

→ Flux Expansion and Homogenization - Non-Identities  
cont'd

- so far, have encountered:

- S-P reconnection  $\Rightarrow$  weak dissipation  
( $R_m \gg 1$ ) has strong effect of ~~weak~~  
singularity - BOUNDARY LAYER
- Taylor Hypothesis  $\Rightarrow$  small flux tubes  
destroyed by stochasticity, leaving  
 $\int d^3x \underline{A} \cdot \underline{B}$  as robust invariant

∴ diffusion dissipation most effective  
at breaking freezing-in on small  
scales

Another example:  $\left\{ \begin{array}{l} \text{singular behavior in} \\ 2D, \text{ closed-streamline flow} \end{array} \right.$

→ Homogenization Theory  $\rightarrow$   $\left\{ \begin{array}{l} \text{Prandtl, Batchelor} \\ \text{Weiss} \\ \text{Rhines, Young} \end{array} \right.$

newly  $\omega$  evolution for  
 $\underline{v} \cdot \underline{v} = 0$

$$\frac{\partial \omega}{\partial t} + \underline{v} \cdot \nabla \omega = \omega \cdot \nabla \underline{v} + \nu \nabla^2 \omega$$

$$2D \rightarrow \underline{\omega} \cdot \nabla \underline{v} = 0 \quad \omega = \omega(\frac{z}{l})$$

$$\underline{v} = \nabla \phi \times \hat{z}$$

E. S. A. Davidson] - C.P.  
2004

then  $\partial_t \omega + \underline{\sigma} \phi \times \underline{\Sigma} \cdot \underline{\sigma} \omega = \gamma \nabla^2 \omega$

more generally scalar  $\underline{z}$ :  $\left\{ \begin{array}{l} \text{active} \\ \text{or} \\ \text{passive} \end{array} \right.$

$$\partial_t \underline{z} + \underline{\sigma} \phi \times \underline{\Sigma} \cdot \underline{\sigma} \underline{z} = \gamma \nabla^2 \underline{z}$$

New:  $f \rightarrow \infty, \quad \partial_t \underline{z} \rightarrow 0$

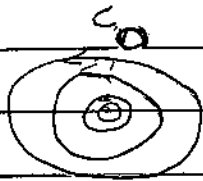
$$\underline{\sigma} \phi \times \underline{\Sigma} \cdot \underline{\sigma} \underline{z} = \gamma \nabla^2 \underline{z}$$

$\gamma \rightarrow 0 \quad \underline{\sigma} \phi \times \underline{\Sigma} \cdot \underline{\sigma} \underline{z} = 0$

$\frac{1}{2} \rho \frac{v}{r} \rightarrow \infty$   
like

$$\underline{z} = \underline{z}(\phi)$$

ie. bounded domain, closed streamline solution



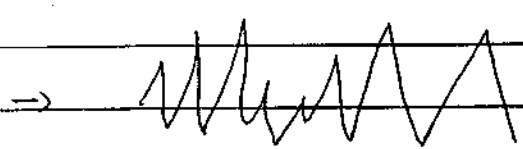
$\rightarrow \underline{z} = \underline{z}(\phi(r))$  is arbitrary solution

can develop arbitrarily fine scale  $\underline{z}(\phi)$

$\rightarrow$  closed streamlines  $\Rightarrow$  perfect memory

$\rightarrow$  " fine scale structure develops, no inter-streamline communication, & persists


ie.



ie. tag each streamline arbitrarily

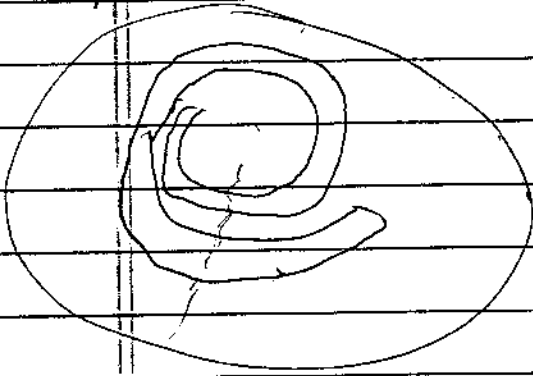
$\sim$  no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature!" 3.  
 Landau & Lifshitz

→ v.e. of 

→ blob in concentric shear flow

blow-up



→ non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for  $\nu \neq 0$   
 $Re, Re_0 \gg 1$

instead of arbitrarily fine scale structure

must have:  $u(\phi) \rightarrow \text{const}$   
 as  $\phi \rightarrow \pm \infty$

the small  $\nu$   
 $\Rightarrow$  global behavior

$\Rightarrow$  i.e. finite  $\nu$  at large  $Re \Rightarrow$

**•** vorticity homogenization,  $w \rightarrow \text{const}$  within  $C_0$

$\Rightarrow$  highly singular behavior!

$\nu = 0 \Rightarrow$  Euler Eqn. (2D)  $\rightarrow w = w(\phi)$  solns

$\nu \neq 0 \Rightarrow$  large  $Re$  2D Navier-Stokes

Eqn.  $\rightarrow w = \text{const}$  solns

Note contrast!

Issues:

→ how long to homogenization? (what means asymptotic)

→ where is  $\nabla \omega \neq 0 \Rightarrow$  boundary layer thickness?

→ analogy in MHD - Flux Expansion

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{J} \quad \underline{v} = \underline{\sigma \phi} \times \underline{\hat{z}}$$

$$\underline{B} = \underline{\sigma A} \times \underline{\hat{z}}$$

$$-\frac{1}{c} \partial_t A - \nabla \phi + (\underline{\sigma \phi} \times \underline{\hat{z}}) \times (\underline{\sigma A} \times \underline{\hat{z}}) = \eta \underline{J}$$

$\underline{\hat{z}} \cdot ( )$

$$\Rightarrow -\frac{1}{c} \partial_t A_z - \nabla \phi \cdot \underline{\hat{z}} + \underline{\hat{z}} \cdot [(\underline{\sigma \phi} \times \underline{\hat{z}}) \cdot \underline{\hat{z}}] \underline{\sigma A}$$

$$= (\underline{\sigma \phi} \times \underline{\hat{z}}) \cdot \underline{\sigma A} = \eta \underline{J}$$

$$\therefore \partial_t A + \underline{\nabla \phi} \times \underline{\hat{z}} \cdot \underline{\sigma A} = \eta \nabla^2 A$$

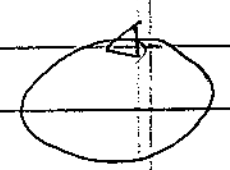
$$\Rightarrow \text{2D convection} \begin{cases} \nabla \cdot \underline{v} = 0 \\ \eta \neq 0 \end{cases}$$

⇒ expect  $\sigma A = 0$ , except boundaries  $t \rightarrow \infty$ .

→ envelope is "Flux expulsion"

(E) Prandtl - Batchelor Theorem

\* G. Batchelor, JFM 1 177 (1956) (posted)  
 P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)  
 JFM 133 130 '83  
 J. Pedlosky, "Ocean Circulation Theory"  
 see also Springer 1996, esp. 3.8.



Prandtl - Batchelor Theorem

Thm 1 Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline  $C_0$ . Then, if diffusive dissipation,

i.e. 
$$\partial_t \omega + \nabla \phi \times \vec{s} \cdot \nabla \omega = \nabla \cdot (\nu \nabla \omega)$$

then, vorticity  $\rightarrow$  uniform (homogenization), as  $f \rightarrow \infty$ , within  $C_0$ .

M.B.: finite  $\nu \Rightarrow$  radically different final state

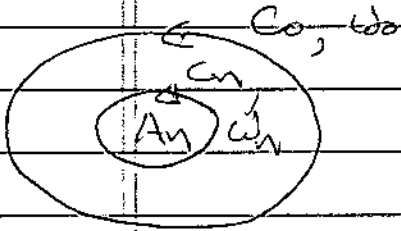
(B) no comment on "how long"

$$\nabla \phi \times \underline{\Sigma} \cdot \nabla \omega = \underline{\nabla} \cdot \underline{r} \nabla \omega$$

for stationarity

[note  $f \rightarrow \infty$  define  
 $r \rightarrow 0$ ]

- choose arbitrary closed  $C_n$  within  $C_0$ .  
Here  $C_n$  a streamline



n.b. - assume simply connected  
region, i.e. no holes  
- stationarity  $\Rightarrow$   
 $\omega$  constant along streamlines

$$\therefore \omega \rightarrow \omega_0 \text{ on } C_0 \text{ (ultimately } C_0 \text{ satis b.c.)}$$

$$\omega \rightarrow \omega_n \text{ on } C_n$$

if  $A_n$  is area enclosed by  $C_n$

$$\int_{A_n} d^3x \underline{v} \cdot \underline{\nabla} \omega = \int_{A_n} d^3x \underline{\nabla} \cdot (\underline{r} \nabla \omega)$$

but

$$\int_{A_n} d^3x \underline{v} \cdot \underline{\nabla} \omega = \int_{A_n} d^3x \underline{\nabla} \cdot (\underline{v} \omega)$$

$$= \int_{C_n} d\ell \hat{n} \cdot (\underline{v} \omega)$$

→ streamline,  $C_n$

now

$\vec{v}$   
 $\vec{n}_{C_n}$  → normal to streamline  $C_n$

$$\int_{C_n} d\ell (\vec{n}_{C_n} \cdot \vec{v}) \omega = 0$$

as  $\vec{v}$  is along streamline

$$0 = \int_V d^3x \nabla \cdot (v \nabla \omega)$$

$$= \int_{C_n} v d\ell \vec{n}_{C_n} \cdot \nabla \omega$$

now in stationary state, must have  $\omega \rightarrow$  const along streamline

$$\omega = \omega(\phi)$$

$$\text{so } \omega_{C_n} = \omega(\phi_n)$$

$$0 = v \int_{C_n} d\ell \vec{n}_{C_n} \cdot \nabla \phi_n \frac{d\omega}{d\phi_n}$$

$$= v \frac{d\omega}{d\phi_n} \int_{C_n} d\ell \vec{n}_{C_n} \cdot \nabla \phi_n$$

but

$$\Gamma^* = \int d\ell \cdot v$$

$$= \int d\ell \cdot (\nabla\phi \times \hat{\ell})$$

$$= \int (\hat{\ell} \times \vec{u}) \cdot (\nabla\phi \times \hat{\ell})$$

$$= - \int d\ell (\nabla\phi \cdot \vec{u}) = - \int d\ell (\nabla\phi \cdot \vec{n})$$

$$0 = v \frac{\partial \omega}{\partial \phi_n} \Gamma_n$$

$$\partial\omega / \partial \phi_n = 0$$

but  $\phi_n$  arbitrary  $\Rightarrow \partial\omega / \partial \phi = 0$ , all  $\phi$

arbitrary  $\Rightarrow$  no variation from line to line

$\Rightarrow$   $\omega$  homogenized !

so, expect  $\partial\omega$  larger at bounding contour  $C_0$

$\partial\omega \rightarrow 0$  within  $\Rightarrow \partial\omega$  held at boundary



Some Comments:

⇒ Homogenization theory looks 'magical' → caveat emptor

i.e.

- 1.) note assumptions of
  - $t \rightarrow \infty$  ⇒ time asymptotic
  - $\Omega = \Omega(\epsilon)$  ⇒ concentric streamlines

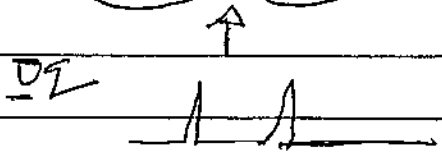
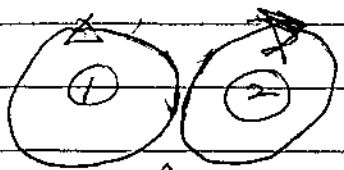


how long to achieve configuration

- 2.) simply connected domain → annulus?

- 3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → (interference of boundary layers?)



⇒ straining interaction  
 ⇒ ② 'staps' ① etc.

4.) Key Assumptions:

→ closed, bounding streamline  
(viscous dissipation

i.e. can envision:

→ exact streamline, molecular viscosity

or

→ coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of

→ total vorticity

→ mean/coarse-grained vorticity

→ time scales different

→  $\frac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \Rightarrow R_{\omega} \gg 1$  (L)

- to establish concentric circulation lines

then

- diffusion across to homogenize → but slow!!

$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$

i.e.  $Re \gg 1$

or equivalently  $\frac{VL}{D} \gg 1$

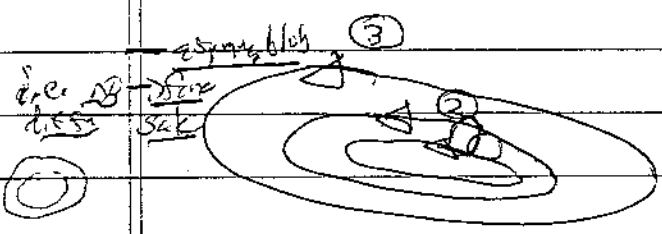
i.e.  $(Re)_{\text{eff}} \gg 1$  ~~cell~~  $\rho \gg 1$

related: - essential idea is that  $q$  constant along streamlines established on fast ( $\sim \tau_c$ ) scale

- dissipation homogenizer on slower ( $\sim \tau_D$ ) time scale (but this is slow...)

→ What are the time scales? -  $\left\{ \begin{array}{l} \text{slow} \\ \text{resolve} \\ \text{slow time} \\ \text{scale problem} \end{array} \right.$

- useful to consider differentially rotating sheared flow within closed pattern



$v_1 \neq v_2 \neq v_3$   
need blob with finite  $ly$


what is the mixing time scale?

① shear, shear scale...

② key: synergism between  $\left\{ \begin{array}{l} \text{shear} \\ \text{diffusion} \end{array} \right.$

c.f. { H. Burglar, P.H. Diamond, P.W. Terry  
Phys Fluids (B2), 1, 1990  
(first noted by G.I. Taylor)

Mixing Shear Dispersion

i.e. compare  time  $\Delta t \rightarrow$

radial diffusion

(a) pure diffusion

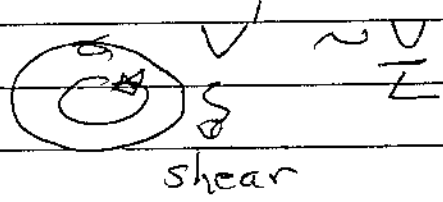
$$1/r \sim D/t^2$$

$$\langle dr^2 \rangle \sim Dt$$

$$D_r \sim \langle V_r^2 \rangle t$$

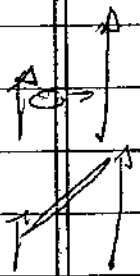
→ used diffusion any radial scattering process (unspecified)

(b) diffusion shear hybrid



now  $\frac{dr}{dt} = \frac{V}{r}$

→ random walk



$$r\theta = y$$

$$\frac{dy}{dt} = V_y(r)$$

→ streaming

$$\frac{d}{dt} dy = \left( \frac{\partial V_y}{\partial r} \right) dr$$

$$dy = \int \left( \frac{\partial V_y}{\partial r} \right) dr dt$$

$$\langle dy^2 \rangle \sim \left( \frac{\partial V_y}{\partial r} \right)^2 \langle dr^2 \rangle t^2$$

$$\langle dr^2 \rangle \sim Dt$$

shear dispersion

$$\langle dy^2 \rangle \sim \left( \frac{\partial V_y}{\partial r} \right)^2 Dt^3$$

→ hybrid decorrelation  $\langle dy^2 \rangle \sim t^3$

scale of comparison



$\langle \sigma_y^2 \rangle \sim L_y^2 \Rightarrow$  arbitrary  $1/3$

$1/\tau_{mix} = \left( \frac{\partial v_y}{\partial x} \right)^2 \frac{D}{L_y^2}$

$\sim \left( \frac{V_0}{L_y} \right)^2 \frac{D}{L_y^2}$

$\sim \frac{V_0}{L_y} \left( \frac{D}{V L_y} \right)$

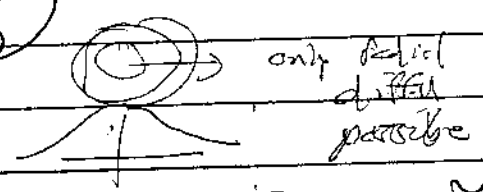
$1/\tau_{mix} \sim \frac{1}{\tau_c} (Re)^{1/3}$

$\left\{ \begin{array}{l} Re \gg 1 \\ \text{by construction} \\ \rightarrow \text{consistent} \checkmark \end{array} \right.$

so have:

→ mixing/homogenization on hybrid time scale → time to come to symmetric state

$1/\tau_{mix} = 1/\tau_c \left( \tau_c/\tau_0 \right)^{1/3}$



only radial diff possible

→  $\frac{1}{\tau_c} > \frac{1}{\tau_{mix}} > \frac{1}{\tau_0}$  time to uniformize vs  $\tau_0$

⇒ PV homogenization most relevant to closed eddies with sheared rotation

Some Points

i.) Time scales

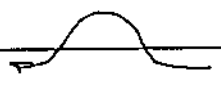
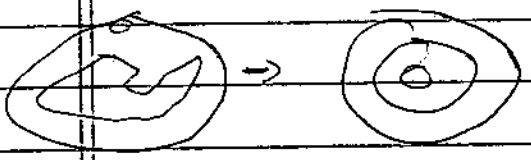
have  $Re, Pe \gg 1 \Rightarrow \frac{\tau_D}{\tau_c} \gg 1$

but  $\tau_{mix} < \tau_D$   $\tau_{mix} \sim Re^{1/3} \tau_c \sim \frac{\tau_D}{Re^{2/3}}$

time to establish  $\rightarrow$  time to homogeneity

azimuthally symmetric state

ie



but radially profiled

ii.) Point of theorem is global impact of small dissipation.

iii.) Interesting to note that P-B theorem applies to both active, passive scalar.

ii) Observe: all that is really required for applicability of theory is:

- incompressible advection - 2D:  $\nabla \phi \times \hat{z} \cdot \nabla$
- closed streamlines  $\rightarrow \begin{cases} \text{fine} \\ \text{coarse} \end{cases}$
- diffusive dissipation  $\rightarrow \begin{cases} \text{molecular} \\ \text{eddy} \end{cases}$

d.e. can apply to magnetic potential, as noted previously, i.e.

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \nabla A = \mu \nabla^2 A$$

2D MHD  $\frac{u}{B} = \sigma \phi \times \hat{z}$   
 $E + v \times B = \mu J$

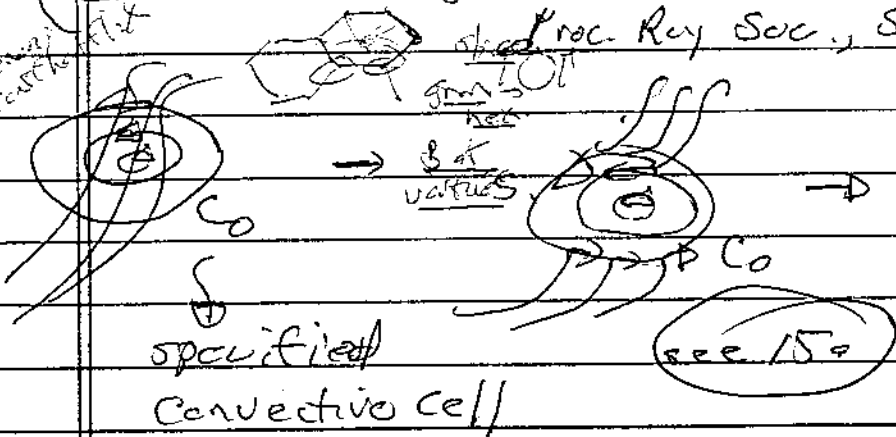
$$\frac{\partial A}{\partial t} - \sigma \phi + (\sigma \phi \times \hat{z}) \cdot (\nabla A \times \hat{z}) = \mu J$$

$$\frac{\partial A}{\partial t} + (\sigma \phi \times \hat{z}) \cdot \nabla A - (\sigma \phi \times \hat{z}) \cdot \nabla A = \mu J$$

→ Famous problem of Flux Expulsion

(i.e. N. Weiss)

Max  
Miles  
at Earth



Proc. Roy. Soc., Series A 293, 310, 1966

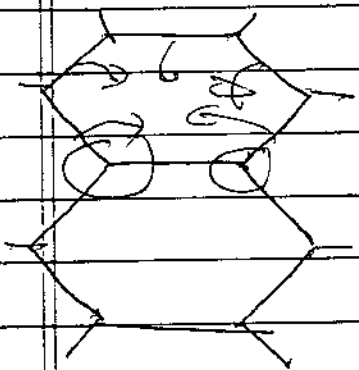
$\nabla A \neq 0$  within  
 → B expelled to boundary (i.e.  $B = \nabla A \times \hat{z}$ )  
 →  $B \circ$  in cell.

(has magnetoconvection as aim)

→ obvious that above argument can be recycled, so

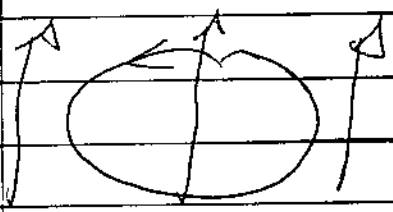
$$\nabla A / \sigma \phi = 0, \text{ for all } \phi \text{ on } C_0$$

c.e. top view  $\rightarrow$  solar granulation



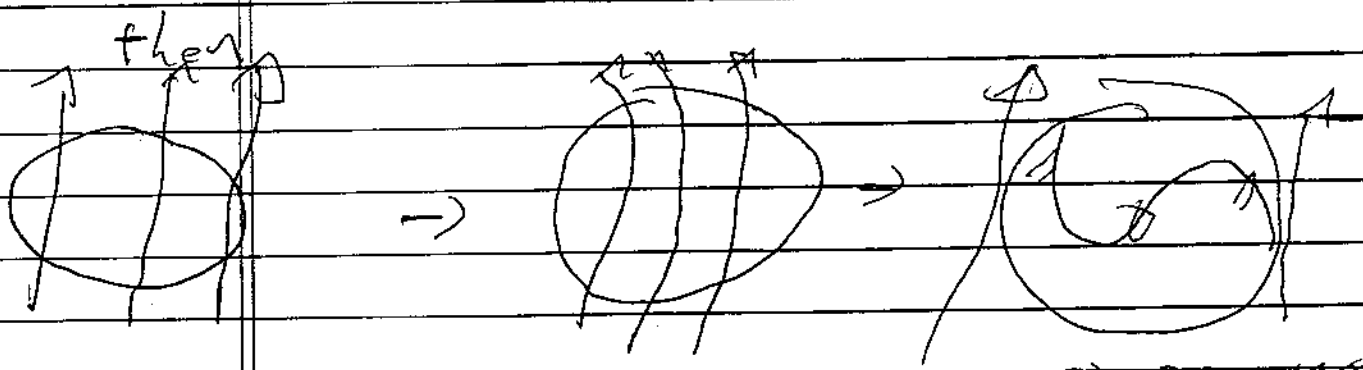
$\sim$  hexagonal pattern  
 $\sim$  field strength at vertices  
 $\sim$  expulsion

suggests  $\rightarrow$  side view  $\rightarrow$  toy problem of flux expulsion



$\rightarrow$  c.e.

cell in uniform field



see Weiss, Proc. Roy Soc. A 293, 310-1966

also Moffatt, 3-7-3.10



→ here requirement is  $R_m = \frac{LV}{\eta} \gg 1$

and

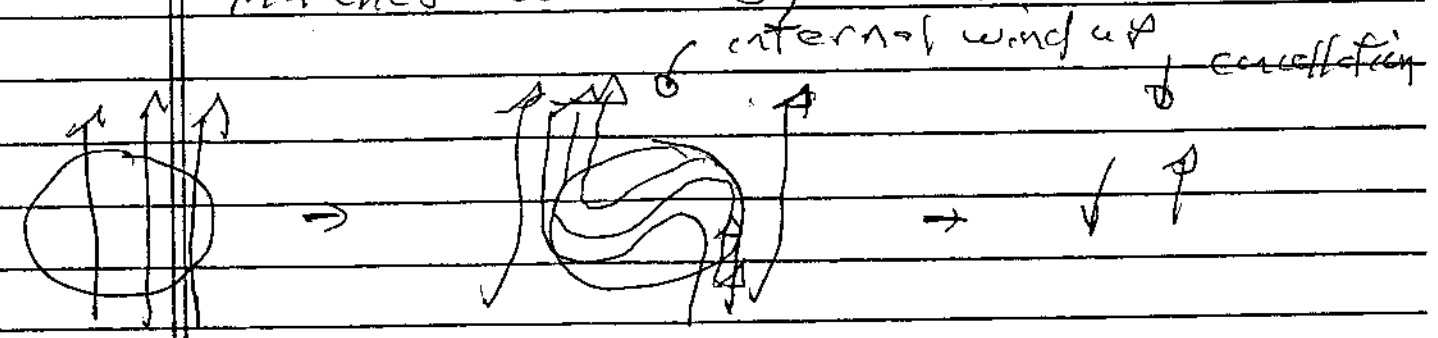
$$\frac{1}{\rho} \frac{d\rho}{dt} \sim \frac{1}{\tau_{mix}} \sim \frac{V_0}{L_0} (R_m)^{-1/3}$$

time scale for flux expulsion

→ physically, can see relevant time scale by noting:

- ~~wind up~~ must conserve volume/mass
- wind up must conserve flux
- irreversibility sets in when  $R_m \sim 1$

⇒ dissipation of field local matches drive by wind-up

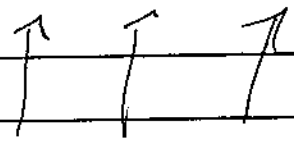


internal cancellation → { Field expulsion  
Flux homogenization

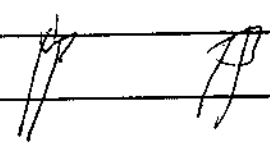
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- field stretched/compressed in  $\eta$ -wards

$$\eta l = L_0$$



$$l = L_0/\eta$$



- Flux (vertical) conserved, so

so  $B_{\text{alt}} \sim l B$  (B T upon flux conservation)

$\Rightarrow B \sim n B_0$

$\rightarrow$  now expect freezing-in lost when compressed field

$(Rm)_{\text{eff}} \sim \tau l \Rightarrow \frac{V B_0}{L_0} \sim n \frac{B}{l^2} \sim n \frac{n B_0}{L_0^2/n^2}$

$\Rightarrow n^3 \sim \left( \frac{V_0 L_0}{\eta} \right) \Rightarrow n \sim Rm^{1/3}$

$\left\{ \begin{array}{l} \text{thickness of layer} \\ \frac{V B}{L} \sim \frac{n B}{l^2} \end{array} \right. \quad \left. \begin{array}{l} \text{\# of turns to} \\ \text{render boundary} \\ \text{diffuse} \end{array} \right. \quad \frac{\delta}{L} \sim \eta / Rm$

but: - diffusion in boundary layer  $\Leftrightarrow$  homogenization within

so -  $n \sim Rm^{1/3} \Rightarrow$  # turns for homogenization

-  $\tau_{\text{hom}} \sim \tau_c Rm^{1/3} \rightarrow$  time  $\checkmark$

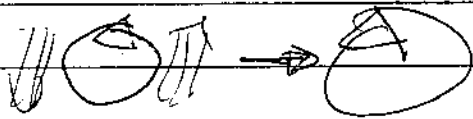
agree above

⇒ expect flux expelled from closed cell

→ field strongest at boundary

→ possibly explain why strongest cells in 2D

magneto convection often reach a state independent of  $B$  while neighbours quenched.



Now: ⇒ why care about this?  
Why homogenization important

→ statistical mechanics

① identifies a trend; i.e. in spirit of Taylor Theory ( $E$  may minimized s/t

$\int \underline{A} \cdot \underline{B} d^3x$  conserved), homogenization

theory identifies a trend, i.e.

if  $F$  — conserved locally by  
2D flow,  $\nabla \cdot \underline{v} = 0$   
diffused  
— enclosed

⇒  $F$  homogenized

② trend applies to ~~non passive~~ → verticality  
~~passive~~ →  $\int C$

In particular

③ trend severely constrains form of  
verticality flux, flow evolution

i.e. zonal flows → 2D closed streamline  
flows

→ Do zonal flows tend homogenize PV?

→ if raise (i.e. emission), what scale  
selected?