection rate $\bar\epsilon\equiv\langle\kappa(\boldsymbol{\nabla}\theta)^2\rangle=\frac{1}{2}\Phi(0)$. This illustrates the aforementioned phenomenon of the "dissipative anomaly": the limit $\kappa\to 0$ of the mean dissipation rate is nonzero despite the explicit κ factor in its definition. The "convective interval" $r_d\ll r_{12}\ll l$ widens up at increasing Péclet number. There, one may drop the diffusive term in Eq. (131) and thus obtain

$$\langle(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1+\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)\theta_1\theta_2\rangle\approx\Phi(0). \tag{132}$$

The expression (4) may be derived from the general flux relation (132) by the additional assumption of isotropy. The relation (132) states that the mean flux of θ^2 stays constant within the convective interval and expresses analytically the downscale scalar cascade. For the velocity scaling $\Delta_r v\propto r^\alpha$, dimensional arguments suggest that $\Delta_r\theta\propto r^{(1-\alpha)/2}$ (Obukhov, 1949; Corrsin, 1951). This relation gives a proper qualitative understanding that the degrees of roughness of the scalar and the velocity are complementary, yet it suggests a wrong scaling for the scalar structure functions of order higher than the second one; see Sec. III.C.1.

Let us now derive the exact Lagrangian expressions for the scalar correlation functions. The scalar field along the Lagrangian trajectories $\mathbf{R}(t)$ changes as

$$\frac{d}{dt}\theta(\mathbf{R}(t),t)=\varphi(\mathbf{R}(t),t). \tag{133}$$

The Nth-order scalar correlation $\langle\Pi_{p=1}^N\theta(\mathbf{r}_n,t)\rangle\equiv C_N(\underline{\mathbf{r}},t)$ is then given by

$$C_N(\underline{\mathbf{r}},t)=\left\langle \int_0^t \varphi(\mathbf{R}_1(s_1),s_1)ds_1\cdots\int_0^t \varphi(\mathbf{R}_N(s_N),s_N)ds_N\right\rangle, \tag{134}$$

with the Lagrangian trajectories satisfying the final conditions $\mathbf{R}_n(t)=\mathbf{r}_n$. For the sake of simplicity, we have written down the expression for the case where the scalar field was vanishing at the initial time. If some of the distances among the particles get below the diffusive scale, the molecular noises in the Lagrangian trajectories become relevant and the averaging of Eq. (134) over their statistics is needed.

The average over the Gaussian pumping in Eq. (134) gives for the pair-correlation function

$$C_2(\mathbf{r}_{12},t)=\left\langle \int_0^t \Phi(\mathbf{R}_{12}(s)/l)ds\right\rangle. \tag{135}$$

The function Φ essentially restricts the integration to the time interval where R_{12} is smaller than the injection length l. If the Lagrangian trajectories separate, the pair correlation reaches at long times the stationary form given by the same formula with $t=\infty$. Simply speaking, the stationary pair correlation function of a tracer is proportional to the average time that two particles spent in the past within the correlation scale of the pumping (Falkovich and Lebedev, 1994). Similarly, the pair structure function $S_2(r)=\langle(\theta_1-\theta_2)^2\rangle 2[C_2(\mathbf{0})-C_2(\mathbf{r})]$ is proportional to the time it takes for two coinciding particles to separate to a distance r. This is proportional to $r^{1-\alpha}$ for a scale-invariant velocity statistics with $\Delta_r v \propto r^{\alpha}$, see Eq. (38), so that $S_2(r)$ is in agreement with the Obukhov-Corrsin dimensional prediction.

Higher-order equal-time correlation functions are expressed similarly by using the Wick rule to average over the Gaussian forcing:

$$C_{2n}(\underline{\mathbf{r}},t)=\left\langle \int_0^t \Phi[\mathbf{R}_{12}(s_1)]ds_1\cdots \times\int_0^t \Phi[\mathbf{R}_{(2n-1)2n}(s_n)]ds_n\right\rangle+\cdots, \tag{136}$$

where the remaining average is over the velocity and the molecular noise ensemble and the dots stand for the other possible pairings of the $2n$ points. The correlation functions may be obtained from the generating functional

$$\left\langle \exp\left[i\int \theta(\mathbf{r},t)\chi(\mathbf{r})d\mathbf{r}\right]\right\rangle =\exp\left[-\frac{1}{2}\int_0^t ds\int\int \Phi[\mathbf{R}_{12}(s)]\chi(\mathbf{r}_1)\chi(\mathbf{r}_2)d\mathbf{r}_1\, d\mathbf{r}_2\right]. \tag{137}$$

They probe the statistics of times spent by fluid particles at distances R_{ij} smaller than l. In nonsmooth flows, the correlation functions at small scales, $r_{ij}\ll l$, are dominated by the single-point contributions, corresponding to initially coinciding particles. This is the signature of the explosive separation of the trajectories. To pick up a strong dependence on the positions $\underline{\mathbf{r}}$, one has to study the structure functions which are determined by the time differences between different initial configurations. Conversely, the correlation functions at scales larger than l are strongly dependent on the positions, as it will be shown in Sec. III.B.3.

1. Passive scalar in a spatially smooth velocity

In the rest of Sec. III.B, all the scales are supposed much smaller than the viscous scale of turbulence so that we may assume the velocity field to be spatially smooth and use the Lagrangian description developed in Secs. II.B and II.D. In the Batchelor regime, the backward evolution of the Lagrangian separation vector is given by $\mathbf{R}_{12}(0)=\widetilde{W}(t)^{-1}\mathbf{r}_{12}$ (if we ignore diffusion) and it is dominated by the stretching rate ρ_d at long times. The Eq. (135) takes then the asymptotic form

$$C_2(\mathbf{r},t)\approx\int_0^t ds\int \Phi(e^{-\tilde{\rho}_d(s)}r/l)\widetilde{\mathcal{P}}(\rho_1,\ldots,\rho_d;s)d\rho_1\cdots d\rho_d. \tag{138}$$

The behavior of the interparticle distance crucially depends on the sign of $\tilde{\lambda}_d$. For $\tilde{\lambda}_d<0$, the backward-in-time evolution separates the particles and leads in the limit $t\to\infty$ to a well-defined steady state with the correlation function

$$\langle\theta(t,0)\theta(t,\mathbf{r})\rangle\approx|\tilde{\lambda}_d|^{-1}\Phi(0)\ln(l/r), \tag{139}$$

for $r\lesssim l$. This corresponds to the direct cascade. Conversely, if $\tilde{\lambda}_d>0$ the particles contract and the pair-correlation function grows proportionally to t. Note that the growing part is independent of r. This means that, in a flow contracting backwards in time, tracer fluctuations grow at larger and larger scales, which is a signature of the so-called inverse cascade of a passive tracer.

If the velocity ensemble is time-reversible, as it is the case for the δ-correlated model (57), then $\tilde{\lambda}_i=-\lambda_{d-i+1}$ and λ_1 and $\tilde{\lambda}_d$ have opposite sign. They will thus both change sign at the same value of the degree of compressibility $\wp=d/4$, see Eq. (60). This is peculiar for a short-correlated case and does not hold for an arbitrary velocity statistics. There, the change from stretching to contraction in the forward Lagrangian dynamics does not necessarily correspond to the change in the direction of the cascade for the passive tracer, related to the backward in time Lagrangian dynamics.

2. Direct cascade, small scales

We consider here the case $\tilde{\lambda}_d<0$ (that includes smooth incompressible flows) when the particles do separate backward in time and a steady state exists. We first treat the convective interval of distances between the diffusion scale r_d and the pumping scale l. Deep inside the convective interval where $r\ll l$, the statistics of the passive scalar tends to become Gaussian. Indeed, the reducible part in the $2n$-point correlation function $\langle\Phi(e^{-\tilde{\rho}_d(s_1)}r_{12}/l)\cdots\Phi(e^{-\tilde{\rho}_d(t_n)}r_{(2n-1)2n}/l)\rangle$, see Eqs.

(136) and (138), dominates the irreducible one for $n \ll n_{cr} \approx (|\tilde{\lambda}_d|\tau_s)^{-1}\ln(l/r)$. The reason is that the logarithmic factors are smaller for the irreducible than for the reducible contribution (Chertkov *et al.*, 1995a). The critical order n_{cr} is given by the ratio between the time for the particles to separate from a typical distance r to l and the correlation time τ_s of the stretching rate fluctuations. Since $l \gg r$, the statistics of the passive tracer is Gaussian up to orders $n_{cr} \gg 1$. For the single-point statistics, the scale appearing in the expression of the critical order n_{cr} should be taken as $r = r_d$. The structure functions are dominated by the forced solution rather than the zero modes in the convective interval: $S_{2n} = \langle [\theta(\mathbf{0}) - \theta(\mathbf{r})]^{2n}\rangle \propto \ln^n(r/r_d)$ for $n \ll \ln(r/r_d)$. A complete expression (the forced solution plus the zero modes) for the four-point correlation function in the Kraichnan model can be found in Balkovsky, Chertkov, *et al.* (1995).

Let us show now that the tails of the tracer PDF decay exponentially (Shraiman and Siggia, 1994; Chertkov *et al.*, 1995a; Bernard *et al.*, 1998; Balkovsky and Fouxon, 1999). The physical reasons behind this are transparent and most likely they apply also for a non-smooth velocity. First, large values of the scalar can be achieved only if during a long interval of time the pumping works uninterrupted by stretching events that eventually bring diffusion into play. We are interested in the tails of the distribution, i.e., in intervals much longer than the typical stretching time from r_d to l. Those rare events can be then considered as the result of a Poisson random process and the probability that no stretching occurs during an interval of length t is $\propto \exp(-ct)$. Second, the values achieved by the scalar in those long intervals are Gaussian with variance $\Phi(0)t$. Note that this is also valid for a non-Gaussian and finite-correlated pumping, provided t is larger than its correlation time. By integrating over the length of the no-stretching time intervals with the pumping-produced distribution of the scalar we finally obtain $\mathcal{P}(\theta) \propto \int dt\, \exp[-ct - \theta^2/2\Phi(0)t] \propto \exp[-\theta\sqrt{2c/\Phi(0)}]$. This is valid for $t < L^2/\kappa$ that is for $\theta < \sqrt{c\Phi(0)}L^2/\kappa$. Interval of exponential behavior thus increases with the Peclet number. For a smooth case, the calculations have been carried out in detail and the result agrees with the previous arguments. Experimental data in Jullien *et al.* (2000) confirm both the logarithmic form of the correlation functions and the exponential tails of the scalar PDF. In some experimental setups the aforementioned conditions for the exponential tails are not satisfied and a different behavior is observed; see, for example, Jayesh and Warhaft (1991). The physical reason is simple to grasp. The injection correlation time in those experiments is given by L/V, where L is the velocity integral scale and V is the typical velocity. The no-stretching times involved in the tail of the scalar distribution are of the order of W/V, where W is the width of the channel where the experiment is performed. Our previous arguments clearly require $W \gg L$. As the width of the channel is increased, the tails indeed tend to become exponential.

For a δ-correlated strain, the calculation of the generating functional of the scalar correlators may be reduced to a quantum-mechanical problem. In the Batchelor regime and in the limit of vanishing κ, the exponent in the generating functional (137) may indeed be rewritten as

$$-\int_0^t V_\chi[\tilde{W}(s)]ds = -\frac{1}{2}\int_0^t ds \int\int \Phi[\tilde{W}^{-1}(s)\mathbf{r}_{12}]\chi(\mathbf{r}_1)\chi(\mathbf{r}_2)d\mathbf{r}_1\, d\mathbf{r}_2. \tag{140}$$

Recall that the matrices $\tilde{W}$ form a stochastic process describing a diffusion on GL(d) [or on SL(d) in the incompressible case] with a generator $\tilde{M}$. The above formula may thus be interpreted as the Feynman-Kac expression for the integral $\int e^{t(\tilde{M}-V_\chi)}(1,\tilde{W})d\tilde{W}$ of the heat kernel of $\tilde{M}$ perturbed by the positive potential V_χ. As long as the trajectories separate backward in time, i.e., for $\tilde{\lambda}_d < 0$, the generating functional has a stationary limit, given by Eq. (140) with the time integral extending to infinity. The Feynman-Kac formula may be used to find the exponential rate of decay of the PDF $\mathcal{P}(\theta) \propto e^{-b|\theta|}$. As shown by Bernard *et al.* (1998), this involves the quantum-mechanical Hamiltonian $-\tilde{M} - a^2 V_\chi$, where the positive operator $-\tilde{M}$ has its spectrum starting at a strictly positive value and the negative potential tends to produce a bound state as the parameter a is increased. In the incompressible case, the decay rate b is characterized by the property that the ground state of the Hamiltonian has zero energy. For isotropic situations, the potential is only a function of the stretching rates of $\tilde{W}$ and the quantum-mechanical problem reduces to the perturbation of the Calogero-Sutherland Hamiltonian by a potential; see Sec. II.B.2.

3. Direct cascade, large scales

We consider here the scales $r \gg l$ in the steady state established under the condition $\tilde{\lambda}_d < 0$ (Balkovsky *et al.*, 1999). From a general physical viewpoint, it is of interest to understand the properties of turbulence at scales larger than the pumping scale. A natural expectation is to have there an equilibrium equipartition with the effective temperature determined by the small-scale turbulence (Forster *et al.*, 1977; Balkovsky, Falkovich, *et al.*, 1995). The peculiarity of our problem is that we consider scalar fluctuations at scales larger than that of pumping yet smaller than the correlation length of the velocity field. This provides for an efficient mixing of the scalar even at those large scales. Although one can find the simultaneous correlation functions of different orders, it is yet unclear if such a statistics can be described by any thermodynamical variational principle.

The correlation functions of the scalar are proportional to the time spent by the Lagrangian particles within the pumping scale. It follows that the statistics at $r \gg l$ is related to the probabilities of initially distant particles to come close. For spatially smooth random flow, such statistics turns out to be strongly intermittent and non-Gaussian. Another unexpected feature in this regime is a total breakdown of scale invariance: not only the scaling exponents are anomalous and do not grow linearly with the order of the correlation function, but even fixed correlation functions are generally not scale invariant. The scaling exponents depend indeed on the angles between the vectors connecting the points. Note that the large-scale statistics of a scalar is scale-invariant in a nonsmooth velocity, see Balkovsky *et al.* (1999) and Sec. III.C.1.e.

What is the probability for the vector $\mathbf{R}_{12}(t)$, that was once within the pumping correlation length l, to come exactly to the prescribed value $\mathbf{r}$ at time t? The advection makes a sphere of "pumping" volume l^d evolve into an elongated ellipsoid of the same volume. Ergodicity may be assumed provided that the stretching time $\lambda^{-1}\ln(r/l)$ is larger than the strain correlation time. It follows that the probability for two points separated by r to belong to a "piece" of scalar originated from the same pumping sphere behaves as the volume fraction $(l/r)^d$. That gives the law of decrease of the two-point function: $C_2\propto r^{-d}$.

The advection by spatially smooth velocities preserves straight lines. To determine the correlation functions of an arbitrary order when all the points lie on a line, it is enough to notice that the history of the stretching is the same for all the particles. Looking backward in time we may say that when the largest distance among the points was smaller than l, then all the other distances were as well. It follows that the correlation functions for a collinear geometry depend on the largest distance r among the points so that $C_{2n}\propto r^{-d}$. This is true also when different pairs of points lie on parallel lines. Note that the exponent is n independent which corresponds to a strong intermittency and an extreme anomalous scaling. The fact that $C_{2n}\gg C_2^n$ is due to the strong correlation of the points along their common line.

The opposite takes place for noncollinear geometries, namely, the stretching of different nonparallel vectors is generally anticorrelated in the incompressible case due to the volume conservation. The d volume $\epsilon_{i_1 i_2\cdots i_d}R_{12}^{i_1}\cdots R_{1d}^{i_d}$ is indeed preserved for $(d+1)$ Lagrangian trajectories $\mathbf{R}_n(t)$ and, for $d=2$ and any three trajectories, the area $\epsilon_{ij}\mathbf{R}_{12}^i\mathbf{R}_{13}^j$ of the triangle defined by the three particles remains constant. The anticorrelation due to the area conservation may then be easily understood and the scaling for noncollinear geometries at $d=2$ may be determined. Since the area of any triangle is conserved, three points that form a triangle with area $A\gg l^2$ will never come within the pumping correlation length. In the presence of a triple correlator Φ_3 for a non-Gaussian δ-correlated pumping, the triple correlation function of a scalar,

$$\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\theta(\mathbf{r}_3)\rangle=\left\langle\int_0^\infty\Phi_3(\mathbf{R}_{12}(s),\mathbf{R}_{13}(s))ds\right\rangle, \tag{141}$$

is determined by the asymptotic behavior of Φ_3 at $r_{ij}\gg l$. For example, if Φ_3 has a Gaussian tail, then $C_3\propto\exp(-A/l^2)$. On the other hand, the correlation functions decrease as r^{-2} for a collinear geometry. We conclude that C_3 as a function of the angle between the vectors $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ has a sharp maximum at zero and rapidly decreases within an interval of width of the order $l^2/r^2\ll 1$.

Similar considerations apply for the fourth-order correlation function. Note that, unlike for the three-point function, there are now reducible contributions. Consider, for instance, that coming from $\langle\int_0^\infty\int_0^\infty\Phi[\mathbf{R}_{12}(s_1)]\Phi[\mathbf{R}_{34}(s_2)]ds_1\,ds_2\rangle$. Since the area of the polygon defined by the four particles is conserved throughout the evolution, the answer is again crucially dependent on the relation between the area and l^2. The events contributing to the correlation function C_4 are those where, during the evolution, R_{12} became of the order l and then, at some other moment of time, R_{34} reached l. The probability for the first event to happen is l^2/r_{12}^2. When this happens, the area preservation makes $R_{34}\sim r_{12}r_{34}/l$. The probability for this separation to subsequently reduce to l is $\propto l^4/r_{12}^2r_{34}^2$. The total probability can be thus estimated as l^6/r^6, where r is the typical value of the separations r_{ij}. Remark that the naive Gaussian estimation l^4/r^4 is much smaller than the collinear answer and yet much larger than the noncollinear one.

The previous arguments can be readily extended to an arbitrary number of noncollinear pairs. In accordance with Eq. (136), the realizations contributing to the correlation are those where the separations R_{ij} reduce down to l during the evolution process. Suppose that this happens first for R_{12}. Such process was already explained in the consideration of the pair-correlation function and occurs with probability $(l/r_{12})^2$. All the remaining separations will then be larger than their initial values by a factor r_{12}/l, due to the conservation law of the triangular areas. Next, we should reduce, say, R_{34} from $r_{34}r_{12}/l$ down to the integral scale l. Such a process occurs with probability $[l^2/(r_{12}r_{34})]^2$. When this happens, all the other separations are larger than their initial values by a factor r_{34}/l. Repeating the process, we come to the final answer $C_{2n}\propto(l/r)^{4n-2}$, where r is again the typical value of the separations r_{ij}.

The above analysis is easy to generalize for arbitrary geometries. The points are divided into sets consisting of the pairs of points with parallel separations $\mathbf{r}_{ij}$ (more precisely, forming angles smaller than l^2/r^2). The points within a given set behave as a single separation during the Lagrangian evolution. The order n in the previous formulas should then be replaced by the (minimal) number of sets. The estimates obtained above are supported by the rigorous calculations in Balkovsky *et al.* (1999).

In 2D, the area conservation allowed to get the scaling without calculations. This is related to the fact that there is a single Lyapunov exponent. When $d>2$, we have only the conservation of d-dimensional volumes and hence more freedom in the dynamics. For example, the area of a triangle can change during the evolution and the three-point correlation function for a noncollinear geometry is not necessarily suppressed. Nevertheless, the anticorrelation between different Lagrangian trajectories is still present and the three-point exponent is expected to be larger than the naive estimate 2D. The answer for the Kraichnan model $d+(d-1)\sqrt{d/(d-2)}$ is determined by the whole hierarchy of the Lyapunov exponents (Balkovsky *et al.*, 1999). In the limit of large dimensions, the anticorrelation tends to disappear and the answer approaches 2D. The four-point correlation function is also determined by the joint evolution of two distances, which results for large d in the same value of the exponent.

To conclude this section, we briefly comment on the case $\tilde{\lambda}_d > 0$ when the particles approach rather than separate backward in time. Here, an inversion of what has been described for the direct cascade takes place: the scalar correlation functions are logarithmic and the PDF has a wide Gaussian core at $r > l$, while the statistics is strongly non-Gaussian at small scales (Chertkov, Kolokolov, and Vergassola, 1998). Since the scalar fluctuations injected at l propagate upscale, small-scale diffusion is negligible and some large-scale damping (say, by friction) is needed to provide for a steady state, see also Sec. III.E below.

4. Statistics of the dissipation

We now describe the PDF's of the scalar gradients $\boldsymbol{\omega} = \nabla\theta$ and of the dissipation $\epsilon = \kappa\omega^2$ in the steady state of a direct cascade arising under the action of a large-scale pumping. We consider a smooth velocity field, i.e., both the Schmidt/Prandtl and the Péclet numbers are assumed large. As we remarked in Sec. III.A.5, the scalar gradients can be estimated as $\theta e^{-\rho_d}/l$, where θ is the scalar value and $e^{\rho_d} l$ is the smallest (diffusive) scale. The tails of the gradient PDF are controlled by the large values of θ and $-\rho_d$. The statistics of the former depends both on the pumping and the velocity and that of the latter only on the velocity. The key remark for solving the problem was made by Chertkov, Kolokolov, and Vergassola (1997, 1998): since θ and $-\rho_d$ fluctuate on very separated time scales [$|\lambda_d|^{-1}\ln(l/r_d)$ and λ_d^{-1}, respectively], their fluctuations may be analyzed separately. The PDF of the scalar has been shown in Sec. III.B.2 to decay exponentially. On the other hand, large negative values of ρ_d are determined by the tail of its stationary distribution; see Eq. (27). For a Gaussian short-correlated strain this tail is $\propto\exp[-\mathrm{const}\times e^{-2\rho_d}]$ (Chertkov, Kolokolov, and Vergassola, 1998) and the moments of the gradients are $\langle\omega^n\rangle \propto \langle\theta^n\rangle\langle\exp(-n\rho_d)\rangle \propto n^{3n/2}$. The ensuing behavior $\langle\epsilon^n\rangle \propto n^{3n}$ corresponds to a stretched-exponential tail for the PDF of the dissipation

$$\ln\mathcal{P}(\epsilon) \propto -\epsilon^{1/3}. \tag{142}$$

The detailed calculation for the Kraichnan model as well as a comparison with numerical and experimental data can be found in Chertkov, Kolokolov, and Vergassola (1998), Chertkov, Falkovich, and Kolokolov (1998), and Gamba and Kolokolov (1998). The general case of a smooth flow with arbitrary statistics was considered in Balkovsky and Fouxon (1999).

It is instructive to compare the stretched-exponential PDF of the gradients in a steady state with the lognormal PDF described in Sec. III.A.5 for the initial diffusionless growth. Intermittency builds up during the initial stage, i.e., the higher the moment, the faster it grows. On the other hand, the higher the moment, the shorter is the breakdown time of the diffusionless approximation. This time behaves for example as $(n+2)^{-1}$ in the Kraichnan model. Since higher moments stop growing earlier than lower ones, the tails of the PDF become steeper and the intermittency is weaker in the steady state.

C. Passive fields in the inertial interval of turbulence

For smooth velocities, the single-point statistics of the advected quantities could be inferred from the knowledge of the stretching rates characterizing the Lagrangian flow in the infinitesimal neighborhood of a fixed trajectory. This was also true for the multipoint statistics as long as all the scales involved were smaller than the viscous scale of the velocity, i.e., in the Batchelor regime. In this subsection we shall analyze advection phenomena, mostly of scalars, in the inertial interval of scales where the velocities become effectively nonsmooth. As discussed in Sec. II.C, the explosive separation of the trajectories in nonsmooth velocities blows up interparticle separations from infinitesimal to finite values in a finite time. This phenomenon plays an essential role in maintaining the dissipation of conserved quantities nonzero even when the diffusivity $\kappa\to 0$. The statistics of the advected fields is consequently more difficult to analyze, as we discuss below.

1. Passive scalar in the Kraichnan model

The Kraichnan ensemble of Gaussian white-in-time velocities permits an exact analysis of the nonsmooth case and a deeper insight into subtle features of the advection, like intermittency and anomalous scaling. Those aspects are directly related to the collective behavior of the particle trajectories studied in the first part of the review. Important lessons learned from the model will be discussed in next subsections in a more general context.

a. Hopf equations

The simplifying feature associated to the Kraichnan velocities is a reduction of the corresponding Hopf equations to a closed recursive system involving only correlators of the advected fields. This is due to the temporal decorrelation of the velocity and the ensuing Markov property of the Lagrangian trajectories. Let us consider, for example, the evolution Eq. (127) for a scalar field. For the Kraichnan model, it becomes a stochastic differential equation. As mentioned in Sec. II.B.2, one may view white-in-time velocities as the scaling limit of ensembles with short time correlations. The very fact that $\mathbf{v}(t)dt$ tends to become of the order $(dt)^{1/2}$ calls for a regularization. For velocity ensembles invariant under time reversal, the relevant convention is that of Stratonovich (see the Appendix). Interpreting Eq. (127) within this convention and applying the rules of stochastic differential calculus, one obtains the equation for the scalar correlation functions:

$$\partial_t C_N(\underline{\mathbf{r}})=\mathcal{M}_N C_N(\underline{\mathbf{r}}) + \sum_{n<m} C_{N-2}(\mathbf{r}_1,\ldots,\underset{\hat{n}}{}\ \underset{\hat{m}}{},\mathbf{r}_N)\Phi(\mathbf{r}_{nm}/l). \tag{143}$$

Here, the differential operator $\mathcal{M}_N$ is the same[9] as in Eq. (69) and it may be formally obtained from the second term on the left-hand side of Eq. (130) by a Gaussian integration by parts. Note the absence of any closure problem for the triangular system of Eqs. (143): once the lower-point functions have been found, the N-point correlation function satisfies a closed equation. For spatially homogeneous situations, the operators $\mathcal{M}_N$ may be replaced by their restrictions $\widetilde{\mathcal{M}}_N$ to the translation-invariant sector. It follows from their definition (70) and the velocity correlation function (48) that Eqs. (143) are then invariant with respect to the rescalings

$$\mathbf{r}\to\lambda\mathbf{r},\quad l\to\lambda l,\quad t\to\lambda^{2-\xi}t,\quad \kappa\to\lambda^{\xi}\kappa,\quad \theta\to\lambda^{-(2-\xi)/2}\theta. \tag{144}$$

This straightforward observation implies scaling relations between the stationary correlators: $C_N(\lambda\mathbf{r};\lambda^{\xi}\kappa,\lambda l)=\lambda^{N(2-\xi)/2}C_N(\mathbf{r};\kappa,l)$.

b. Pair correlator

For the isotropic pair correlation function, Eq. (143) takes the form

$$\partial_t C_2(r)-r^{1-d}\partial_r[(d-1)D_1 r^{d-1+\xi}+2\kappa r^{d-1}]\partial_r C_2(r) =\Phi(r/l); \tag{145}$$

see Eq. (53). The ratio of the advective and the diffusive terms is of order unity at the diffusion scale $r_d \equiv[2\kappa/(d-1)D_1]^{1/\xi}$. For the Kraichnan model, the Péclet number $\mathrm{Pe}\equiv(d-1)D_1 l^{\xi}/2\kappa\gg1$ as we assume the scale of pumping much larger than that of diffusion. The stationary form of Eq. (145) becomes an ordinary differential equation (Kraichnan, 1968) that may be easily integrated with the two boundary conditions of zero at infinity and finiteness at the origin:

$$C_2(r)=\frac{1}{(d-1)D_1}\int_r^{\infty}\frac{x^{1-d}\,dx}{x^{\xi}+r_d^{\xi}}\int_0^x\Phi(y/l)y^{d-1}\,dy. \tag{146}$$

Even without knowledge of the explicit form (146), it is easy to draw from Eq. (145) general conclusions, as for time-correlated velocities. Taking the limit $r\to0$ for $\kappa>0$, we infer the mean scalar energy balance

$$\partial_t\overline{e}+\overline{\epsilon}=\Phi(0)/2, \tag{147}$$

where $\overline{e}=\langle\theta^2\rangle/2$ and $\overline{\epsilon}=\langle\kappa(\nabla\theta)^2\rangle$. In the stationary state, the dissipation balances the injection. On the other hand, for $r\gg r_d$ (or for any $r>0$ in the $\kappa\to0$ limit) we may drop the diffusive term in the Hopf Eq. (145). For $r\ll l$, we thus obtain the Kraichnan model formulation of the Yaglom relation (132) expressing the constancy of the downscale flux:

$$-(d-1)D_1\frac{1}{r^{d-1}}\partial_r r^{d-1+\xi}\partial_r C_2(r,t)\approx\Phi(0). \tag{148}$$

To obtain the balance relation (147) for vanishing κ as the $r\to0$ limit of Eq. (145), one has to define the limiting dissipation field by the operator product expansion

$$\lim_{\kappa\to0}\kappa(\boldsymbol{\nabla}\theta)^2(\mathbf{r})=\frac{1}{2}\lim_{\mathbf{r}'\to\mathbf{r}}d^{ij}(\mathbf{r}-\mathbf{r}')\nabla_i\theta(\mathbf{r})\nabla_j\theta(\mathbf{r}'). \tag{149}$$

The relation (149), encoding the dissipative anomaly, holds in general correlation functions away from other insertions (Bernard *et al.*, 1996).

Let us discuss now the solution (146) in more detail. There are three intervals of distinct behavior. First, at large scales $r\gg l$, the pair-correlation function is given by

$$C_2(r)\approx\frac{1}{(d+\xi-2)(d-1)D_1}\overline{\Phi}l^d r^{2-\xi-d}, \tag{150}$$

where $\overline{\Phi}=\int_0^{\infty}y^{d-1}\Phi(y)dy$. This may be thought of as the Rayleigh-Jeans equipartition $\langle\theta(\mathbf{k})\theta(\mathbf{k}')\rangle=\delta(\mathbf{k}+\mathbf{k}')\overline{\Phi}l^d/\Omega(k)$ with $\Omega(k)\propto k^{2-\xi}$ and the temperature proportional to $\overline{\Phi}l^d$. Note that the right-hand side of Eq. (150) is a zero mode of $\widetilde{\mathcal{M}}_2$ away from the origin: $r^{1-d}\partial_r r^{d-1+\xi}\partial_r r^{2-\xi-d}\propto\delta(r)$. Second, in the convective interval $r_d\ll r\ll l$, the pair correlator is equal to a constant (the genuine zero mode of $\widetilde{\mathcal{M}}_2$) plus an inhomogeneous part:

$$C_2(r)\approx\mathcal{A}_2 l^{2-\xi}-\frac{1}{(2-\xi)d(d-1)D_1}\Phi(0)r^{2-\xi}, \tag{151}$$

where $\mathcal{A}_2=\Phi(0)/(2-\xi)(d+\xi-2)(d-1)D_1$. The leading constant term drops out of the structure function:

$$S_2(r)=2[C_2(0)-2C_2(r)] \approx\frac{2}{(2-\xi)d(d-1)D_1}\Phi(0)r^{2-\xi}. \tag{152}$$

Note that the last expression is independent of both κ and l and it depends on the pumping through the mean injection rate only, i.e., $S_2(r)$ is universal. Its scaling exponent $\zeta_2=2-\xi$ is fixed by the dimensional rescaling properties (144) or, equivalently, by the scaling of the separation time of the Lagrangian trajectories; see Eq. (55). As remarked before, the degrees of roughness of the scalar and the velocity turn out to be complementary: a smooth velocity corresponds to a rough scalar and vice versa. Finally, in the diffusive interval $r\ll r_d$, the pair-correlation function is dominated by a constant and the structure function $S_2(r)\approx(1/2\kappa d)\Phi(0)r^2$. Note

[9]The $n=m$ terms would drop out of the expression for $\mathcal{M}_N$ in the Itô convention for Eq. (127).

that $S_2(r)$ is not analytic at the origin, though. Its expansion in r contains noninteger powers of order higher than the second, due to the nonsmoothness of the velocity down to the smallest scales. The analyticity is recovered if we keep a finite viscous cutoff for the velocity.

In the limit $\kappa\to 0$, the diffusive interval disappears and the pair correlator is given by Eq. (146) with the diffusion scale r_d set to zero. The mean square of the scalar $C_2(0)$ remains finite for finite l but diverges in the $l\to\infty$ limit that exists only for the structure function. Recall from Sec. III.B.1 that $C_2(r)$ has the interpretation of the mean time that two Lagrangian trajectories take to separate from distance r to l. The finite value of the correlation function at the origin is therefore another manifestation of the explosive separation of the Lagrangian trajectories.

The solution (146) for the pair correlator and most of the above discussion remain valid also for $\xi=2$, i.e., for smooth Kraichnan velocities. A notable difference should be stressed, though. In smooth velocities and for $\kappa\to 0$, the mean time of separation of two Lagrangian trajectories diverges logarithmically as their initial distance tends to vanish. The pair correlator has a logarithmic divergence at the origin, implying that $\langle\theta^2\rangle$ is infinite in the stationary state with $\kappa=0$. Indeed, it is the $\partial_t C_2(0)$ term that balances the right-hand side of Eq. (145) at finite times and $r=0$. As the diffusivity vanishes, the variance $\langle\theta^2\rangle$ keeps growing linearly in time with the rate $\Phi(0)$ and the mean dissipation tends to zero: no dissipative anomaly is present at finite times. The anomaly occurs only in the stationary state that takes longer and longer to achieve for smaller r. In mathematical terms: $\lim_{t\to\infty}\lim_{\kappa\to 0}\overline{\epsilon}\neq\lim_{\kappa\to 0}\lim_{t\to\infty}\overline{\epsilon}$, with the left-hand side vanishing and the right-hand side equal to the mean injection rate. The physics behind this difference is clear. The dissipative anomaly for nonsmooth velocities is due to the nonuniqueness of the Lagrangian trajectories; see Sec. II.C. The incompressible version of Eq. (100) implies that, in the absence of forcing and diffusion,

$$\int \theta^2(\mathbf{r}',0)d\mathbf{r}'-\int \theta^2(\mathbf{r}',t)d\mathbf{r}'$$
$$=\int d\mathbf{r}\int p(\mathbf{r},t;\mathbf{R},0|\mathbf{v})[\theta(\mathbf{R},0)-\theta(\mathbf{r},t)]^2\,d\mathbf{R}\geqslant 0. \tag{153}$$

The equality holds if and only if, for almost all $\mathbf{r}$, the scalar is constant on the support of the measure $p(\mathbf{r},t;\mathbf{R},0|\mathbf{v})d\mathbf{R}$ giving the distribution of the initial positions of the Lagrangian trajectories ending at $\mathbf{r}$ at time t. In plain language, $\int\theta^2\,d\mathbf{r}$ is conserved if and only if the Lagrangian trajectories are uniquely determined by the final condition. This is the case for smooth velocities and no dissipation takes place for $\kappa=0$ as long as $\int\theta^2\,d\mathbf{r}$ is finite. When the latter becomes infinite (as in the stationary state), the above inequalities become void and the dissipation may persist in the limit of vanishing κ even for a smooth flow.

For $\xi=0$, Eq. (146) still gives the stationary pair correlation function if $d\geqslant 3$. The distinction between the behavior in the convective and the diffusive regimes disappears. The overall behavior becomes diffusive with the stationary equal-time correlation functions coinciding with those of the forced diffusion $\partial_t\theta=[\frac{1}{2}(d-1)D_1+\kappa]\nabla^2\theta+\varphi$ (Gawędzki and Kupiainen, 1996). In $d=2$, the pair-correlation function has a constant contribution growing logarithmically in time but the structure function does stabilize, as in forced diffusion.

c. Higher correlators and zero modes

Let us consider the evolution of higher-order scalar correlation functions C_N, assumed to decay rapidly in the space variables at the initial time. At long times, the correlation functions will then approach a stationary form given by the recursive relation

$$C_N(\underline{\mathbf{r}})$$
$$=\int G_N(\underline{\mathbf{r}},\underline{\mathbf{R}})\sum_{n<m} C_{N-2}(\mathbf{R}_1,\underset{\hat n\ \hat m}{\ldots},\mathbf{R}_N)\Phi(\mathbf{R}_{nm}/l)d\underline{\mathbf{R}} \tag{154}$$

for even N and vanishing for odd N (by the $\theta\mapsto-\theta$ symmetry). Here, $G_N=\int_0^\infty e^{t\tilde{\mathcal{M}}_N}dt$ are the operators inverse to $-\tilde{\mathcal{M}}_N$. The above formulas give specific solutions of the stationary Hopf Eqs. (143) that, alone, determine solutions only up to zero modes of operators $\mathcal{M}_N$. We are interested in the scaling properties of the stationary correlation function C_N in the convective interval. If the correlation functions were becoming independent of κ and l in this interval (mathematically, if the limits $\kappa\to 0$ and $l\to\infty$ of the functions existed), the scaling behavior would follow from the dimensional relation (144):

$$C_N(\lambda\underline{\mathbf{r}})=\lambda^{(2-\xi)N/2}C_N(\underline{\mathbf{r}}). \tag{155}$$

This would be the Kraichnan-model version of the normal Kolmogorov-Obukhov-Corrsin scaling (Obukhov, 1949; Corrsin, 1951). The $\kappa\to 0$ limit of the stationary correlation functions does exist and the κ dependence drops out of the expressions in the convective interval, as for the pair correlator. The limit is given by the formulas (154) with the $\kappa=0$ versions of G_N (Hakulinen, 2000). Note in passing that the advection preserves any power of the scalar so that dissipative anomalies are present also for orders higher than the second. The existence of the zero diffusivity limit means that possible violations of the normal scaling in the convective interval may only come from a singularity of the limit $l\to\infty$. In fact, this was already the case for C_2, dominated by the constant term that diverged as l increases. The constant dropped out, however, from the pair structure function (152) that did not depend on l and, consequently, scaled dimensionally. Concerning higher-order scalar structure functions, Kraichnan (1994) was the first to argue in favor of their anomalous scaling. His paper steered a renewed interest in the problem which led to the discovery by Chertkov *et al.*

(1995b), Gawędzki and Kupiainen (1995), and Shraiman and Siggia (1995) of a simple mechanism to avoid normal scaling: the domination of the correlation functions by scaling zero modes of the operators $\widetilde{\mathcal{M}}_N$. For small ξ, Gawędzki and Kupiainen (1995) and Bernard *et al.* (1996) showed that in the convective interval

$$C_N(\underline{\mathbf{r}})=\mathcal{A}_N l^{\Delta_N} f_{N,0}(\underline{\mathbf{r}})+C_N'(\underline{\mathbf{r}})+o(l)+[\cdots]. \qquad (156)$$

Above, $f_{N,0}$ is the irreducible isotropic zero mode of scaling dimension $\zeta_{N,0}=(N/2)(2-\xi)-\Delta_N$, see Sec. II.E.3, the term C_N' is scaling with the normal dimension $(N/2)(2-\xi)$, and $[\cdots]$ stands for reducible contributions depending only on a subset of points. The anomalous corrections $\Delta_N=N(N-2)/2(d+2)\,\xi+\mathcal{O}(\xi^2)$ are positive for small ξ; see Eq. (84). A similar result $\Delta_N=N(N-2)/2d+\mathcal{O}(1/d^2)$ was established by Chertkov *et al.* (1995b) and Chertkov and Falkovich (1996) for large space dimensionalities. For $\Delta_N>0$, the first term on the right-hand side of Eq. (156) is dominating the second one for large l or, equivalently, at short distances for $\kappa=0$. The analytic origin of the zero-mode dominance of the stationary correlation functions (154) lies in the asymptotic short-distance expansion (79) of the kernels of G_N (Bernard *et al.*, 1998). The dominant zero mode $f_{N,0}$ is the irreducible term in Eq. (79) with the lowest scaling dimension. The reducible terms $[\cdots]$ drop out of the correlators of scalar differences, e.g., in the N-point structure functions. The latter are dominated by the contribution from $f_{N,0}$:

$$S_N(r)=\langle[\theta(\mathbf{r})-\theta(\mathbf{0})]^N\rangle\propto l^{\Delta_N}r^{\zeta_N}, \qquad (157)$$

with $\zeta_N=\zeta_{N,0}$. The physical meaning of zero-mode dominance is transparent. Any structure function is a difference between the terms with different number of particles coming at the points 1 and 2 at time t, like, for instance, $S_3=3\langle\theta_1^2\theta_2-\theta_1\theta_2^2\rangle$. Under the (backward-in-time) Lagrangian evolution, this difference decreases as $(r/l)^{\zeta_{N,0}}$ because of shape relaxation with the slowest term due to the irreducible zero mode. The structure function is thus given by the total temporal factor $l^{N(2-\xi)/2}$ multiplied by $(r/l)^{\zeta_{N,0}}$.

The scaling exponents in Eq. (157) are universal in the sense that they do not depend on the shape of the pumping correlation functions $\Phi(r)$. The coefficients $\mathcal{A}_N$ in Eq. (156), as well as the proportionality constants in Eq. (157), are, however, nonuniversal. Numerical analysis, see Sec. III.D.2, indicates that the previous framework applies for all $0<\xi<2$ at any space dimensionality, with the anomalous corrections Δ_N continuing to be strictly positive for $N>2$. That implies the small-scale intermittency of the scalar field: the ratios S_{2n}/S_2^n grow as r decreases. At orders $N\gg(2-\xi)d/\xi$, the scaling exponents ζ_N tend to saturate to a constant, see Secs. III.C.2 and III.D.2 below.

In many practical situations the scalar is forced in an anisotropic way. Shraiman and Siggia (1994, 1995) have proposed a simple way to account for the anisotropy. They subtracted from the scalar field an anisotropic background by defining $\theta'(\mathbf{r})=\theta(\mathbf{r})-\mathbf{g}\cdot\mathbf{r}$, with $\mathbf{g}$ a fixed vector. It follows from the unforced Eq. (92) that

$$\partial_t\theta'+\mathbf{v}\cdot\boldsymbol{\nabla}\theta'-\kappa\boldsymbol{\nabla}^2\theta'=-\mathbf{g}\cdot\mathbf{v}, \qquad (158)$$

with the term on the right-hand side giving the effective pumping. In Kraichnan velocities, the translation invariance of the equal-time correlators of θ' is preserved by the evolution with the Hopf equations taking the form

$$\partial_t C_N(\underline{\mathbf{r}})=\widetilde{\mathcal{M}}_N C_N(\underline{\mathbf{r}})+2\sum_{n<m}C_{N-2}(\mathbf{r}_1,\underset{\hat{n}\ \hat{m}}{\ldots},\mathbf{r}_N)g_ig_jD^{ij}(\mathbf{r}_{nm})-\sum_{n,m}g_id^{ij}(\mathbf{r}_{nm})\nabla_{r_m^j}C_{N-1}(\mathbf{r}_1,\underset{\hat{n}}{\ldots},\mathbf{r}_N) \qquad (159)$$

in the homogeneous sector. The stationary correlation functions of θ' which arise at long times if the initial correlation functions decay in the space variables, may be analyzed as before. In the absence of the $\theta'\mapsto-\theta'$ symmetry, the odd correlators are no longer constrained to be zero. Still, the stationary one-point function vanishes so that the scalar mean is preimposed: $\langle\theta(\mathbf{r})\rangle=\mathbf{g}\cdot\mathbf{r}$. For the two-point function, the solution remains the same as in the isotropic case, with the forcing correlation function simply replaced by $2g_ig_jD^{ij}(\mathbf{r})$ and approximately equal to the constant $2D_0\mathbf{g}^2$ in the convective interval. The three-point function is

$$C_3(\underline{\mathbf{r}})=-\int G_3(\underline{\mathbf{r}},\underline{\mathbf{R}})\times\sum_{n,m}g_id^{ij}(\mathbf{R}_{nm})\nabla_{R_m^j}C_2(\mathbf{R}_1,\underset{\hat{n}}{\ldots},\mathbf{R}_3)d\underline{\mathbf{R}}. \qquad (160)$$

The dimensional scaling would imply that $C_3(\lambda\underline{\mathbf{r}})=\lambda^{3-\xi}C_3(\underline{\mathbf{r}})$ in the convective interval since $\boldsymbol{\nabla}C_2(\mathbf{r})$ scales there as $r^{1-\xi}$. Instead, for ξ close to 2, the three-point function is dominated by the angular momentum $j=1$ zero mode of $\mathcal{M}_3$ with scaling dimension $2+o(2-\xi)$, as shown by Pumir *et al.* (1997). A similar picture arises from the perturbative analysis around $\xi=0$ (Pumir, 1996, 1997), around $d=\infty$ (Gutman and Balkovsky, 1996), and from the numerical study of the whole interval of ξ values for $d=2$ and $d=3$ (Pumir, 1997); see Sec. II.E.5. As will be discussed in Sec. III.F, the zero-mode mechanism is likely to be responsible for the experimentally observed persistence of the anisotropies, see, e.g., Warhaft (2000).

It is instructive to analyze the limiting cases $\xi=0,2$ and $d=\infty$ from the viewpoint of the statistics of the scalar. Since the field at any point is the superposition of contributions brought from d directions, it follows from the central limit theorem that the scalar statistics becomes Gaussian as the space dimensionality d increases. In the case $\xi=0$, an irregular velocity field acts like Brownian motion. The corresponding turbulent transport process is normal diffusion and the Gaussianity of the scalar statistics follows from that of the input. What is general in both the previous limits is that the degree of Gaussianity, as measured, say, by the flatness S_4/S_2^2, is scale independent. Conversely, we have seen in Sec. III.B.1 that $\ln(l/r)$ is the parameter of Gaussianity in the Batchelor limit with the statistics becoming Gaussian at small scales whatever the input statistics. The key here is in the temporal rather than the spatial behavior. Since the stretching in a smooth velocity field is exponential, the cascade time is growing logarithmically as the scale decreases. That leads to the essential difference: at small yet nonzero ξ/d the degree of non-Gaussianity increases downscales, while at small $(2-\xi)$ it first decreases downscales until $\ln(l/r)\approx 1/(2-\xi)$ and then it starts to increase. Note that the interval of decrease grows as the Batchelor regime is approached. Already that simple reasoning suggests that the perturbation theory is singular in the limit $\xi\to 2$, which is formally manifested in the quasisingularities of the many-point correlation functions for collinear geometries (Balkovsky, Chertkov, *et al.*, 1995).

The anomalous exponents determine also the moments of the dissipation field $\epsilon=\kappa(\nabla\theta)^2$. A straightforward analysis of Eq. (154) indicates that $\langle\epsilon^n\rangle=c_n\bar{\epsilon}^n(l/r_d)^{\Delta_{2n}}$ (Chertkov *et al.*, 1995b; Chertkov and Falkovich, 1996), where $\bar{\epsilon}$ is the mean dissipation rate. The dimensionless constants c_n are determined by the fluctuations of the dissipation scale and, most likely, they are of the form n^{qn} with yet unknown q. In the perturbative domain $n\ll(2-\xi)d/\xi$, the anomalies Δ_{2n} are a quadratic function of the order and the corresponding part of the dissipation PDF is close to lognormal (Chertkov and Falkovich, 1996). The form of the distant tails of the PDF is still unknown.

d. Operator product expansion

While the irreducible zero mode dominates the respective structure function, all the zero modes may be naturally incorporated into an operator product expansion (OPE) of the scalar correlation functions. There has been many attempts to use this powerful tool of quantum field theory (Wilson, 1969) in the context of turbulence; see Adzhemyan *et al.* (1989, 1999), Eyink (1993), and Polyakov (1993, 1995). We briefly describe here a general direction for accomplishing that for the problem of scalar advection (Chertkov and Falkovich, 1996; Adzhemyan *et al.*, 1998; Zamolodchikov *et al.*, 2000). Let $\{\mathcal{O}_a\}$ be a set of local observables (which contains all spatial derivatives of any field already included). The existence of OPE presumes that

$$\mathcal{O}_a(\mathbf{r})\mathcal{O}_b(\mathbf{r}')=\sum_k C^c_{ab}(\mathbf{r}-\mathbf{r}')\mathcal{O}_c(\mathbf{r}'), \tag{161}$$

which is understood as the following relations among the correlation functions:

$$\langle\mathcal{O}_a(\mathbf{r})\mathcal{O}_b(\mathbf{r}')\cdots\rangle=\sum_k C^c_{ab}(\mathbf{r}-\mathbf{r}')\langle\mathcal{O}_c(\mathbf{r}')\cdots\rangle. \tag{162}$$

The sum represents the correlation function in the left-hand side if $|\mathbf{r}-\mathbf{r}'|$ is small enough. Renormalization symmetry $\varphi\to\Lambda\varphi$, $\theta\to\Lambda\theta$ allows one to classify the operators (fluctuating fields) by degrees: $\mathcal{O}^{(n)}$ has degree n if $\mathcal{O}^{(n)}\to\Lambda^n\mathcal{O}^{(n)}$ under the transformation. The OPE conserves the degree and is supposed to be scale invariant in the convective interval. This means that one may choose a basis of the observables in such a way that $\mathcal{O}_a$ has "dimension" d_a, and the OPE is invariant under the transformation $\mathcal{O}_a(\mathbf{r})\to\lambda^{d_a}\mathcal{O}_a(\lambda\mathbf{r})$ so that its coefficient functions scale: $C^c_{ab}[\lambda(\mathbf{r}-\mathbf{r}')]=\lambda^{d_c-d_a-d_b}C^c_{ab}(\mathbf{r}-\mathbf{r}')$. Besides, functions C^c_{ab} are supposed to be pumping independent with the whole dependence on pumping carried by the expectation values $\langle\mathcal{O}_c\rangle\propto l^{-d_c}$. The scale invariance can be ("spontaneously") broken at the level of correlation functions if some of the fields with nonzero dimension develop nonzero expectation values. The dimension of θ^N is $N(\xi-2)/2$. The operators can be organized into strings, each with the primary operator Θ_a with the lowest dimension d_a and its descendants with the dimensionalities $d_a+n(2-\xi)$. A natural conjecture is that there is one-to-one correspondence between the primary operators of degree N and the zero modes of $\widetilde{M}_N$. The dimensions of such primaries are minus the anomalous dimensions Δ's of the zero modes f_a and are therefore negative. By fusing $N-1$ times one gets $\theta_1\cdots\theta_N=\Sigma f_a(\mathbf{r}_1,\ldots,\mathbf{r}_N)\Theta_a(\mathbf{r}_N)+\cdots$, where the dots include the derivatives and descendants of Θ_a.

For $n=2$, one has only one primary field θ^2, and its descendents $|\nabla^m\theta|^2$. For $n=4$, there is an infinity of primaries. Only θ^4, ϵ^2, and $\epsilon\theta\nabla^2\theta-d^2\epsilon/(d^2+2)$ have nonzero expectation values. The operators with zero expectation values correspond to the operators with more derivatives than twice the degree (that is with the order of the angular harmonic in the respective zero mode being larger than the number of particles, in terms of Sec. II.E). Building an OPE explicitly and identifying its algebraic nature remains a task for the future.

e. Large scales

The scalar correlation functions at scales larger than that of the pumping decay by power laws. The pair-correlation function is given by Eq. (150). Recall that applying $\mathcal{M}_2$ on it, we obtain a contact term $\propto\delta(\mathbf{r})$. Concerning higher-order correlation functions, straight lines are not preserved in a nonsmooth flow and no strong angular dependencies of the type encountered in the smooth case are thus expected. To determine the scaling behavior of the correlation functions, it is therefore enough to focus on a specific geometry. Consider, for instance, the equation $\widetilde{\mathcal{M}}_4C_4(\underline{\mathbf{r}})=\Sigma\chi(r_{ij})C_2(r_{kl})$ for the fourth-order function. A convenient geometry to analyze is that with one distance among the points, say r_{12}, much smaller than the other r_{1j}, whose typical value is R. At the dominant order in r_{12}/R, the solution of the equation is $C_4\propto C_2(r_{12})C_2(R)\sim(r_{12}R)^{2-\xi-d}$. Similar arguments apply to arbitrary orders. We conclude that the scalar statistics at $r\gg l$ is scale invariant, i.e., $C_{2n}(\lambda\underline{\mathbf{r}})=\lambda^{n(2-\xi-d)}C_{2n}(\underline{\mathbf{r}})$ as $\lambda\to\infty$. Note that the statistics is generally non-Gaussian when the distances between the points are comparable. As ξ increases from

0 to 2, the deviations from the Gaussianity starts from zero and reach their maximum for the smooth case described in Sec. III.B.3.

f. General pumping

The fact that the scalar correlation functions in the convective interval are dominated by zero modes indicates that the hypotheses of Gaussianity and δ correlation of the pumping are not crucial. The purpose of this subsection is to give some more details on how they might be relaxed. The new point to be taken into account is that the pumping has now a finite correlation time τ_p and irreducible contributions are present. The situation with the second order is quite simple. The injection rate of θ^2 is $\langle\phi\theta\rangle$ and its value defines the mean dissipation rate $\bar{\epsilon}$ at the stationary state. The only difference is that its value cannot be estimated *a priori* as $\Phi(0)$. Let us then consider the behavior at higher orders, whose typical example is the fourth. Its general flux relation, derived similarly as Eq. (131), reads

$$\langle(\mathbf{v}_1\cdot\nabla_1+\mathbf{v}_2\cdot\nabla_2)\theta_1^2\theta_2^2\rangle+\kappa\langle\theta_1\theta_2[\nabla_1^2+\nabla_2^2]\theta_1\theta_2\rangle =\langle\varphi_1\theta_1\theta_2^2+\varphi_2\theta_2\theta_1^2\rangle. \tag{163}$$

Taking the limit of coinciding points, we get the production rate of θ^4. It involves the usual reducible contribution $3\bar{\epsilon}C_2(0)$ and an irreducible one. The ratio of the two is estimated as $C_2(0)/\tau_p$. The non-Gaussianity of the pumping is irrelevant as long as τ_p is smaller than the time for the particles to separate from the diffusive to the integral scale. The smooth and nonsmooth cases need to be distinguished. For the former, the separation time is logarithmically large and the previous condition is always satisfied. Indeed, the reducible part of the injection rate in Eq. (163) necessarily contains $2\langle\theta_1\theta_2\rangle[\langle\varphi_1\theta_2\rangle+\langle\varphi_2\theta_1\rangle]\simeq 2\bar{\epsilon}^2\tilde{\lambda}_d^{-1}\ln(l/r_{12})$. Since the correlation function grows as r_{12} decreases, one can always neglect the constant irreducible contribution for small enough separations. Similarly, the input rate of all even moments up to $N\simeq\ln(l/r_d)$ is determined by $\bar{\epsilon}$. The fact that the fluxes of higher integrals are not constant in the convective interval was called the effect of "distributed pumping" in Falkovich (1994) and Falkovich and Lebedev (1994).

In the nonsmooth case, the cascade time is finite and the irreducible contributions might be relevant. They affect the statistics of the scalar yet, of course, not the scaling of the zero modes. The fourth-order correlation function C_4 acquires, for example, extra terms proportional to $\Sigma r_{ij}^{2-\xi}$. They contribute to the fourth-order cumulant but not to the structure function S_4. The existence of those extra terms in the correlation function affects the matching conditions at the pumping scale, though. We conclude that the numerical coefficients $\mathcal{A}_N$ in the structure functions $S_N=\mathcal{A}_N r^{\zeta_N} l^{\Delta_N}$ generally depend on all the irreducible pumping contributions of order $m\leqslant N$.

2. Instanton formalism for the Kraichnan model

Since the perturbative approaches in Sec. II.E.5 are all limited to finite orders, it is natural to look for alternative methods to capture the scaling exponents in the nonperturbative domain $N\xi\gg(2-\xi)d$. As in many other instances in field theory or statistical physics, such a nonperturbative formalism is expected to result from a saddle-point technique applied to the path integral controlling the statistics of the field. Physically, that would correspond to finding some optimal fluctuation responsible for a given structure function. Not any structure function can be found by this approach but only those with $N\gg 1$, related to the PDF tails which are indeed controlled by rare events. This is a general idea of the instanton formalism (see, e.g., Coleman, 1977) adapted for turbulence by Falkovich *et al.* (1996). The case in question is so complicated though that an effective analysis (carried out by Balkovsky and Lebedev, 1998) is possible only with yet another large parameter, $(2-\xi)d\gg 1$, which guarantees that the Lagrangian trajectories are almost deterministic. The relation between $N\xi$ and $(2-\xi)d$ is now arbitrary so one is able to describe both the perturbative and nonperturbative domains. Unfortunately, a straightforward application of this approach to the path integral over the velocity field does not work because of a usual problem in saddle-point calculations: the existence of a soft mode makes the integrand nondecaying in some direction in the functional space. One ought to integrate over the soft mode before the saddle-point approximation is made. Balkovsky and Lebedev identified the soft mode as that responsible for the slow variations of the direction of the main stretching. Since the structure functions are determined only by the modulus of the distance then an effective integration over the soft mode simply corresponds to passing from the velocity to the absolute value of the Lagrangian separation as the integration variable in the path integral. This can be conveniently done by introducing the scalar variable

$$\eta_{12}\equiv(2-\xi)^{-1}\partial_t R_{12}^{2-\xi}=R_{12}^{-\xi}R_{12}^i(v_1^i-v_2^i). \tag{164}$$

For the Kraichnan velocity field, η_{12} has the nonzero mean $\langle\eta_{12}\rangle=-D$ and the variance

$$\langle\langle\eta_{12}(t_1)\eta_{34}(t_2)\rangle\rangle=\frac{2D}{d}q_{12,34}\delta(t_1-t_2). \tag{165}$$

The explicit dependence of the $q_{12,34}$ function on the particle distances will not be needed here and can be found in the original paper (Balkovsky and Lebedev, 1998). Any average over the statistics of the Lagrangian distances can be written in terms of a Martin-Siggia-Rose path integral $\int\mathcal{D}R\,\mathcal{D}m\exp(\imath\mathcal{I}_R)$, with the action

$$\mathcal{I}_R=\int_{-\infty}^0 dt\int d\mathbf{r}_1\,d\mathbf{r}_2\,m_{12}\times\left[\left(\frac{\partial_t R_{12}^{2-\xi}}{2-\xi}+D\right)+\frac{iD}{d}d\mathbf{r}_3\,d\mathbf{r}_4\,q_{12,34}m_{34}\right]. \tag{166}$$

The auxiliary field conjugated to R_{12} is denoted by $m_{12} \equiv m(t,\mathbf{r}_1,\mathbf{r}_2)$. Note that the second (nonlinear) term in Eq. (166) vanishes both as $(2-\xi)d \to \infty$ and $\xi \to 0$. The moments of any linear functional of the scalar $\vartheta = \int d\mathbf{r}\, \beta(\mathbf{r})\theta(\mathbf{r})$ are then expressed as

$$\langle |\vartheta|^N \rangle = \int \frac{dy\, d\vartheta}{2\pi} \int \mathcal{D}R\mathcal{D}m\, e^{i\mathcal{I}_R - \mathcal{F}_\lambda - iy\vartheta + N \ln|\vartheta|}, \tag{167}$$

where $\mathcal{F}_\lambda = (y^2/2) \int dt\, d\mathbf{r}_1\, d\mathbf{r}_2\, \chi(R_{12})\beta(\mathbf{r}_1)\beta(\mathbf{r}_2)$. To obtain the structure functions, one should in principle take for β differences of δ functions. This would, however, bring diffusive effects into the game. To analyze the scaling behavior, it is in fact enough to consider any observable where the reducible components in the correlation functions are filtered out. A convenient choice is $\beta(\mathbf{r}_1) = \delta_\Lambda(\mathbf{r}_1 - \mathbf{r}/2) - \delta_\Lambda(\mathbf{r}_1 + \mathbf{r}/2)$, where the smeared function $\delta_\Lambda(\mathbf{r})$ has a width Λ^{-1} and satisfies the normalization condition $\int d\mathbf{r}\, \delta_\Lambda(\mathbf{r}) = 1$. The diffusive effects may be disregarded provided the width is taken much larger than r_d.

The saddle-point equations for the integral (167) are

$$\partial_t R_{12}^{2-\xi} = -(2-\xi)D\left[1 + \frac{2i}{d}\int d\mathbf{r}_3\, d\mathbf{r}_4\, q_{12,34} m_{34}\right], \tag{168}$$

$$\begin{aligned} -iR_{12}^{1-\xi}\partial_t m_{12} = {} & \frac{2D}{d}\int d\mathbf{r}_3\, d\mathbf{r}_4 \frac{\partial q_{12,34}}{\partial R_{12}} \\ & \times [m_{12}m_{34} + 2m_{13}m_{24}] \\ & + \frac{y^2}{2}\chi'(R_{12})\beta(\mathbf{r}_1)\beta(\mathbf{r}_2), \end{aligned} \tag{169}$$

with the two extremal conditions on the parameters ϑ and y:

$$\vartheta = iy \int dt\, d\mathbf{r}_1\, d\mathbf{r}_2\, \chi(R_{12})\beta(\mathbf{r}_1)\beta(\mathbf{r}_2), \quad iy = N/\vartheta. \tag{170}$$

The two boundary conditions are $R_{12}(t=0) = |\mathbf{r}_1 - \mathbf{r}_2|$ and $m_{12} \to 0$ as $t \to -\infty$. The variables R_{12} and m_{12} are *a priori* two fields, i.e., they depend on both t and $\mathbf{r}$. In fact, the problem can be shown to reduce effectively to two degrees of freedom: R_-, describing the separation of two points, and R_+, describing the spreading of a cloud of size Λ around a single point. It follows from the analysis in Balkovsky and Lebedev (1998) that there are two different regimes, depending on the order of the moments considered. At $N < (2-\xi)d/(2\xi)$ the values of R_+ and R_- are very close during most of the evolution and different fluid particles behave similarly. For higher moments, R_+ and R_- differ substantially throughout the evolution. The fact that different groups of fluid particles move in a very different way might be interpreted as the signature of the strong fronts in the scalar field that are discussed in Sec. III.F. The final result for the scaling exponents is

$$\zeta_N = N(2-\xi)/2 - 2\xi N^2/2d \quad \text{at } N < (2-\xi)d/(2\xi), \tag{171}$$

$$\zeta_N = (2-\xi)^2 d/(8\xi) \quad \text{at } N > (2-\xi)d/(2\xi). \tag{172}$$

These expressions are valid when the fluctuations around instanton give negligible contribution which requires $N \gg 1$ and $(2-\xi)d \gg 1$, while the relation between d and N is arbitrary. The exponents depend quadratically on the order and then saturate to a constant. The saturation for the Kraichnan passive scalar model had been previously inferred from a qualitative argument by Yakhot (1997) and from an upper bound on ζ_N by Chertkov (1997). The relevance of the phenomenon of saturation for generic scalar turbulence is discussed in Sec. III.F.

3. Anomalous scaling for magnetic fields

Magnetic fields transported by a Kraichnan velocity field display anomalous scaling already at the level of the second-order correlation functions. For a scalar, $\langle \theta^2 \rangle$ is conserved and its flow across the scales fixes the dimensional scaling of the covariance found in Sec. III.C.1.b. For a magnetic field, this is not the case. The presence of an anomalous scaling for the covariance $\mathcal{C}_2^{ij}(\mathbf{r},t) = \langle B^i(\mathbf{r},t)B^j(\mathbf{0},t)\rangle$ becomes quite intuitive from the Lagrangian standpoint. We have seen in Sec. II.E that the zero modes are closely related to the geometry of the particle configurations. For a scalar field, the single distance involved in the two-particle separation explains the absence of anomalies at the second order. The magnetic-field Eq. (95) for $\kappa = 0$ is the same as for a tangent vector, i.e., the separation between two infinitesimally close particles. Although $\mathcal{C}_2^{ij}$ involves again two Lagrangian particles, each of them is now carrying its own tangent vector. In other words, we are somehow dealing with a four-particle problem, where the tangent vectors bring the geometric degrees of freedom needed for the appearance of nontrivial zero modes. The compensation mechanism leading to the respective two-particle integral of motion is now due to the interplay between the interparticle distance and the angular correlations of the vectors carried by the particles. The attractive feature of the problem is that the anomaly can be calculated nonperturbatively.

Specifically, consider Eq. (95) for the solenoidal magnetic field $\mathbf{B}(\mathbf{r},t)$ and assume the Kraichnan correlation function (48) for the Gaussian incompressible velocity. We first analyze the isotropic sector (Vergassola, 1996). The first issue to be addressed is the possibility of a stationary state. For this to happen, there should be no dynamo effect, i.e., an initial condition should relax to zero in the absence of injection. Let us show that this is the case for $\xi < 1$ in 3D. As for a scalar, the δ correlation of the velocity leads to a closed equation for the pair-correlation function $\partial_t \mathcal{C}_2^{ij} = \mathcal{M}_{ij}^{kl}\mathcal{C}_2^{kl}$. The isotropy and the solenoidality of the magnetic field permit us to write $\mathcal{C}_2^{ij}(\mathbf{r},t)$ in terms of its trace $H(\mathbf{r},t)$ only. This leads to an imaginary time Schrödinger equation for the "wave function" $\psi(r,t) = (\kappa + D_1 r^\xi)/r \int_0^r H(\rho,t)\rho^2\, d\rho$ (Kazantsev, 1968). The energy eigenstates $\psi(r)e^{-Et}$, into which $\psi(r,t)$ may be decomposed, satisfy the stationary equation

$$\frac{d^2\psi(r)}{dr^2}+m(r)[E-U(r)]\psi(r)=0 \tag{173}$$

of a quantum particle of variable mass $m(r)$ in the potential $U(r)$. The presence of a dynamo effect is equivalent to the existence of negative energy levels. Since the mass is everywhere positive, it is enough to look for bound states in the effective potential $V=mU$. The detailed expressions of the mass and the potential can be found in Vergassola (1996). Here, it is enough to remark that $V(r)$ is repulsive at small scales and has a quadratic decay in the inertial range with a prefactor $2-3/2\xi-3/4\xi^2$. It is known from quantum-mechanics textbooks that the threshold for bound states in an attractive potential $-c/r^2$ is $c=1/4$. The absence of a dynamo effect for $\xi<1$ follows immediately from the expression of the potential prefactor.

The stability just described implies that, in the presence of a forcing term in Eq. (95), the magnetic-field covariance will relax to a time-independent expression at long times. We may then study the spatial scaling properties at the stationary state. The energy E in Eq. (173) should be set to zero and we should add the corresponding forcing term in the first equation. Its precise form is not important here as any anomalous scaling is known to come from the zero modes of the operator $M=d^2/dr^2-V(r)$. The behavior $-c/r^2$ of the potential in the inertial range implies that the operator is scale invariant and its two zero modes behave as power laws with exponents $1/2[1\pm\sqrt{1-4c}]$. The zero mode with the smaller exponent is not acceptable due to a singular behavior in the dissipation range of the corresponding correlation function. The remaining zero mode dominates the inertial-range behavior $H(r)\propto r^{\gamma_2}$ with

$$\gamma_2=-(3+\xi)/2+3/2\sqrt{1-\xi(\xi+2)/3}. \tag{174}$$

Note that dimensional arguments based on a constant flux of A^2 (with the vector potential defined by $\mathbf{B}=\nabla\times\mathbf{A}$) would give $\gamma_2=-\xi$. This is indeed the result in the limits of small ξ and large space dimensionality and for the 2D case. For the latter, the vector potential is reduced indeed to a scalar whose equation coincides with the advection-diffusion Eq. (92). Note that $\gamma_2\leqslant-\xi$, that is the zero mode provides for correlation functions that are $(l/r)^{-\gamma_2-\xi}$ times larger than what dimensional arguments would suggest.

The zero mode dominating the stationary two-point function of $\mathbf{B}$ is preserved by the unforced evolution. In other words, if $\mathcal{C}_2^{ij}(\mathbf{r},0)$ is taken as the zero mode of $\mathcal{M}$ then the correlation function does not change with time. A counterpart to that is the existence of a statistical Lagrangian invariant that contains both the distance between the fluid particles and the values of the fields: $I(t)=\langle B^k(\mathbf{R}_1)B^l(\mathbf{R}_2)Z^+_{kl}(\mathbf{R}_{12})\rangle$ (Celani and Mazzino, 2000). Here Z^+_{kl} is a zero mode of the operator adjoint to $\mathcal{M}$. The scaling dimension of such zero mode is $\gamma_2+2>0$. The appearance of the adjoint operator has a simple physical reason. To calculate the correlation functions of the magnetic field, the tangent vectors attached to the particles evolve forward in time while, as for the scalar, their trajectories should be traced backward in time. Adjoint objects naturally appear when we look for invariants where all the quantities run in the same direction of time. The conservation of $I(t)$ is due to the power-law growth of Z^+ with time being offset by the decorrelation between the directions of the $\mathbf{B}$ vectors along the separating trajectories.

Up to now, we have been considering the covariance in the isotropic sector. The scaling exponents in the nonisotropic sectors can also be calculated nonperturbatively (Lanotte and Mazzino, 1999; Arad, Biferale, and Procaccia, 2000). The problem is analogous to that for the scalar in Sec. II.E.5, solved by the expression (88). Here, the calculation is more involved since in each sector j,m of SO(3) there are nine independent tensors into which $\mathcal{C}_2^{ij}$ may be decomposed. Their explicit expression may be found in (Arad *et al.*, 1999). As in the isotropic case, it is shown that no dynamo takes place for $\xi<1$ and the scaling properties at the stationary state can then be calculated. For odd j, one has, for example,

$$\gamma_2^j=-(3+\xi)/2+1/2\sqrt{1-10\xi+\xi^2+2j(j+1)(\xi+2)}. \tag{175}$$

The expression for the other sectors can be found in Lanotte and Mazzino (1999) and Arad, Biferale, and Procaccia (2000). An important point (we shall come back to it in Sec. III.F) is that the exponents increase with j: the more anisotropic the contribution, the faster it decays going toward the small scales.

Higher-order correlation functions of the magnetic field also obey closed equations. Their analysis proceeds along the same lines as for the scalar, with the additional difficulty of the tensorial structure. The extra terms in the equations for the correlation functions of the magnetic field come from the presence of the stretching term $\mathbf{B}\cdot\nabla\mathbf{v}$. Since the gradient of the velocity appears, they are all proportional to ξ. On the other hand, the stabilizing effective-diffusivity terms do not vanish as $\xi\to0$. We conclude that the forced correlation functions at any arbitrary yet finite order will relax to time-independent expressions for sufficiently small ξ. It makes then sense to consider their scaling properties at the stationary state, as it was done in (Adzhemyan and Antonov, 1998). For the correlation functions $\langle[\mathbf{B}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{0},t)]^N\rangle\propto r^{\gamma_N}$, the perturbative expression in ξ reads $\gamma_N=-N\xi-2N(N-1)\xi/(d+2)+\mathcal{O}(\xi^2)$, demonstrating the intermittency of the magnetic-field distribution.

Let us conclude by discussing the behavior of the magnetic helicity $\langle\mathbf{A}\cdot\mathbf{B}\rangle$, considered in Borue and Yakhot (1996). This quantity is conserved in the absence of the molecular diffusivity, as can be easily verified using Eq. (95) and the equation for $\mathbf{A}$:

$$\partial_t\mathbf{A}=\mathbf{v}\times\mathbf{B}-\nabla\phi+\kappa\nabla^2\mathbf{A}. \tag{176}$$

The function ϕ may be fixed by the choice of a specific gauge, e.g., $\nabla\cdot\mathbf{A}=0$. The spatial behavior of the helicity correlation functions $\langle\mathbf{A}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{0},t)\rangle$ is derived using Eqs. (95) and (176) and averaging over the Gaussian velocity with the variance (48). The resulting equation coincides with that for the scalar covariance (145), implying the dimensional scaling $r^{2-\xi}$ and a constant helicity flux. We conclude that it is possible to have coexistence of normal and anomalous scaling for different components of the correlation tensor of a given order. Note also that the helicity correlation functions relax to a stationary form even for $\xi>1$, i.e., when the magnetic correlation functions do not. The increase of the magnetic-field magnitude is indeed accompanied by a modification of its orientation and the quasi-orthogonality between **A** and **B** ensures the stationarity of the helicity correlation functions. For a helical velocity, considered in Rogachevskii and Kleeorin (1999), the magnetic and the helicity correlators are coupled via the so-called α effect, see, e.g., Moffatt (1978), and the system is unstable in the limit $\kappa\to 0$ considered here.

D. Lagrangian numerics

The basic idea of the Lagrangian numerical strategy is to calculate the scalar correlation functions using the particle trajectories. The expressions (101) and (134) naturally provide for such a Lagrangian Monte Carlo formulation: the N Lagrangian trajectories are generated by integrating the stochastic Eqs. (5), the right-hand side of Eqs. (101) and (134) is calculated for a large ensemble of realizations and averaged over it. If we are interested in correlation functions of finite order, the Lagrangian procedure involves the integration of a few differential equations. This is clearly more convenient than having to deal with the partial differential equation for the scalar field. The drawback is that quantities involving a large number of particles, such as the tails of the PDF's, are not accessible. Once the correlation functions have been measured, their appropriate combinations will give the structure functions. For the second-order $S_2(r,t)$ two different configurations of particles are needed. One corresponding to $\langle\theta^2\rangle$, where the particles are at the same point at time t, and another one corresponding to $\langle\theta(\mathbf{r})\theta(0)\rangle$, where they are spaced by $\mathbf{r}$. For the $2n$th-order structure function, $n+1$ particle configurations are needed.

Another advantage of the Lagrangian method is that it gives direct access to the scaling in l of the structure functions (157), that is, to the anomalous dimensions $\Delta_N=N\zeta_2/2-\zeta_N$. The quantities (101) and (134) for various l's can indeed be calculated along the same Lagrangian trajectories. That is more efficient than measuring the scaling in r, i.e., changing the final positions of the particles and generating a new ensemble of trajectories.

1. Numerical method

The Lagrangian method as presented up to now might be applied to any velocity field. The situation with the Kraichnan model is simpler in two respects. First, the velocity statistics is time reversible and the Lagrangian trajectories can be generated forward in time. Second, the velocity fields at different times are independent. Only $(N-1)d$ random variables are needed at each time step, corresponding to the velocity increments at the location of the N particles. The major advantage is that there is no need to generate the whole velocity field. Finite-size effects, such as the space periodicity for pseudospectral methods, are thus avoided.

The Lagrangian trajectories for the Kraichnan model are conveniently generated as follows. The relevant variables are the interparticle separations, e.g., $\mathbf{R}_{nN}=\mathbf{R}_n-\mathbf{R}_N$ for $n=1,\ldots,N-1$. Their equations of motion are easily derived from Eq. (5) and conveniently discretized by the standard Euler-Itô scheme of order 1/2 (Kloeden and Platen, 1992),

$$\mathbf{R}_{nN}(t+\Delta t)-\mathbf{R}_{nN}(t)=\sqrt{\Delta t}(\mathbf{V}_n+\sqrt{2\kappa}\mathbf{W}_n), \tag{177}$$

where Δt is the time step. The quantities $\mathbf{V}_n$ and $\mathbf{W}_n$ are d-dimensional Gaussian, independent random vectors generated at each time step. Both have zero mean and their covariance matrices follow directly from the definition (48) of the Kraichnan velocity correlation:

$$\begin{aligned}\langle V_n^i V_m^j\rangle &= d^{ij}(\mathbf{R}_{nN})+d^{ij}(\mathbf{R}_{mN})-d^{ij}(\mathbf{R}_{nN}-\mathbf{R}_{mN}),\\ \langle W_n^i W_m^j\rangle &= (1+\delta_{nm})\delta^{ij}.\end{aligned} \tag{178}$$

The most convenient numerical procedure to generate the two sets of vectors is the classical Cholesky decomposition method (Ralston and Rabinowitz, 1978). The covariance matrices are triangularized in the form MM^T and the lower triangular matrix M is then multiplied by a set of $(N-1)d$ Gaussian, independent random variables with zero mean and unit variance. The resulting vectors have the appropriate correlations.

Various possibilities to extract the anomalous scaling exponents are available for the Kraichnan model. The straightforward one, used in Frisch *et al.* (1998), is to take the forcing correlation close to a step function (equal to unity for $r/l<1$ and vanishing otherwise). The correlation functions (134) involve then the products of the average residence times of couples of particles at distances smaller than l. An alternative method is based on the shape dynamics discussed in Sec. II.E.4. Measuring first-exit times and not residence times gives an obvious advantage in computational time. As stressed in Gat *et al.* (1998), the numerical problem here is to measure reliably the contributions of the irreducible zero modes, masked by the fluctuations of the reducible ones. The latter were filtered out by Celani *et al.* (1999) taking various initial conditions and combining appropriately the corresponding first-exit times. A relevant combination for the fourth order is for example $\mathcal{T}_l(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})-4\mathcal{T}_l(\mathbf{r}_0,\mathbf{0},\mathbf{0},\mathbf{0})+3\mathcal{T}_l(\mathbf{r}_0,\mathbf{r}_0,\mathbf{0},\mathbf{0})$, where $\mathcal{T}_l$ is the first time the size of the particle configuration reaches l.

2. Numerical results

We shall now present the results for the Kraichnan model obtained by the Lagrangian numerical methods just discussed.

The fourth-order anomaly $2\zeta_2-\zeta_4$ vs the exponent ξ of the velocity field is shown in Fig. 5 for both 2D and 3D (Frisch *et al.*, 1998, 1999). A few remarks are in order. First, the comparison between the 3D curve and the prediction $4\xi/5$ for small ξ's provides direct support

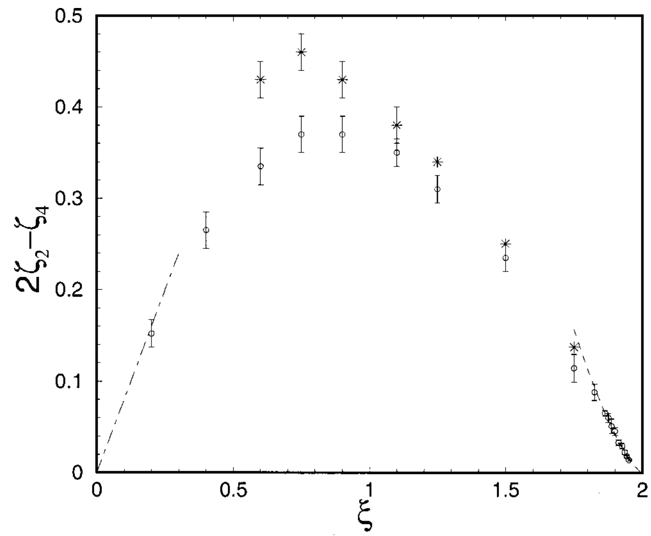

FIG. 5. The fourth-order anomalous exponent $2\zeta_2-\zeta_4$ of the scalar field vs the roughness parameter ξ of the velocity field in the Kraichnan model. The circles and the stars refer to the three-dimensional and the two-dimensional cases, respectively. The dashed lines are the perturbative predictions for small ξ and $2-\xi$ in 3D.

for the perturbation theory discussed in Sec. II.E.5. Similar numerical support for the expansion in $1/d$ has been obtained in Mazzino and Muratore-Ginanneschi (2001). Second, the curve close to $\xi=2$ is fitted with reasonable accuracy by $a(2-\xi)+b(2-\xi)^{3/2}$ for $a=0.06$ and $b=1.13$. That is compatible with an expansion in powers of $(2-\xi)^{1/2}$ (Shraiman and Siggia, 1996), where the first term is ruled out by the Schwartz inequality $\zeta_4\leqslant 2\zeta_2=2(2-\xi)$. Third, remark that the anomalies are stronger in 2D than in 3D and their maximum shifts towards smaller ξ as the dimension decreases. The former remark is in agreement with the general idea which emerged in previous sections that intermittency is associated with the particles staying close to each other for times longer than expected. It is indeed physically quite sensible that those processes are favored by lowering the space dimensionality. The second remark can be qualitatively interpreted as follows (Frisch *et al.*, 1998). Consider scalar fluctuations at a given scale. The smaller-scale components of the velocity act like an effective diffusivity whilst its larger-scale components affect the scalar as in the Batchelor regime. Neither of them leads to any anomalous scaling of the scalar. Those nonlocal interactions are dominant as $\xi\to 0$ and $\xi\to 2$, respectively. For intermediate values of ξ the velocity components having a scale comparable to that of the scalar fluctuations become important and intermittency is produced. The strongest anomalies are attained when the relevant interactions are mostly local. To qualitatively explain how the maximum of the anomalies moves with the space dimensionality, it is then enough to note that the effective diffusivity increases with d but not the large-scale stretching. As for the dependence on the order of the moments, the maximum moves toward smaller ξ as N increases, see the 3D curves for the sixth-order anomaly $3\zeta_2-\zeta_6$ (Mazzino and Muratore-Ginanneschi, 2001). It is indeed natural that higher moments are more sensitive to multiplicative effects due to large-scale stretching than to additive effects of small-scale eddy diffusivity.

Let us now discuss the phenomenon of saturation, i.e., the fact that ζ_N tend to a constant at large N. The orders where saturation is taking place are expected to increase with ξ and diverge as $\xi\to 0$. It is then convenient to consider small values of $2-\xi$. On the other hand, approaching the Batchelor limit too closely makes nonlocal effects important and the range of scales needed for reliable measurements becomes huge. A convenient tradeoff is that considered in Celani *et al.* (2000) with the 3D cases $2-\xi=0.125$, 0.16, and 0.25. For the first value of ξ it is found there that the fourth- and the sixth-order exponents are the same within the error bars. The curves for the other ξ values show that the order of saturation increases with $2-\xi$, as expected. How do those data constrain the ζ_N curve? It follows from the Hölder inequalities that the curve for $N>6$ must lie below the straight line joining ζ_4 and ζ_6. Furthermore, from the results in Sec. III.A.3 we know that the spatial scaling exponents in the forced and the decaying cases are the same and independent of the scalar initial conditions. For the unforced Eq. (92), the maximum value of θ cannot increase with time. Taking an initial condition with a finite maximum value, we have the inequality $S_N(r,t)\leqslant(2\max\theta)^p S_{N-p}(r,t)$. We conclude that the ζ_N curve cannot decrease with the order. The presence of error bars makes it, of course, impossible to state rigorously that the ζ_N curves tend to a constant. It is, however, clear that the combination of the numerical data in Celani *et al.* (2000) and the theoretical arguments discussed in Sec. III.C.2 leaves little doubt about the saturation effect in the Kraichnan model. The situation with an arbitrary velocity field is the subject of Sec. III.F.

E. Inverse cascade in the compressible Kraichnan model

The uniqueness of the Lagrangian trajectories discussed in Sec. II.D for the strongly compressible Kraichnan model has its counterpart in an inverse cascade of the scalar field, that is, in the appearance of correlations at larger and larger scales. Moreover, the absence of dissipative anomaly allows us to calculate analytically the statistics of scalar increments and to show that intermittency is suppressed in the inverse cascade regime. In other words, the scalar increment PDF tends at long times to a scale-invariant form.

Let us first discuss the simple physical reasons for those results. The absence of a dissipative anomaly is an immediate consequence of the expression (101) for the scalar correlation functions. If the trajectories are unique, particles that start from the same point will remain together throughout the evolution and all the moments $\langle\theta^N\rangle(t)$ are preserved. Note that the conservation laws are statistical: the moments are not dynamically conserved in every realization, but their average over the velocity ensemble are.

In the presence of pumping, fluctuations are injected and a flux of scalar variance toward the large scales is established. As explained in Sec. III.B.3, scalar correlation functions at very large scales are related to the probability for initially distant particles to come close. In strongly compressible flow, the trajectories are typically contracting, the particles tend to approach and the distances will reduce to the forcing correlation length l (and smaller) for long enough times. Strong correlations at larger and larger scales are therefore established as time increases, signaling the inverse cascade process.

The absence of intermittency is due to the fact that the Nth-order structure function is dynamically related to a two-particle process. Correlation functions of the Nth order are generally determined by the evolution of N-particle configurations. However, the structure functions involve initial configurations with just two groups of particles separated by a distance r. The particles explosively separate in the incompressible case and we are immediately back to an N-particle problem. Conversely, the particles that are initially in the same group remain together if the trajectories are unique. The only relevant degrees of freedom are then given by the intergroup separation and we are reduced to a two-particle dynamics. It is therefore not surprising that the scaling behaviors at the various orders are simply related in the inverse cascade regime.

Specifically, let us consider the equations for the single-point moments $\langle \theta^N \rangle(t)$. Since the moments are conserved by the advective term, see Eq. (101), their behavior in the limit $\kappa \to 0$ (nonsingular now) is the same as for the equation $\partial_t \theta = \varphi$. It follows that the single-point statistics is Gaussian, with $\langle \theta^2 \rangle$ coinciding with the total injection $\Phi(0)t$ by the forcing.

The equation for the structure functions $S_N(r,t)$ is derived from Eq. (143). That was impossible in the incompressible case since the diffusive terms could not be expressed in terms of S_N (another sign of the dissipative anomaly). No such anomaly exists here so we can disregard the diffusion term and simply derive the equation for S_N from Eq. (143). This is the central technical point allowing for the analytical solution. The equations at the various orders are recast in a compact form via the generating function $Z(\lambda;r,t) = \langle e^{i\lambda \Delta_r \theta} \rangle$ for the scalar increments $\Delta_r \theta = \theta(\mathbf{r},t) - \theta(\mathbf{0},t)$. The equation for the generating function is

$$\partial_t Z(\lambda;\mathbf{r},t) = M Z(\lambda;\mathbf{r},t) + \lambda^2 [\Phi(0) - \Phi(r/l)] Z(\lambda;\mathbf{r},t), \tag{179}$$

where the operator M was defined in Eq. (62) and $Z = 1$ at the initial time. Note that M is the restriction of $\mathcal{M}_2$, signaling the two-particle nature of the dynamics at any order. The stationary solution for Z depends on the pumping, but two different regions with a universal behavior can be identified.

1. Large scales $r \gg l$

In this region, $\Phi(r)$ in Eq. (179) can be neglected and the generating function is an eigenfunction of M with eigenvalue $\lambda^2 \Phi(0)$. Introducing a new variable $r^{(2-\xi)/2}$, Eq. (179) is transformed into a Bessel form and solved analytically. The solution takes a scale-invariant form $Z(|\lambda| r^{(2-\xi)/2})$ whose detailed expression may be found in Gawędzki and Vergassola (2000). It follows that $S_2(r) \propto r^{2-\xi}$. The associated scalar flux is calculated as in Eq. (148) and turns out to be constant in the upscale direction, the footprint of the inverse cascade. The scale-invariant form of the generating function signals that no anomalous scaling is present in the inverse cascade regime. As we shall discuss in Sec. IV.B.2, the phenomenon is not accidental and other physical systems with inverse cascades share the same property. Note that, despite its scale invariance, the statistics of the scalar field is strongly non-Gaussian. The expression for the scalar increment PDF is obtained by the Fourier transform of the generating function. The tails of the PDF decay algebraically with the power $-(2b+1)$, where $b = 1 + (\gamma - d)/(2-\xi)$ and γ was defined in Eq. (62). The slow decay of the PDF renders moments of order $N > 2b$ divergent (as $t^{N/2-b} r^{b(2-\xi)}$) when time increases. A special case is that of smooth velocities $\xi = 2$, considered in Chertkov, Falkovich, and Kolokolov (1998). The PDF of scalar increments reduces then to the same form as in the direct cascade at small scales. For amplitudes smaller than $\ln(r/l)$ the PDF is Gaussian. The reason is the same as in Sec. III.B.2: the time to reach the integral scale l from an initial distance $r \gg l$ is typically proportional to $\ln(r/l)$; the fluctuations of the travel time are Gaussian. For larger amplitudes, the PDF has an exponential tail whose exponent depends on the whole hierarchy of the Lyapunov exponents, as in the smooth incompressible case.

2. Small scales $r \ll l$

Contrary to large scales, the scalar increments are now strongly intermittent. The structure functions of integer orders are all dominated by the zero mode of the M operator scaling as $r^{b(2-\xi)}$, with b defined in the previous paragraph. The exponent of the constant flux solution $\propto r^{2-\xi}$ crosses that of the zero mode at the threshold of compressibility $b=1$ for the inverse cascade.

Let us now consider the role of an infrared cutoff in the inverse cascade dynamics. The natural motivation is the quasistationarity of the statistics: due to the excitation of larger and larger scales, some observables do not reach a stationary form. It is therefore of interest to analyze the effects of physical processes, such as friction, acting at very large scales. The corresponding equation of motion is

$$\partial_t \theta + \mathbf{v} \cdot \boldsymbol{\nabla} \theta + \alpha \theta - \kappa \boldsymbol{\nabla}^2 \theta = \varphi, \tag{180}$$

and we are interested in the limit $\alpha \to 0$. For nonsmooth velocities, the friction and the advection balance at a scale $\eta_f \sim \alpha^{-1/(2-\xi)}$, much larger than l as $\alpha \to 0$. The smooth case $\xi = 2$ is special as no such scale separation is present and it will be considered at the end of the section. For nonsmooth flows, the energy is injected at the integral scale l, transferred upwards in the inertial range and finally extracted by friction at the scale η_f.

The friction term in Eq. (180) is taken into account by noting that the field $\exp(\alpha t)\theta$ satisfies the usual passive scalar equation with a forcing $\exp(\alpha t)f$. It follows that the previous Lagrangian formulas can be carried over by just introducing the appropriate weights. The expression (154) for the N-point correlation function becomes, for instance,

$$C_N(\mathbf{r},t)=\int_0^t ds\int e^{-(t-s)N\alpha}P_N^{t,s}(\mathbf{r};\mathbf{r}')\times\sum_{n<m} C_{N-2}(\mathbf{r}'_1,\ldots,\underset{\hat{n}\hat{m}}{\ldots},\mathbf{r}'_N;s)\Phi(|\mathbf{r}'_n-\mathbf{r}'_m|/l)d\mathbf{r}'. \tag{181}$$

From Eq. (181), one can derive the equations for $\langle\theta^N\rangle(t)$ and the structure functions $S_N(r,t)$ and analyze them (Gawędzki and Vergassola, 2000). The single-point moments are finite and the scalar distribution is Gaussian with $\langle\theta^2\rangle=\Phi(0)/2\alpha$. The structure functions of order $N<2b$ are not affected by friction. The orders that were previously diverging are now finite and they all scale as $r^{b(2-\xi)}$ in the inertial range, with a logarithmic correction for $N=2b$. The algebraic tails that existed without friction are replaced by an exponential fall off for amplitudes larger than $\sqrt{\Phi(0)/\alpha}$. The saturation of the exponents comes from the fact that the moments of order $N/2\geqslant b$ are all dominated by the contribution near the cut.

Let us conclude by considering the smooth case $\xi=2$, where the velocity increments scale linearly with the distance. The advective term $\mathbf{v}\cdot\boldsymbol{\nabla}$ has zero dimension, like the friction term. As first noted by Chertkov (1998, 1999), this naturally leads to an anomalous scaling and intermittency. Let us consider, for example, the second-order correlation function $C_2(r,t)=\langle\theta(\mathbf{r},t)\theta(\mathbf{0},t)\rangle$. Its governing equation is derived from Eq. (181):

$$\partial_t C_2=MC_2+\Phi(r)C_2-2\alpha C_2, \tag{182}$$

with the same M operator as in Eq. (179). At large scales $r\gg l$, the forcing term is negligible and we look for a stationary solution. Its nontrivial decay $C_2\propto(r/l)^{-\Delta_2}$ is due to the zero mode arising from the balance between the M operator and the friction term:

$$\Delta_2=\frac{1}{2}\left[\sqrt{(\gamma-d)^2+\frac{8\alpha}{(d-1)(1+2\wp)}}-(\gamma-d)\right], \tag{183}$$

where γ is as in Eq. (62) and $\wp$ is the compressibility degree of the velocity. The notation Δ_2 is meant to stress that dimensional arguments would predict an exponent zero. Higher-order connected correlation functions also exhibit anomalous decay laws. Similar mechanisms for anomalous scaling and intermittency for the 2D direct enstrophy cascade in the presence of friction are discussed in Sec. IV.B.1.

F. Lessons for general scalar turbulence

The results for spatially nonsmooth flows have mostly been derived within the framework of the Kraichnan model. Both the forcing and the velocity were Gaussian and short correlated in time. As discussed in Sec. III.C.1, the conditions on the forcing are not crucial and may be easily relaxed. The scaling properties of the scalar correlation functions are universal with respect to the forcing, i.e., independent of its details, while the constant prefactors are not. The situation with the velocity field is more interesting and nontrivial. Even though a short-correlated flow might in principle be produced by an appropriate forcing, all the cases of physical interest have a finite correlation time. The very existence of closed equations of motion for the particle propagators, which we heavily relied upon, is then lost. It is therefore natural to ask about the lessons drawn from the Kraichnan model for scalar turbulence in the generic situation of finite-correlated flows. The existing numerical evidence is that the basic mechanisms for scalar intermittency are quite robust: anomalous scaling is still associated with statistically conserved quantities and the expansion (75) for the multiparticle propagator seems to carry over. The specific statistics of the advecting flow affects only quantitative details, such as the numerical values of the exponents. The general consequences for the universality of the scalar statistics and the decay of the anisotropies are presented in what follows. We also show that the phenomenon of saturation, discussed in Secs. III.C.2 and III.D.2 for the Kraichnan model, is quite general.

A convenient choice of the velocity $\mathbf{v}$ to investigate the previous issues is a 2D flow generated by an inverse energy cascade (Kraichnan, 1967). The flow is isotropic, it has a constant upscale energy flux and is scale invariant with exponent 1/3, i.e., without intermittency corrections as shown both by experiments and numerical simulations (Smith and Yakhot, 1993; Paret and Tabeling, 1998; Boffetta *et al.*, 2000). The inverse cascade flow is thus akin to the Kraichnan ensemble, but its correlation times are finite.

Let us first discuss the preserved Lagrangian structures. The simplest nontrivial case to analyze anomalous scaling is the third-order correlation function $C_3(\mathbf{r})=\langle\theta(\mathbf{r}_1,t)\theta(\mathbf{r}_2,t)\theta(\mathbf{r}_3,t)\rangle$. The function is nonzero only if the symmetry $\theta\mapsto-\theta$ is broken, which often happens in real systems via the presence of a mean scalar gradient $\langle\theta\rangle=\mathbf{g}\cdot\mathbf{r}$. The function $C_3(\underline{\mathbf{r}})$ depends then on the size, the orientation with respect to $\hat{\mathbf{g}}$ and the shape of the triangle defined by $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. For a scalar field advected by the 2D in-verse energy cascade flow, the dependence on the size R of the triangle is a power law with anomalous exponent $\zeta_3=1.25$, smaller than the dimensional prediction 5/3 (Celani *et al.*, 2000). To look for statistical invariants under the Lagrangian dynamics, let us take a translation-invariant function $f(\underline{\mathbf{r}})$ of the N points $\mathbf{r}_n$ and define its Lagrangian average as in Eq. (72), i.e., as the average of the function calculated along the Lagrangian trajectories. In the 2D inverse energy cascade, the distances grow as $|t|^{3/2}$ and the Lagrangian average of a scaling function of positive degree σ is expected to grow as $|t|^{3\sigma/2}$. The numerical evidence presented in Celani and Vergassola (2001) is that the anomalous scaling is again due to statistical integrals of motion: the Lagrangian average of the anomalous part of the correlation functions remains constant in time. The shape of the figures identified by the tracer particles plays again a crucial role: the growth $\propto|t|^{3\zeta_3/2}$ of the size factor R^{ζ_3} in $C_3(\mathbf{r})$ is compensated by its shape dependence. As we indicated in Sec. II.E.4, anomalous scaling reflects slowed-down separations among subgroups of particles and the fact that triangles with large aspect ratios live much longer than expected. It is also immediate to provide an example of slow mode, see Eq. (75), for the two-particle dynamics. The Lagrangian average of $(\mathbf{g}\cdot\mathbf{r}_{12})$ is obviously preserved as its time derivative is proportional to $\langle(\mathbf{v}_1-\mathbf{v}_2)\rangle=0$. Its first slow mode is given by $(\mathbf{g}\cdot\mathbf{r}_{12})r_{12}^{2/3}$, whose Lagrangian average is found to grow as $|t|$ (that is much slower than $|t|^{5/2}$) at large times. In the presence of a finite volume and boundaries, the statistical conservation laws hold as intermediate asymptotic behaviors, as explicitly shown in Arad *et al.* (2001).

An important consequence is about the decay rate of the anisotropies. As already mentioned, isotropy is usually broken by the large-scale injection mechanisms. One of the assumptions in the Oboukhov-Corrsin reformulation of the Kolmogorov 1941 theory for the passive scalar is that the statistical isotropy of the scalar is restored at sufficiently small scales. Experiments do not confirm those expectations. Consider, for example, the case where a mean scalar gradient $\mathbf{g}$ is present. A quantitative measure of the degree of anisotropy is provided by odd-order structure functions or by odd-order moments of $\hat{\mathbf{g}}\cdot\nabla\theta$. All these quantities are identically zero for isotropic fields. The predictions of the Oboukhov-Corrsin theory for the anisotropic situations are the following. The hyperskewness of the scalar increments $S_{2n+1}(r)/S_2^{n+1/2}(r)$ should decay with the separation as $r^{2/3}$. The corresponding behavior of the scalar gradient hyperskewness with respect to the Péclet number should be $\mathrm{Pe}^{-1/2}$. In fact, the previous quantities are experimentally found to remain constant or even to increase with the relevant parameter (Gibson *et al.*, 1977; Mestayer, 1982; Sreenivasan, 1991; Mydlarski and Warhaft, 1998). There is therefore no restoration of isotropy in the original Kolmogorov sense and the issue of the role of anisotropies in the small-scale scalar statistics is naturally raised (Sreenivasan, 1991; Warhaft, 2000). The analysis of the same problem in the Kraichnan model is illuminating and permits to clarify the issue in terms of zero modes and their scaling exponents. For isotropic velocity fields, the correlation functions may be decomposed according to their angular momentum j, as in Sec. II.E.5. Each of those contributions is characterized by a scaling exponent ζ_N^j. The general expectation, confirmed in all the situations where the explicit calculations could be performed, is that $\zeta_N^{j\neq0}>\zeta_N^{j=0}$ and that the exponents increase with j. As their degree of anisotropy increases, the contributions are less and less relevant at small scales. Note that, in the presence of intermittency, the inequality $\zeta_N^{j\neq0}<(N/2)\,\zeta_2$ is still possible and anisotropies might then have dramatic effects for observables whose isotropic dominant contribution is vanishing, such as $S_{2n+1}/S_2^{n+1/2}$. Rather than tending to zero, they may well increase while going toward the small scales, blatantly violating the restoration of isotropy in the original Kolmogorov sense. Remark that no violation of the hierarchy in j is implied though. In other words, the degree of anisotropy of every given moment does not increase as the scale decreases; if, however, one measures odd moments in terms of the appropriate power of the second one (as is customary in phenomenological approaches) then the degree of anisotropy may grow downscales. The previous arguments are quite general and compatible with all the existing experimental and numerical data for passive scalar turbulence. For Navier-Stokes turbulence, the use of the rotation symmetries and the existence of a hierarchy among the anisotropic exponents were put forward and exploited in Arad *et al.* (1998, 1999). Numerical evidence for persistence of anisotropies analogous to those of the scalar fields was first presented in Pumir and Shraiman (1995). Kraichnan model with an anisotropic velocity statistics has been considered by Adzhemyan, Antonov, Hnatich, and Novikov (2001).

Let us now discuss the phenomenon of saturation. A snapshot of the scalar field advected by the 2D inverse cascade flow is shown in Fig. 6. A clear feature is that strong scalar gradients tend to concentrate in sharp fronts separated by large regions where the variations are weak. The scalar jumps across the fronts are of the order of $\theta_{rms}=\sqrt{\langle\theta^2\rangle}$, i.e., comparable to the largest values of the field itself. Furthermore, the minimal width of the fronts reduces with the dissipation scale, pointing to their quasidiscontinuous nature. If the probability of having such θ_{rms} jumps across a separation r decreases as r^{ζ_∞}, then phenomenological arguments of the multifractal type suggest a saturation to the asymptotic value ζ_∞; see Frisch (1995). The presence of fronts in scalar turbulence is a very well established fact, both in experiments (Gibson *et al.*, 1977; Mestayer, 1982; Sreenivasan, 1991; Mydlarski and Warhaft, 1998) and in numerical simulations (Holzer and Siggia, 1994; Pumir, 1994). It is shown in the latter work that fronts are formed in the hyperbolic regions of the flow, where distant particles are brought close to each other. The other important remark is that fronts appear also in the Kraichnan model (Fairhall *et al.*, 1997; Vergassola and Mazzino, 1997; Chen and Kraichnan, 1998), despite the δ correlation of the velocity. What matters for bringing distant particles close to each other are indeed the effects cumulated in time. The integral of a δ-correlated random process behaves as a Brownian motion in time, whose sign is known to have strong persistence properties (Feller, 1950). Even a δ-correlated flow might then efficiently compress the particles (locally) and this naturally explains how fronts may be formed in the Kraichnan model. It is also clear from the previous arguments that a finite correlation time favors the formation of fronts and that the Kraichnan model is somehow the most un-

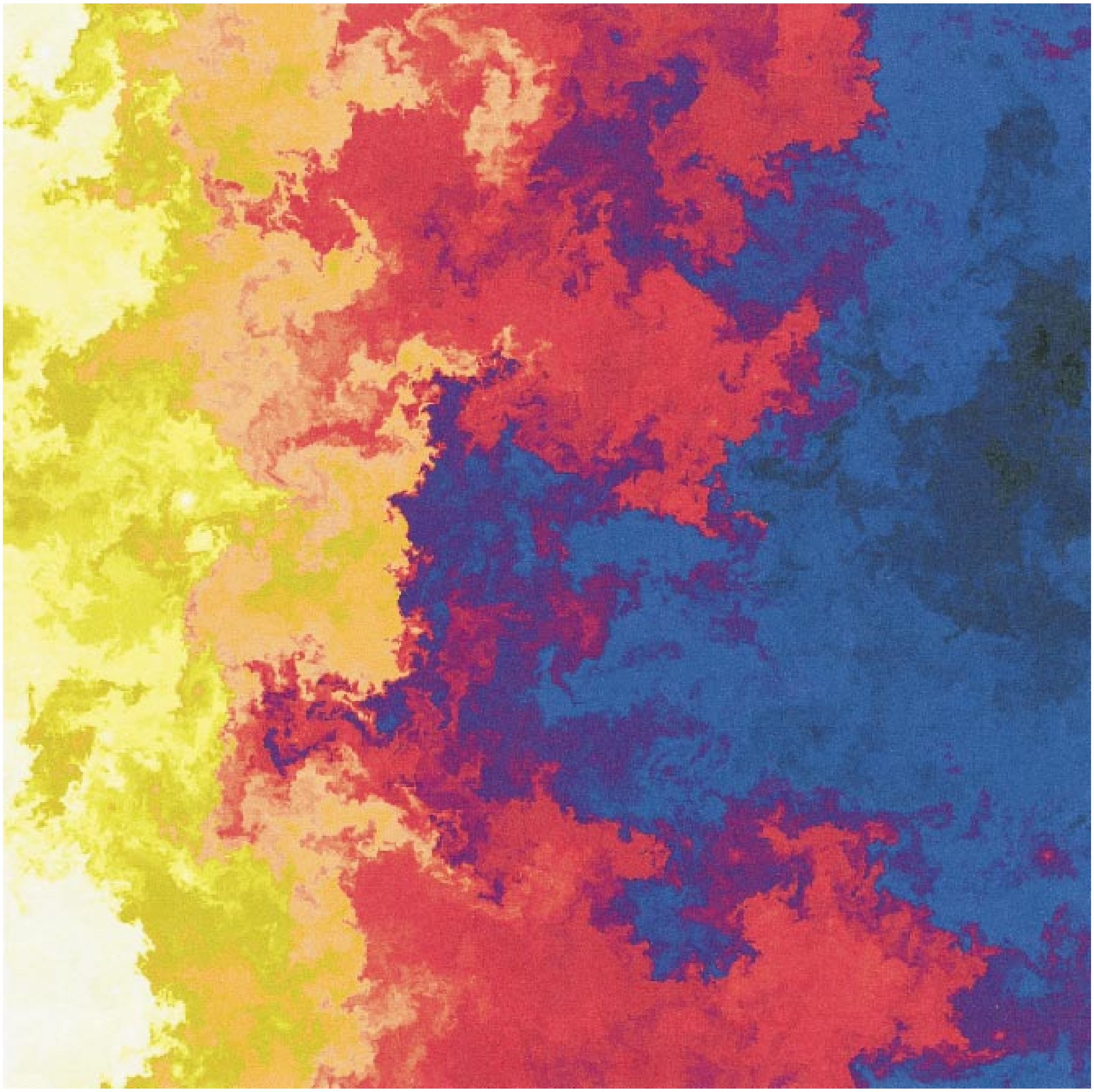

FIG. 6. A typical snapshot of a scalar field transported by a turbulent flow.

favorable case in this respect. The fact that fronts still form and that the saturation takes place points to generality of the phenomenon for scalar turbulence. The order of the moments and the value where the ζ_N curve flattens out might depend on the statistics of the advecting velocity, but the saturation itself should generally hold. Direct evidence for the advection by a 2D inverse cascade flow is provided in (Celani *et al.*, 2000, 2001). Saturation is equivalent to the scalar increment PDF taking the form $\mathcal{P}(\Delta_r\theta)=r^{\zeta_\infty}q(\Delta_r\theta/\theta_{rms})$ for amplitudes larger than θ_{rms}. The tails at various r can thus be all collapsed by plotting $r^{-\zeta_\infty}\mathcal{P}$, as shown in Fig. 7. Note finally that the saturation exponent ζ_∞ coincides with the fractal codimension of the fronts, see Celani *et al.* (2001) for a more detailed discussion.

As far as the compressible Kraichnan model is concerned, applying even qualitative predictions requires much more care than in the incompressible case. Indeed, the compressibility of the flow makes the sum of the Lyapunov exponents nonzero and leads to the permanent growth of density perturbations described in Sec. III.A.4. In a real fluid, such growth is stopped by the back reaction of the density on the velocity, providing for a long-time memory of the divergence $\nabla\cdot\mathbf{v}$ of the velocity along the Lagrangian trajectory. This shows that some characteristics of the Lagrangian velocity may be considered short correlated (like the off-diagonal components of the strain tensor), while others are long correlated (like the trace of the strain).

In summary, the situation with the Kraichnan model

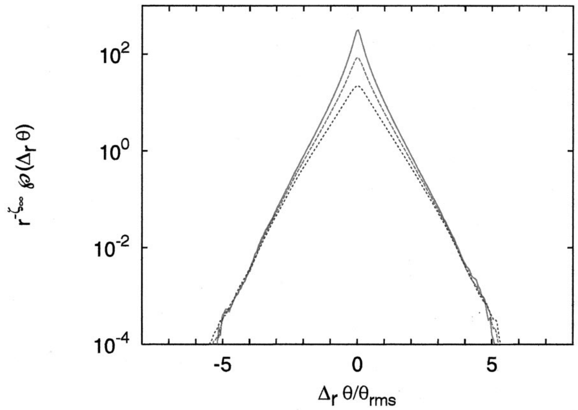

FIG. 7. The PDF's $\mathcal{P}(\Delta_r\theta)$ of the scalar increments $\Delta_r\theta = \theta(\mathbf{r}) - \theta(\mathbf{0})$ for three values of r inside the inertial range of scales, multiplied by the factor $r^{-\zeta_\infty}$. The observed collapse of the curves implies the saturation of the scaling exponents of the scalar structure functions.

and general passive scalar turbulence is much like the motto at the beginning of the review. The interest was originally stirred by the closed equations of motion for the correlation functions and the possibility of deriving an explicit formula for the anomalous scaling exponents, that are quite specific features. Actually, the model turned out to be much richer and capable of capturing the basic properties of the Lagrangian tracer dynamics in generic turbulent flow. The major lessons drawn from the model, such as the statistical integrals of motion, the geometry of the particle configurations, the dynamics in smooth flow, the importance of multipoint correlations, the persistence of anisotropies, all seem to have quite general validity.

IV. BURGERS AND NAVIER-STOKES EQUATIONS

All the previous sections were written under the assumption that the velocity statistics (whatever it is) is given. In this section, we shall describe what one can learn about the statistics of the velocity field itself by considering it in the Lagrangian frame. We start from the simplest case of Burgers turbulence whose inviscid version describes a free propagation of fluid particles, while viscosity provides for a local interaction. We then consider an incompressible turbulence where the pressure field provides for a nonlocal interaction between infinitely many particles.

A. Burgers turbulence

The d-dimensional Burgers equation (Burgers, 1974) is a pressureless version of the Navier-Stokes Eq. (1):

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} - \nu\nabla^2\mathbf{v} = \mathbf{f} \tag{184}$$

for irrotational (potential) velocity $\mathbf{v}(\mathbf{r},t)$ and force $\mathbf{f}(\mathbf{r},t)$. It is used to describe a variety of physical situations from the evolution of dislocations in solids to the formation of large scale structures in the universe, see, e.g., Krug and Spohn (1992) and Shandarin and Zel'dovich (1989). Involving a compressible $\mathbf{v}$, it allows for a meaningful (and nontrivial) one-dimensional case that describes small-amplitude perturbations of velocity, density, or pressure depending on a single spatial coordinate in the frame moving with sound velocity, see, e.g., Landau and Lifshitz (1959). Without force, the evolution described by Eq. (184) conserves total momentum $\int \mathbf{v}\, d\mathbf{r}$. By the substitutions $\mathbf{v} = \boldsymbol{\nabla}\phi$ and $\mathbf{f} = \boldsymbol{\nabla} g$ the Burgers equation is related to the KPZ equation

$$\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 - \nu\nabla^2\phi = g, \tag{185}$$

governing an interface growth (Kardar *et al.*, 1986).

Already the one-dimensional case of Eq. (184) illustrates the themes that we discussed in previous sections: turbulent cascade, Lagrangian statistics, and anomalous scaling. Under the action of a large-scale forcing (or in free decay of large-scale initial data) a cascade of kinetic energy towards the small scales takes place. The nonlinear term provides for steepening of negative gradients and the viscous term causes energy dissipation in the fronts that appear this way. In the limit of vanishing viscosity, the energy dissipation stays finite due to the appearance of velocity discontinuities called shocks. The Lagrangian statistics is peculiar in such an extremely nonsmooth flow and can be closely analyzed even though it does not correspond to a Markov process. Forward and backward Lagrangian statistics are different, as it has to be in an irreversible flow. Lagrangian trajectories stick to the shocks. That provides for a strong interaction between the particles and results in an extreme anomalous scaling of the velocity field. A tracer field passively advected by such a flow undergoes an inverse cascade.

At vanishing viscosity, the Burgers equation may be considered as describing a gas of particles moving in a force field. Indeed, in the Lagrangian frame defined for a regular velocity by $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R},t)$, relation (184) becomes the equation of motion of noninteracting unit-mass particles whose acceleration is determined by the force

$$\ddot{\mathbf{R}} = \mathbf{f}(\mathbf{R},t). \tag{186}$$

In order to find the Lagrangian trajectory $\mathbf{R}(t;\mathbf{r})$ passing at time zero through $\mathbf{r}$ it is then enough to solve the second-order Eq. (186) with the initial conditions $\mathbf{R}(0) = \mathbf{r}$ and $\dot{\mathbf{R}}(0) = \mathbf{v}(\mathbf{r},0)$. For sufficiently short times such trajectories do not cross and the Lagrangian map $\mathbf{r} \mapsto \mathbf{R}(t;\mathbf{r})$ is invertible. One may then reconstruct $\mathbf{v}$ at time t from the relation $v(\mathbf{R}(t),t) = \dot{\mathbf{R}}(t)$. The velocity stays potential if the force is potential. At longer times, however, the particles collide creating velocity discontinuities, i.e., shocks. The nature and the dynamics of the shocks may be understood by treating the inviscid equation as the limit of the viscous one. Positive viscosity removes the singularities. As is well known, the KPZ Eq. (185) may be linearized by the Hopf-Cole substitution $Z = \exp[-\phi/2\nu]$ that gives rise to the heat equation in an external potential:

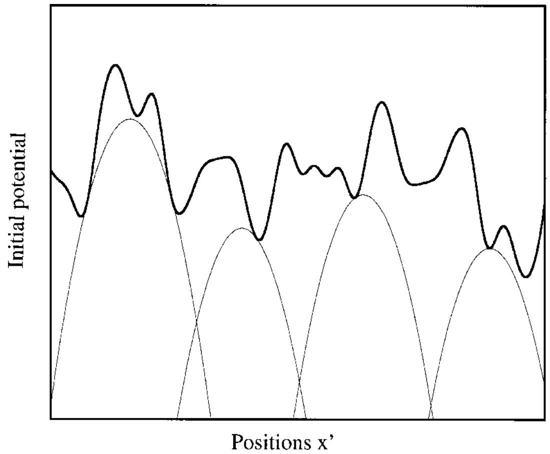

FIG. 8. Geometric construction of the Hopf-Cole inviscid solution of the 1D Burgers equation. The inverted parabola $C-(1/2t)(r-r')^2$ is moved upwards until the first contact point with the profile of the initial potential $\phi(r',0)$. The corresponding height C gives the potential $\phi(r,t)$ at time t. Shocks correspond to positions r where there are several contact points r', as for the first parabola on the left.

$$\left[\partial_t-\nu\nabla^2+\frac{1}{2\nu}g\right]Z=0. \tag{187}$$

The solution of the initial value problem for the latter may be written as the Feynman-Kac path integral

$$Z(\mathbf{r},t)=\int_{\mathbf{R}(t)=\mathbf{r}}\exp\left[-\frac{1}{2\nu}S(\mathbf{R})\right]Z(\mathbf{R}(0),0)\mathcal{D}\mathbf{R} \tag{188}$$

with the classical action $S(\mathbf{R})=\int_0^t[1/2\dot{\mathbf{R}}^2+g(\tau,\mathbf{R})]d\tau$ of a path $\mathbf{R}(\tau)$. The limit of vanishing viscosity selects the least-action path:

$$\phi(\mathbf{r},t)=\min_{\mathbf{R}:\mathbf{R}(t)=\mathbf{r}}\{S(\mathbf{R})+\phi(\mathbf{R}(0),0)\}. \tag{189}$$

Equating to zero the variation of the minimized expression, one infers that the minimizing path $\mathbf{R}(\tau)\equiv\mathbf{R}(\tau;\mathbf{r},t)$ is a solution of Eq. (186) such that $\dot{\mathbf{R}}(0)=\mathbf{v}(\mathbf{R}(0),0)$. Taking the gradient of Eq. (189), one also infers that $\mathbf{v}(\mathbf{r},t)=\dot{\mathbf{R}}(t)$. The above procedure extends the short-time construction of the solutions of the inviscid Burgurs equation to all times at the cost of admitting shocks where $\mathbf{v}$ is discontinuous. The velocity still evolves along the solutions of Eq. (186) with $\dot{\mathbf{R}}(0)$ given by the initial velocity field, but if there are many such solutions reaching the same point at the same moment, the ones that do not realize the minimum (189) should be disregarded. The shocks arise when there are several minimizing paths. At fixed time, shocks are, generically, hypersurfaces which may intersect, have boundaries, corners, etc. (Vergassola *et al.*, 1994; Frisch *et al.*, 1999; Frisch and Bec, 2000). This may be best visualized in the case without forcing where Eq. (189) takes the form

$$\phi(\mathbf{r},t)=\min_{\mathbf{r}'}\left\{\frac{1}{2t}(\mathbf{r}-\mathbf{r}')^2+\phi(\mathbf{r}',0)\right\} \tag{190}$$

with a clear geometric interpretation: $\phi(\mathbf{r},t)$ is the height C of the inverted paraboloid $C-(1/2t)(\mathbf{r}-\mathbf{r}')^2$ touching but not overpassing the graph of $\phi(\mathbf{r}',0)$. The points of contact between the paraboloid and the graph correspond to $\mathbf{r}'$ in Eq. (190) on which the minimum is attained. Shock loci are composed of those $\mathbf{r}$ for which there are several such contact points.

The simplest situation occurs in one dimension with $\mathbf{r}\equiv x$. Here, shocks are located at points x_i where the velocity jumps, i.e., where the two-sided limits $v^\pm(x,t)\equiv\lim_{\epsilon\to 0}v(x\pm\epsilon,t)$ are different. The limits correspond to the velocities of two minimizing paths that do not cross except at x_i. The shock height $s_i=v^+(x_i,t)-v^-(x_i,t)$ has to be negative. Once created shocks never disappear but they may merge so that they form a tree branching backward in time. For no forcing, the shock positions x_i correspond to the inverse parabolas $C-(1/2t)(x_i-x')^2$ that touch the graph of $\phi(x',0)$ in (at least) two points $x_i'^{\pm}$, $x_i'^{+}>x_i'^{-}$, such that $v^\pm(x_i,t)=(x_i-x_i'^{\pm})/t$; see Fig. 8. Below, we shall limit our discussion to the one-dimensional case that was most extensively studied in the literature; see Burgers (1974), Woyczyński (1998) and E and Vanden Eijnden (2000a). In particular, the asymptotic long-time large-distance behavior of freely decaying initial data with Gaussian finitely correlated velocities or velocity potentials has been intensively studied. In the first case, the asymptotic solution for $x^{3/2}t^{-1}=\mathcal{O}(1)$ has the form (190) with $\phi(x',0)$ representing a Brownian motion. In the second case, the solution settles under the diffusive scaling with a logarithmic correction $x^2t^{-1}\ln^{1/4}x=\mathcal{O}(1)$ to the form

$$\phi(x,t)=\min_j\left\{\frac{1}{2t}(x-y_j)^2+\phi_j\right\}, \tag{191}$$

where (y_j,ϕ_j) is the Poisson point process with intensity $e^\phi\,dy\,d\phi$. In both cases explicit calculations of the velocity statistics have been possible, see, respectively, Burgers (1974), Frachebourg and Martin (2000), Kida (1979), and Woyczyński (1998). Other asymptotic regimes of decaying Gaussian initial data were analyzed in Gurbatov *et al.* (1997).

The equation of motion of the one-dimensional shocks $x_i(t)$ is easy to obtain even in the presence of forcing. To this aim, note that along the shock there are two minimizing solutions determining the same function ϕ and that

$$\frac{d}{dt}\phi(x_i,t)=\partial_t\phi(x_i,t)+\dot{x}_i\partial_x\phi(x_i,t)$$
$$=-\frac{1}{2}v^\pm(x_i,t)^2+g(x_i,t)+\dot{x}_iv^\pm(x_i,t). \tag{192}$$

Equating both expressions, we infer that $\dot{x}_i=\frac{1}{2}[v^+(x_i,t)+v^-(x_i,t)]\equiv\overline{v}(x_i,t)\equiv\overline{v}_i$, i.e., that the shock speed is the mean of the velocities on both sides of the shock. The crucial question for the Lagrangian description of the Burgers velocities is what happens with the fluid particles after they reach shocks where their equation of motion $\dot{x}=v(x,t)$ becomes ambiguous. The question may be easily answered by considering the inviscid case as a limit of the viscous one where shocks become steep fronts with large negative velocity gradients. It is easy to see that the Lagrangian particles are trapped within such fronts and keep moving with them. We should then define the inviscid Lagrangian trajectories as solutions of the equation $\dot{x}=\overline{v}(x,t)$, with $\dot{x}$ understood as the right derivative. Indeed, the solutions of that equation clearly move with the shocks after reaching them. In other words, the two particles arriving at the shock from the right and the left at a given moment aggregate upon the collision. Momentum is conserved so that their velocity after the collision is the mean of the incoming ones (recall that the particles have unit mass) and is equal to the velocity of the particles moving with the shock that have been absorbed at earlier times. Note that in the presence of shocks the Lagrangian map becomes many-to-one, compressing whole space intervals into the shock locations.

It is not difficult to write field evolution equations that take into account the presence of shocks (Vol'pert, 1967; Bernard and Gawędzki, 1998; E and Vanden Eijnden, 2000a). We shall do it for local functions of the velocity of the form $e^{\lambda v(x,t)}$. For positive viscosity, these functions obey the equation of motion

$$(\partial_t+\lambda\partial_\lambda\lambda^{-1}\partial_x-\lambda f)e^{\lambda v}=-\lambda^2\epsilon(\lambda), \tag{193}$$

where $\epsilon(\lambda)=\nu[(\partial_x v)^2-\lambda^{-2}\partial_x^2]e^{\lambda v}$ is the contribution of the viscous term in Eq. (184). In the limit $\nu\to 0$ the dissipation becomes concentrated within the shocks. Using the representation $e^{\lambda v(x,t)}=e^{\lambda v^+(x,t)}\theta[x-x_i(t)]+e^{\lambda v^-(x,t)}\theta[x_i(t)-x]$ around the shocks, it is easy to check that Eq. (193) still holds for $\nu=0$ with

$$\epsilon(\lambda;x,t)=\sum_i F(\overline{v}_i,s_i)\,\delta[x-x_i(t)], \tag{194}$$

concentrated at shock locations. The "form factor" $F(\overline{v},s)=-2e^{\lambda\overline{v}}\lambda^{-1}\partial_\lambda\lambda^{-1}\sinh(\lambda s/2)$.

When the forcing and/or initial data are random, the equation of motion (193) induces the Hopf evolution equations for the correlation functions. For example, in the stationary homogeneous state $\langle fe^{\lambda v}\rangle=\lambda\langle\epsilon(\lambda)\rangle$. Upon expanding into powers of λ, this relates the single-point expectations $\langle v^n\rangle$ to the shock statistics. The first of these relations says that the average force should vanish and the second gives the energy balance $\langle fv\rangle=\overline{\epsilon}_v$, where the mean energy dissipation rate $\overline{\epsilon}_v=\langle\epsilon(0)\rangle=-\rho\langle s^3\rangle/12$ and $\rho=\langle\Sigma_i\delta[x-x_i(t)]\rangle$ is the mean shock density. Similarly, for the generating function of the velocity increments $\langle\exp(\lambda\Delta v)\rangle$ with $\Delta v=v_1-v_2$, one obtains

$$\lambda^2\partial_\lambda\lambda^{-2}\partial_1\langle e^{\lambda\Delta v}\rangle-\lambda\langle\Delta f e^{\lambda\Delta v}\rangle$$
$$=-\lambda^2\langle\epsilon(\lambda)_1 e^{-\lambda v_2}+e^{\lambda v_1}\epsilon(-\lambda)_2\rangle. \tag{195}$$

For a Gaussian force with $\langle f(x,t)f(0,0)\rangle=\delta(t)\chi(x/L)$, where L is the injection scale, $\langle\Delta f e^{\lambda\Delta v}\rangle=[\chi(0)-\chi(\Delta x/L)]\langle e^{\lambda\Delta v}\rangle$. For a spatially smooth force, the second term in Eq. (195) tends to zero when the separation Δx (taken positive) shrinks. In the same limit, the quantity inside the expectation value on the right-hand side tends to a local operator concentrated on shocks, as in Eq. (194), but with the height-dependent form factor $-4e^{\lambda s/2}\lambda^{-1}\partial_\lambda\lambda^{-1}\sinh(\lambda s/2)=-\partial_\lambda\lambda^{-2}(e^{\lambda s}-1-\lambda s)$. Comparing the terms, one infers that for $N=3,4\ldots$,

$$\lim_{x_2\to x_1}\partial_1\langle(\Delta v)^N\rangle=\rho\langle s^N\rangle. \tag{196}$$

The first of these equalities is the one-dimensional version of the Kolmogorov flux relation (2). The higher ones express the fact that, for small Δx, the higher moments of velocity increments are dominated by the contribution from a single shock of height s occurring with probability $\rho\Delta x$. That implies the anomalous scaling of the velocity structure functions

$$S_N(\Delta x)\equiv\langle(\Delta v)^N\rangle=\rho\langle s^N\rangle\Delta x+o(\Delta x), \tag{197}$$

that is a signature of an extreme intermittency. In fact, due to the shock contributions, all moments of velocity increments of order $p\geqslant 1$ scale with exponents $\zeta_p=1$. In contrast, for $0\leqslant p\leqslant 1$, $\zeta_p=p$ since the fractional moments are dominated by the regular contributions to velocity. Indeed, denoting by ξ the regular part of the velocity gradient, one obtains (E *et al.*, 1997; E, Khanin, *et al.*, 2000; E and Vanden Eijnden, 2000a)

$$\langle(|\Delta v|)^p\rangle=\langle|\xi|^p\rangle(\Delta x)^p+o[(\Delta x)^p]. \tag{198}$$

The shock contribution proportional to Δx is subleading in that case.

The Lagrangian interpretation of these results can be based on the fact that the velocity is a Lagrangian invariant of the unforced inviscid system. In the presence of the force,

$$v(x,t)=v(x(0),0)+\int_0^t f(x(s),s)ds \tag{199}$$

along the Lagrangian trajectories. The velocity is an active scalar and the Lagrangian trajectories are evidently dependent on the force that drives the velocity. One cannot write a formula like Eq. (135) obtained by two independent averages over the force and over the trajectories. Nevertheless, the main contribution to the distance-dependent part of the two-point function $\langle v(x,t)v(x',t)\rangle$ is due, for small distances, to realizations with a shock in between the particles. It is insensitive to a large-scale force and hence approximately proportional to the time that the two Lagrangian trajectories ending at x and x' take to separate backwards to the injection scale L. With a shock in between x and x' at time t, the initial backward separation is linear so that the second-order structure function becomes proportional to Δx, in accordance with Eq. (197). Other structure functions may be analyzed similarly and give the same linear dependence on the distance (all terms involve at most two trajectories). This mechanism of the anomalous scaling is similar to that of the collinear case in Sec. III.B.3.

Following numerical observations of Chekhlov and Yakhot (1995, 1996), a considerable effort has been invested to understand the shape of the PDF's $\mathcal{P}(\Delta v)$ of the velocity increments and $\mathcal{P}(\xi)$ of the regular part of $\partial_x v$. The stationary expectation of the exponential of ξ satisfies for a white-in-time forcing the relation

$$(\lambda \partial_\lambda^2 - \partial_\lambda - D\lambda^2)\langle e^{\lambda\xi}\rangle = \langle \rho_\lambda \rangle \tag{200}$$

that may established the same way as Eq. (195). Here $D = -\frac{1}{2}\chi''(0)L^{-2}$ and ρ_λ is an operator supported on shocks with the form-factor $(s/2)\,(e^{\lambda\xi^+} + e^{\lambda\xi^-})$. For the PDF of ξ this gives the identity (E and Vanden Eijnden, 1999, 2000a)

$$(D\partial_\xi^2 + \xi^2\partial_\xi + 3\xi)\mathcal{P}(\xi) + \int \frac{d\lambda}{2\pi i} e^{-\lambda\xi}\langle \rho_\lambda\rangle = 0. \tag{201}$$

Various closures for this equation or for Eq. (195) have been proposed (Polyakov, 1995; Bouchaud and Mézard, 1996; Boldyrev, 1997; Gotoh and Kraichnan, 1998). They all give the right tail $\propto e^{-\xi^3/(3D)}$ of the distribution, as determined by the first two terms with derivatives in Eq. (201), with the power-law prefactors depending on the details of the closure. That right tail was first obtained by Feigel'man (1980) and also reproduced for $\mathcal{P}(\Delta v)$ by an instanton calculation in Gurarie and Migdal (1996). The instanton appears to be a solution of a deterministic Burgers equation with a force linear in space, see also Balkovsky *et al.* (1997). The right tail may be also understood from the stochastic equation along the trajectory $(d^2/dt^2)\,\Delta x = \Delta f$. For Δx much smaller than the injection scale L, the force increment may be linearized $\Delta f = \sigma(t)\Delta x$. In the first approximation, $\sigma(t)$ is a white noise and we obtain a problem familiar from the one-dimensional localization in a δ-correlated potential. In particular, $z = (d/dt)\,(\Delta x)/\Delta x$ satisfies $\dot{z} = \sigma(t) - z^2$. This is a Langevin dynamics in the unbounded from below potential $\propto z^3$ (Bouchaud and Mézard, 1996). Upon conditioning against escape to $z = -\infty$, one gets a stationary distribution for z with the right tail $\propto e^{-z^3/(3D)}$ and the left tail $\propto z^{-2}$. In reality, due to the shock creation, $\Delta f/\Delta x \approx \partial_x f(x(t),t)$ is a white noise only if we fix the value of v at the end point of the Lagrangian trajectory. This introduces subtle correlations which effect the power-law factors in $\mathcal{P}(\xi)$, in particular its left tail $\propto|\xi|^{-\alpha}$. Large negative values of ξ appear in the vicinity of preshocks, as stressed in (E *et al.*, 1997) where the value $\alpha = 7/2$ was argued for, based on a geometric analysis of the preshocks, the birth points of the shock discontinuities. E and Vanden Eijnden (2000a), have proved by analysis of realizability conditions for the solutions of Eq. (201) that $\alpha > 3$ and made a strong case for $\alpha = 7/2$, assuming that shocks are born with vanishing heights and that preshocks do not accumulate. The exponent $\alpha = 7/2$ was subsequently found in the decaying case with random large scale initial conditions both in 1D (Bec and Frisch, 2000) and in higher dimensions (Frisch *et al.*, 2001), and in the forced case when the forcing consists of deterministic large scale kicks repeated periodically (Bec *et al.*, 2000). In those cases, the statistics of the shocks and their creation process are easier to control than for the δ-correlated forcing. The numerical analysis of the latter case clearly confirms, however, the prediction $\alpha = 7/2$. As to the PDF $\mathcal{P}(\Delta v)$ of the velocity increment in the decaying case and, possibly, in the forced case, it exhibits a crossover from the behavior characteristic for the velocity gradients to the one reproducing the behaviors (197) and (198). Note that the single-point velocity PDF also has cubic tails $\ln\mathcal{P}(v) \propto -|v|^3$ (Avellaneda *et al.*, 1995; Balkovsky *et al.*, 1997). The same is true for white-driven Navier-Stokes equation and may be generalized for a non-Gaussian force (Falkovich and Lebedev, 1997).

The Lagrangian picture of the Burgers velocities allows for a simple analysis of advection of scalar quantities carried by the flow. In the inviscid and diffusionless limit, the advected tracer satisfies the evolution equation

$$\partial_t\theta + \overline{v}\partial_x\theta = \varphi, \tag{202}$$

where φ represents an external source. As usual, the solution of the initial value problem is given in terms of the PDF $p(x,t;y,0|v)$ to find the backward Lagrangian trajectory at y at time 0, given that at later time t it passed by x; see Eq. (100). Except for the discrete set of time t shock locations, the backward trajectories are uniquely determined by x. As a result, a smooth initial scalar will develop discontinuities at shock locations but no stronger singularities. Since a given set of points $(x_1,\ldots,x_N) \equiv \underline{x}$ avoids the shocks with probability 1, the joint backward PDF's of N trajectories $\mathcal{P}_N(\underline{x};\underline{y};-t)$, see Eq. (65), should be regular for distinct x_n and should possess the collapse property (67). This leads to the conservation of $\langle\theta^2\rangle$ in the absence of scalar sources and to the linear pumping of the scalar variance into a soft mode when a stationary source is present. Such behavior corresponds to an inverse cascade of the passive scalar, as in the strongly compressible phase of the Kraichnan model discussed in Sec. (III.E).

The Burgers velocity itself and all its powers constitute an example of advected scalars. Indeed, the equation of motion (193) may be also rewritten as

$$(\partial_t + \overline{v}\partial_x - \lambda f)e^{\lambda v} = 0 \tag{203}$$

from which the relation (202) for $\theta = v^n$ and $\varphi = nfv^{n-1}$ follows. Of course, v^n are active scalars so that in the random case their initial data, the source terms, and the Lagrangian trajectories are not independent, contrary to the case of passive scalars. That correlation makes the unlimited growth of $\langle v^2\rangle$ impossible: the larger the value of local velocity, the faster it creates a shock and dissipates the energy. The difference between active and passive tracers is thus substantial enough to switch the direction of the energy cascade from inverse for the passive scalar to direct for the velocity.

As usual in compressible flows, the advected density n satisfies the continuity equation

$$\partial_t n + \partial_x(\overline{v})n = \varphi \tag{204}$$

different from Eq. (202) for the tracer. The solution of the initial value problem is given by the forward Lagrangian PDF $p(y,0;x,t|v)$; see Eq. (100). Since the trajectories collapse, a smooth initial density will become singular under the evolution, with δ-function contributions concentrating all the mass from the regions compressed to shocks by the Lagrangian flow. Since the trajectories are determined by the initial point y, the joint forward PDF's $\mathcal{P}_N(\underline{y};\underline{x};t)$ should have the collapse property (67) but they will also have contact terms in x_n's when the initial points y_n are distinct. Such terms signal a finite probability of the trajectories to aggregate in the forward evolution, the phenomenon that we have already met in the strongly compressible Kraichnan model discussed in Sec. II.D. The velocity gradient $\partial_x v$ is an example of an (active) density satisfying Eq. (204) with $\varphi = \partial_x f$.

In Bernard and Bauer (1999), the behavior of the Lagrangian PDF's and the advected scalars summarized above have been established by a direct calculation in freely decaying Burgers velocities with random Gaussian finitely correlated initial potentials ϕ.

B. Incompressible turbulence from a Lagrangian viewpoint

As we learned from the study of passive fields, treating the dissipation is rather easy as it is a linear mechanism. The main difficulty resides in proper understanding advection. For incompressible turbulence, the problem is even more complicated than for the Burgers equation due to spatial nonlocality of the pressure term. The Euler equation may indeed be written as the equation

$$\ddot{\mathbf{R}} = \mathbf{f}(\mathbf{R},t) - \nabla P, \tag{205}$$

for the Lagrangian trajectories $\mathbf{R}(t;\mathbf{r})$ where $\mathbf{f}$ is the external force and the pressure field is determined by the incompressibility condition $\nabla^2 P = -\nabla \cdot [\mathbf{v} \cdot \nabla \mathbf{v}]$ with $\mathbf{v} = \dot{\mathbf{R}}$ and the spatial derivatives taken with respect to $\mathbf{R}$. The inversion of the Laplace operator in the previous relation brings in nonlocality via the kernel decaying as a power law. We thus have a system of infinitely many particles interacting strongly and nonlocally. In such a situation, any attempt at an analytic description looks unavoidably dependent on possible simplifications in limiting cases. The natural parameter to exploit for the incompressible Euler equation is the space dimensionality, varying between 2 and infinity. The two-dimensional case indeed presents important simplifications since the vorticity is a scalar Lagrangian invariant of the inviscid dynamics, as we shall discuss hereafter. The opposite limit of the infinite-dimensional Euler equation is very tempting for some kind of mean-field approximation to the interaction among the fluid particles but nobody has derived it yet. The level of discussion in this section is thus naturally different from the rest of the review: as a consistent theory is absent, we present a set of particular arguments and remarks that, on one hand, make contact with the previously discussed subjects, and, on the other hand, may inspire further progress.

1. Enstrophy cascade in two dimensions

The Euler equation in any even-dimensional space has an infinite set of integrals of motion besides energy. One may indeed show that the determinant of the matrix $\omega_{ij} = \nabla_j v^i - \nabla_i v^j$ is the non-negative density of an integral of motion, i.e., $\int (\det \omega) d\mathbf{r}$ is conserved for any function F. The quadratic invariant $\int (\det \omega)^2 d\mathbf{r}$ is called enstrophy. In the presence of an external pumping ϕ injecting energy and enstrophy, it is clear that both quantities may flow throughout the scales. If both cascades are present, they cannot go in the same direction: the different dependence of energy and enstrophy on the scale prevents their fluxes to be both constant in the same interval. A finite energy dissipation would imply an infinite enstrophy dissipation in the limit $\nu \to 0$. The natural conclusion is that, given a single pumping at some intermediate scale and assuming the presence of two cascades, the energy and the enstrophy flow toward the large and the small scales, respectively (Kraichnan, 1967; Batchelor, 1969). This is indeed the case for the two-dimensional case.

In this section, we shall focus on the 2D direct enstrophy cascade. The basic knowledge of the Lagrangian dynamics presented in Secs. II.B.1 and III.B is essentially everything needed. Vorticity in 2D is a scalar and the analogy between vorticity and passive scalar was noticed by Batchelor and Kraichnan already in the 1960s. Vorticity is not passive though and such analogies may be very misleading, as it is the case for vorticity and magnetic field in 3D and for velocity and passive scalar in 1D Burgers. The basic flux relation for the enstrophy cascade is analogous to Eq. (132):

$$\langle (\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\omega_1 \omega_2 \rangle = \langle \varphi_1 \omega_2 + \varphi_2 \omega_1 \rangle = P_2. \tag{206}$$

The subscripts indicate the spatial points $\mathbf{r}_1$ and $\mathbf{r}_2$ and the pumping is assumed to be Gaussian with $\langle \varphi(\mathbf{r},t)\phi(\mathbf{0},0)\rangle = \delta(t)\Phi(r/L)$ decaying rapidly for $r > L$. The constant $P_2 \equiv \Phi(0)$, having dimensionality time^{-3}, is the input rate of the enstrophy ω^2. Equation (206) states that the enstrophy flux is constant in the inertial range, that is for r_{12} much smaller than L and much larger than the viscous scale. A simple power counting suggests that the velocity differences and the vorticity scale as the first and the zeroth power of r_{12}, respectively. That fits the idea of a scalar cascade in a spatially smooth velocity: scalar correlation functions are indeed logarithmic in that case, as it was discussed in Sec. III.B.

Even though one can imagine hypothetical power-law vorticity spectra (Saffman, 1971; Moffatt, 1986; Polyakov, 1993), one can argue that they are structurally unstable (Falkovich and Lebedev, 1994). Indeed, imagine for a moment that the pumping at L produces the spectrum $\langle (\omega_1 - \omega_2)^p \rangle \propto r_{12}^{\zeta_p}$ at $r_{12} \ll L$. Regularity of the Euler equation in 2D requires $\zeta_p > 0$, see, e.g., Eyink (1995), and references therein. In the spirit of the stability theory of Kolmogorov spectra (Zakharov *et al.*, 1992), let us add an infinitesimal pumping at some l in the inertial interval producing a small yet nonzero flux of enstrophy. Small perturbations $\delta\omega$ obey the equation $\partial_t\delta\omega+(v\nabla)\delta\omega+(\delta v\nabla)\omega=\nu\nabla^2\delta\omega$. Here, δv is the velocity perturbation related to $\delta\omega$. The perturbation $\delta\omega$ has the typical scale l while vorticity ω, associated with the main spectrum, is concentrated at L when $\zeta_2>0$. The third term can be neglected as it is $(l/L)^2$ times smaller than the second one. Therefore $\delta\omega$ behaves as a passive scalar convected by main turbulence, i.e., the Batchelor regime from Sec. III.B takes place. The correlation functions of the scalar are logarithmic in this case for any velocity statistics. The perturbation in any vorticity structure function thus decreases downscales slower than the contribution of the main flow. That means that any hypothetical power-law spectrum is structurally unstable with respect to the pumping variation. Stability analysis cannot of course describe the spectrum downscales where the perturbations is getting comparable to the main flow. It is logical though to assume that since only the logarithmic regime may be neutrally stable, it represents the universal small-scale asymptotics of steady forced turbulence. Experiments (Rutgers, 1998; Paret *et al.*, 1999; Jullien *et al.*, 2000) and numerical simulations (Borue, 1993; Gotoh, 1998; Bowman *et al.*, 1999) are compatible with that conclusion.

The physics of the enstrophy cascade is thus basically the same as that for a passive scalar: a fluid blob embedded into a larger-scale velocity shear is extended along the direction of a positive strain and compressed along its negative eigendirection; such stretching provides for the vorticity flux toward the small scales, with the rate of transfer proportional to the strain. The vorticity rotates the blob decelerating the stretching due to the rotation of the axes of positive and negative strain. One can show that the vorticity correlators are indeed solely determined by the influence of larger scales (that give exponential separation of the fluid particles) rather than smaller scales (that would lead to a diffusive growth as the square root of time). The subtle differences from the passive scalar case come from the active nature of the vorticity. Consider, for example, the relation (135) expressing the fact that the correlation function of a passive scalar is essentially the time spent by the particles at distances smaller than L. The passive nature of the scalar makes Lagrangian trajectories independent of scalar pumping which is crucial in deriving Eq. (135) by two independent averages. For an active scalar, the two averages are coupled since the forcing affects the velocity and thus the Lagrangian trajectories. In particular, the statistics of the forcing along the Lagrangian trajectories $\phi[\mathbf{R}(t)]$ involves nonvanishing multipoint correlations at different times. Falkovich and Lebedev (1994) argued that, as far as the dominant logarithmic scaling of the correlation functions is concerned, the active nature of vorticity simply amounts to the following: the field can be treated as a passive scalar, but the strain acting on it must be renormalized with the scale. Their arguments are based on the analysis of the infinite system of equations for the variational derivatives of the vorticity correlation functions with respect to the pumping and the relations between the strain and the vorticity correlation functions. The law of renormalization is then established as follows, along the line suggested earlier by Kraichnan (1967, 1971, 1975). From Eq. (12), one has the dimensional relation that *times* behave as $\omega^{-1}\ln(L/r)$. Furthermore, by using the relation (135) for the vorticity correlation function, one has $\langle\omega\omega\rangle\propto P_2\times$time. Combining the two previous relations, the scaling $\omega\sim[P_2\ln(L/r)]^{1/3}$ follows. The consequences are that the distance between two fluid particles satisfies $\ln(R/r)\sim P_2^{1/2}t^{3/2}$, and that the pair-correlation function $\langle\omega_1\omega_2\rangle\sim[P_2\ln(L/r_{12})]^{2/3}$. Note that the fluxes of higher powers ω^{2n} are not constant in the inertial range due to the same phenomenon of "distributed pumping" discussed in Sect. III.C.1.f. The vorticity statistics is thus determined by the enstrophy production rate alone.

It is worth stressing that the logarithmic regime described above is a small-scale asymptotics of a steady turbulence. Depending on the conditions of excitation and dissipation, different other regimes can be observed either during an intermediate time or in an intermediate interval of scales. First, a constant friction that provides for the velocity decay rate α prevails (if present) over viscosity at scales larger than $(\nu/\alpha)^{1/2}$. At such scales, the vorticity correlation functions are expected to behave as power laws rather than logarithmically, very much like for the passive scalar as described in Chertkov (1999) and in Sec. III.E. Indeed, the advective and the friction terms $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ and $-\alpha\mathbf{v}$ have again the same dimension for a smooth velocity. Nontrivial scaling is therefore expected, including for the second-order correlation function (and hence for the energy spectrum). The difficulty is of course that the system is now nonlinear and exact closed equations, such as those in Sec. III.E, are not available. For theoretical attempts to circumvent that problem by some approximations, not quite controlled yet, see Bernard (2000) and Nam *et al.* (2000). Second, strong large-scale vortices often exist with their (steeper) spectrum masking the enstrophy cascade in some intermediate interval of scales (Legras *et al.*, 1988).

2. On the energy cascades in incompressible turbulence

The phenomenology of the energy cascade suggests that the energy flux $\bar{\epsilon}$ is a major quantity characterizing the velocity statistics. It is interesting to understand the difference between the direct and the inverse energy cascades from the Lagrangian perspective. The mean Lagrangian time derivative of the squared velocity difference is as follows:

$$\left\langle\frac{d(\Delta\mathbf{v})^2}{dt}\right\rangle=2\langle\Delta\mathbf{v}\Delta\mathbf{f}+\nu(2\mathbf{v}\cdot\nabla^2\mathbf{v}-\mathbf{v}_1\cdot\nabla^2\mathbf{v}_2-\mathbf{v}_2\cdot\nabla^2\mathbf{v}_1)\rangle. \tag{207}$$

The right-hand side coincides with minus twice the flux and this gives the Lagrangian interpretation of the flux relations. In the 2D inverse energy cascade, there is no energy dissipative anomaly and the right-hand side in the inertial range is determined by the injection term $4\langle\mathbf{f}\cdot\mathbf{v}\rangle$. The energy flux is negative (directed upscale) and the mean Lagrangian derivative is positive. On the contrary, in the 3D direct cascade the injection terms cancel and the right-hand side becomes equal to $-4\nu\langle(\nabla\mathbf{v})^2\rangle$. The mean Lagrangian derivative is negative while the flux is positive (directed downscale). This is natural as a small-scale stirring causes opposite effects with respect to a small-scale viscous dissipation. The negative sign of the mean Lagrangian time derivative in 3D does not contradict the fact that any couple of Lagrangian trajectories eventually separates and their velocity difference increases. It tells, however, that the squared velocity difference between two trajectories generally behaves in a nonmonotonic way: the transverse contraction of a fluid element makes initially the difference between the two velocities decrease, while eventually the stretching along the trajectories takes over (Pumir and Shraiman, 2000).

The Eulerian form of Eq. (207) is the generalization of 4/5 law (2) for the d-dimensional case: $\langle(\Delta_r v)^3\rangle = -12\bar{\epsilon}r/d(d+2)$ if $\Delta_r v$ is the longitudinal velocity increment and $\bar{\epsilon}$ is positive for the direct cascade and negative for the inverse one. Since the average velocity difference vanishes, a negative $\langle(\Delta_r v)^3\rangle$ means that small longitudinal velocity differences are predominantly positive, while large ones are negative. In other words, in 3D if the longitudinal velocities of two particles differ strongly then the particles are likely to attract each other; if the velocities are close, then the particles preferentially repel each other. The opposite behavior takes place in 2D, where the third-order moment of the longitudinal velocity difference is positive. Another Lagrangian meaning of the flux law in 3D can be appreciated by extrapolating it down to the viscous interval. Here, $\Delta_r\mathbf{v} \approx \sigma\mathbf{r}$ and the positivity of the flux is likely to be related to the fact that the negative Lyapunov exponent is the largest one (in absolute value) in 3D incompressible turbulence.

If one assumes (after Kolmogorov) that $\bar{\epsilon}$ is the only pumping-related quantity that determines the statistics then the separation between the particles $\mathbf{R}_{12} = \mathbf{R}(t;\mathbf{r}_1) - \mathbf{R}(t;\mathbf{r}_2)$ has to obey the already mentioned Richardson law: $\langle R_{12}^2\rangle \propto \bar{\epsilon}t^3$. The equation for the separation immediately follows from the Euler Eq. (205): $\partial_t^2\mathbf{R}_{12} = \mathbf{f}_{12} - \nabla P_{12}$. The corresponding forcing term $\mathbf{f}_{12} \equiv \mathbf{f}(\mathbf{R}(t;\mathbf{r}_1)) - \mathbf{f}(\mathbf{R}(t;\mathbf{r}_2))$ has completely different properties for an inverse energy cascade in 2D than for a direct energy cascade in 3D. For the former, R_{12} in the inertial range is much larger than the forcing correlation length. The forcing can therefore be considered short correlated both in time and in space. Was the pressure term absent, one would get the separation growth: $\langle R_{12}^2\rangle/\bar{\epsilon}t^3 = 4/3$. The experimental data by Jullien *et al.* (1999) give a smaller numerical factor $\simeq 0.5$, which is quite natural since the incompressibility constrains the motion. What is, however, important to note is that already the forcing term prescribes the law $\langle R_{12}^2\rangle \propto t^3$ consistent with the scaling of the energy cascade. Conversely, for the direct cascade the forcing is concentrated at the large scales and $f_{12} \propto R_{12}$ in the inertial range. The forcing term is thus negligible and even the scaling behavior comes entirely from the advective terms (the viscous term should be accounted as well). Another amazing aspect of the 2D inverse energy cascade can be inferred if one considers it from the viewpoint of vorticity. Enstrophy is transferred toward the small scales and its flux at the large scales (where the inverse energy cascade is taking place) vanishes. By analogy with the passive scalar behavior at the large scales discussed in Sec. III.C.1, one may expect the behavior $\langle\omega_1\omega_2\rangle \propto r_{12}^{1-\alpha-d}$, where α is the scaling exponent of the velocity. The self-consistency of the argument dictated by the relation $\omega = \nabla\times\mathbf{v}$ requires $1-\alpha-d = 2\alpha-2$ which indeed gives the Kolmogorov scaling $\alpha = 1/3$ for $d=2$. Experiments (Paret and Tabeling, 1997, 1998) as well as numerical simulations (Smith and Yakhot, 1993; Boffetta *et al.*, 2000) indicate that the inverse energy cascade has a normal Kolmogorov scaling for all measured correlation functions. No consistent theory is available yet, but the previous arguments based on the enstrophy equipartition might give an interesting clue. To avoid misunderstanding, note that in considering the inverse cascades one ought to have some large-scale dissipation (like bottom and wall friction in the experiments with a fluid layer) to avoid the growth of condensate modes on the scale of the container. Another example of inverse cascade is that of the magnetic vector potential in two-dimensional magnetohydrodynamics, where the numerical simulations also indicate that intermittency is suppressed (Biskamp and Bremer, 1994). The generality of the absence of intermittency for inverse cascades and its physical reasons is still an open problem. The only inverse cascade fully understood is that of the passive scalar in Sec. III.E, where the absence of anomalous scaling was related to the uniqueness of the trajectories in strongly compressible flow. That explanation applies neither to 2D Navier-Stokes nor to magnetohydrodynamics since the scalar is active in both cases. Qualitatively, it is likely that the scale invariance of an inverse cascade is physically associated to the growth of the typical times with the scale. As the cascade proceeds, the fluctuations have indeed time to get smoothed out and not multiplicatively transferred as in the direct cascades, where the typical times decrease in the direction of the cascade.

An interesting phenomenological Lagrangian model of 3D turbulence based on the consideration of four particles was introduced by Chertkov, Pumir, and Shraiman (1999). As far as an anomalous scaling observed in the 3D energy cascade is concerned, the primary target is to understand the nature of the statistical integrals of motion responsible for it. Note that the velocity exponent $\sigma_3 = 1$ and experiments demonstrate that $\sigma_p \to ap$ as $p \to 0$ with a exceeding 1/3 beyond the measurement error; see Sreenivasan and Antonia (1997) and the references therein. The convexity of σ_p means then that $\sigma_2 > 2/3$. In other words, already the pair-correlation function should be determined by some nontrivial conservation law (like for magnetic fields in Sec. III.C.3).

V. CONCLUSIONS

This review is intended to bring home to the reader two main points: the power of the Lagrangian approach to fluid turbulence and the importance of statistical integrals of motion for systems far from equilibrium.

As it was shown in Secs. II and III, the Lagrangian approach allows for a systematic description of most important aspects of particle and field statistics. In a spatially smooth flow, Lagrangian chaos and exponentially separating trajectories are generally present. The associated statistics of passive scalar and vector fields is related to the statistics of large deviations of stretching and contraction rates in a way that is well understood. The theory is quite general and it finds a natural domain of application in the viscous range of turbulence. The most important open problem here seems to be the understanding of the back reaction of the advected field on the velocity. That would include an account of the buoyancy force in inhomogeneously heated fluids, the saturation of the small-scale magnetic dynamo and the polymer drag reduction. In nonsmooth velocities, pertaining to the inertial interval of developed turbulence, the main Lagrangian phenomenon is the intrinsic stochasticity of the particle trajectories that accounts for the energy depletion at short distances. This phenomenon is fully captured in the Kraichnan model of nonsmooth time-decorrelated velocities. To exhibit it for more realistic nonsmooth velocities and to relate it to the hydrodynamical evolution equations governing the velocity field remains an open problem. The spontaneous stochasticity of Lagrangian trajectories enhances the interaction between fluid particles leading to intricate multiparticle stochastic conservation laws. Here, there are open problems already in the framework of the Kraichnan model. First, there is the issue of whether one can build an operator product expansion, classifying the zero modes and revealing their possible underlying algebraic structure, both at large and small scales. The second class of problems is related to a consistent description of high-order moments of scalar, vector and tensor fields. In the situations where the amplitudes of the fields are growing, this would be an important step towards a description of feedback effects.

Our inability to derive the Lagrangian statistics directly from the Navier-Stokes equations of motion for the fluid particles is related to the fact that the particle coupling is strong and nonlocal due to pressure effects. Some small parameter for perturbative approaches, like those discussed for the Kraichnan model, has often been sought for. We would like to stress, however, that most strongly coupled systems, even if local, are not analytically solvable and that not all measurable quantities may be derived from first principles. In fluid turbulence, it seems more important to reach basic understanding of the underlying physical mechanisms, than it is to find out the numerical values of the scaling exponents. Such an understanding has been achieved in passive scalar and magnetic fields through the statistical conservation laws. We consider the notion of statistical integrals of motion to be of central importance for fluid turbulence and general enough to apply to other systems in nonequilibrium statistical physics. Indeed, nondimensionally scaling correlation functions appearing in such systems should generally be dominated by terms that solve dynamical equations in the absence of forcing (zero modes). As we explained throughout the review, for passively advected fields, such terms describe conservation laws that are related to the geometry either of the configuration of particles (for the scalar) or of the particle-plus-field configurations (for the magnetic field). It is a major open problem to identify the appropriate configurations for active and nonlocal cases. New particle-tracking methods (La Porta *et al.*, 2001) open promising experimental possibilities in this direction. An investigation of geometrical statistics of fluid turbulence by combined analytical, experimental and numerical methods aimed at identifying the underlying conservation laws is a challenge for future research.

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APPENDIX: REGULARIZATION OF STOCHASTIC INTEGRALS

For the δ-correlated strain, Eq. (12) becomes a stochastic differential equation. Let us present here a few elementary facts about such equations for that simple case. The differential Eq. (12) is equivalent to the integral equation

$$\mathbf{R}(t)=\mathbf{R}(0)+\int_0^t \sigma(s)ds\,\mathbf{R}(s), \tag{A1}$$

where $\mathbf{R}\equiv\mathbf{R}_{12}$. The right-hand side involves a stochastic integral whose distinctive feature is that $\sigma(t)dt$ is of order $(dt)^{1/2}$, as indicated by the relation $\langle[\int_0^t\sigma_{ij}(s)ds]^2\rangle \propto t$. Such integrals require second-order manipulations of differentials and, in general, are not unambiguously defined without the choice of a defining convention. The most popular are the Itô, the Stratonovich, and the anti-Itô ones. Physically, different choices reflect finer details of the strain correlations wiped out in the white-noise scaling limit, like the presence or the absence of time reversibility of the velocity distribution. The Itô, Stratonovich, and anti-Itô versions of the stochastic integral in Eq. (A1) are given by the limits over partitions of the time interval of different Riemann sums:

$$\int_0^t \sigma(s)ds\,\mathbf{R}(s) = \begin{cases} \lim \sum_n \int_{t_n}^{t_{n+1}} \sigma(s)ds\,\mathbf{R}(t_n) \\ \lim \sum_n \int_{t_n}^{t_{n+1}} \sigma(s)ds\,\tfrac{1}{2}[\mathbf{R}(t_n)+\mathbf{R}(t_{n+1})] \\ \lim \sum_n \int_{t_n}^{t_{n+1}} \sigma(s)ds\,\mathbf{R}(t_{n+1}), \end{cases} \tag{A2}$$

respectively, where $0=t_0<t_1<\cdots<t_N=t$. It is not difficult to compare the different choices. For example, the difference between the second and the first one is

$$\begin{aligned} &\tfrac{1}{2}\lim \sum_n \int_{t_n}^{t_{n+1}} \sigma(s)ds[\mathbf{R}(t_{n+1})-\mathbf{R}(t_n)] \\ &= \tfrac{1}{2}\lim \sum_n \left[\int_{t_n}^{t_{n+1}} \sigma(s)ds\right]^2 \mathbf{R}(t_n) \\ &= \int_0^t \widetilde{C}\mathbf{R}(s)ds, \end{aligned} \tag{A3}$$

where $\widetilde{C}_{il}=C_{ijjl}$ (sum over j) and the last equality is a consequence of the central limit theorem that suppresses the fluctuations in $\lim \Sigma_n[\int_{t_n}^{t_{n+1}}\sigma(s)ds]^2$. Similarly, the difference between the anti-Itô and the Itô procedure is twice the latter expression. In other words, the Stratonovich and anti-Itô versions of Eq. (12) are equivalent, respectively, to the Itô stochastic equations

$$d\mathbf{R} = \begin{cases} [\sigma(t)dt + \widetilde{C}\,dt]\mathbf{R} \\ [\sigma(t)dt + 2\widetilde{C}\,dt]\mathbf{R}. \end{cases} \tag{A4}$$

Given the stochastic Eq. (12) with a fixed convention, its solution can be obtained by iteration from Eq. (A1) and has the form (13) with $W(t)$ given by Eq. (16) and the integrals interpreted with the same convention. Note that the value of $\langle \int_0^t \sigma(s)ds \int_0^s \sigma(s')ds' \rangle$ depends on the choice of the convention: it vanishes for the Itô one and is equal to $\widetilde{C}t$ for the Stratonovich and twice that for the anti-Itô ones. The conventions are clearly related to the time reversibility of the finite-correlated strain before the white-noise scaling limit is taken. For example, time-reversible strains have even two-point correlation functions and produce the Stratonovich value for the above integrals in the white-noise scaling limit. Most of these stochastic subtleties may be forgotten for the incompressible strain where $\widetilde{C}=0$ as follows from Eq. (30) so that the difference between different conventions for Eq. (12) disappears.

We shall often have to consider functions of solutions of stochastic differential equations so one should be aware that the latter behave under such operation in a somewhat peculiar way. For the Itô convention, this is the content of the so-called Itô formula which results from straightforward second-order manipulations of the stochastic differentials and takes in the case of Eq. (12) the form

$$df(\mathbf{R}) = [\sigma(t)dt\,\mathbf{R}]\cdot \boldsymbol{\nabla} f(\mathbf{R}) + C_{ijkl}R^jR^l\nabla_i\nabla_k f(\mathbf{R})dt. \tag{A5}$$

Note the extra second-order term absent in the normal rules of differential calculus. The latter are, however, preserved in the Stratonovich convention. Note that the latter difference may appear even for the incompressible strain.

More general stochastic equations may be treated similarly. For example, Eq. (5) for Kraichnan velocities may be rewritten as an integral equation involving the stochastic integral $\int_0^t \mathbf{v}(\mathbf{R}(s),s)ds$. The latter is defined as

$$\lim \sum_n \int_{t_n}^{t_{n+1}} \frac{1}{2}[\mathbf{v}(\mathbf{R}(t_n),s) + \mathbf{v}(\mathbf{R}(t_{n+1}),s)]ds \tag{A6}$$

in the Stratonovich convention, with the last t_{n+1} (t_n) replaced by t_n (t_{n+1}) in the (anti-) Itô one. The difference is

$$\begin{aligned} &\pm \tfrac{1}{2}\lim \sum_n \int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} (\mathbf{v}(\mathbf{R}(t_n),s')\cdot\boldsymbol{\nabla})\mathbf{v}(\mathbf{R}(t_n),s)ds'\,ds \\ &= \pm \nabla_j D^{ij}(\mathbf{0})t, \end{aligned} \tag{A7}$$

where the left-hand side was replaced by its mean by virtue of the central limit theorem. The last term vanishes if the velocity two-point function is isotropic and parity invariant. Hence the choice of the convention is unimportant here even for the compressible velocities.

It may seem bizarre that the choice of convention in compressible velocities does not matter for individual trajectories but it does for Eq. (12) which describes the evolution of small trajectory differences. It is not difficult to explain this discrepancy (Horvai, 2000). The stochastic equation for $\mathbf{R}_{12}\equiv\mathbf{R}$ leads (in the absence of noise) to the integral equation

$$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t [\mathbf{v}(\mathbf{R}(s)+\mathbf{R}_2(s),s) - \mathbf{v}(\mathbf{R}_2(s),s)]ds. \tag{A8}$$

The difference between the Stratonovich and the Itô conventions for the latter integral is

$$\begin{aligned} &\tfrac{1}{2}\lim \sum_n \int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} \{[\mathbf{v}(\mathbf{R}(t_n)+\mathbf{R}_2(t_n),s') \\ &\quad - \mathbf{v}(\mathbf{R}_2(t_n),s')]\cdot\boldsymbol{\nabla}\}\mathbf{v}(\mathbf{R}(t_n)+\mathbf{R}_2(t_n),s) \\ &\quad + \tfrac{1}{2}\lim \sum_n \int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} \mathbf{v}(\mathbf{R}_2(t_n),s')\cdot\boldsymbol{\nabla}) \\ &\quad \times[\mathbf{v}(\mathbf{R}(t_n)+\mathbf{R}_2(t_n),s) - \mathbf{v}(\mathbf{R}_2(t_n),s)]ds \\ &\quad = \int_0^t \{\nabla_j D^{ij}[\mathbf{R}(s)] - \nabla_j D^{ij}(\mathbf{0})\}ds \\ &\quad\quad + \int_0^t \{\nabla_j D^{ij}(\mathbf{0}) - \nabla_j D^{ij}[\mathbf{R}(s)]\}ds = 0, \end{aligned} \tag{A9}$$

where the two first lines that cancel each other are due to the time dependence of $\mathbf{R}$ and $\mathbf{R}_2$, respectively. If we replace $\mathbf{R}$ by $\epsilon\mathbf{R}$ then, when $\epsilon\to 0$, the right-hand side of Eq. (A8) is replaced by the right-hand side of Eq. (A1) if we use the Itô convention for the stochastic integrals. The similar limiting procedure applied to the (vanishing) difference (A9) does not reproduce the difference between the values of the integral (A1) for different conventions. The latter corresponds to the limit of the first line of Eq. (A9) only and does not reproduce the limit of the term of Eq. (A9) due to the $\mathbf{R}_2$ dependence. The approximation (12) is then valid only within the Itô convention.

As for the PDF (43), it may be viewed as a (generalized) function of the trajectory $\mathbf{R}(t)$. The advection Eq. (45) results then from the equation for $\mathbf{R}(t)$ by applying the standard rules of differential calculus which hold when the Stratonovich convention is used. The Itô rules produce the equivalent Itô form of the equation with an additional second-order term containing the eddy diffusion generator $D_0\nabla^2_{\mathbf{R}}$.

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