ELECTRON MAGNETOHYDRODYNAMICS

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1. GENERAL CONCEPTS

This review deals with a range of plasma phenomena which cannot be described and understood in terms of ordinary magnetohydrodynamics (MHD), but do allow a physically intuitive and relatively simple representation in the approximation of electron magnetohydrodynamics (EMH). EMH is a limiting case of multicomponent MHD in which the motion of the ions can be neglected and the motion of the electrons maintains quasineutrality.

As will be seen below, the EMH case is usually realized when the characteristic scale length is small, the characteristic times are short, and the current flow velocities are large compared to the mass velocity. This often involves a collisionless or weakly collisional plasma. The hydrodynamic description of these plasmas has been in use for some time [1–5] and the general limits for this approach have been established more or less reliably. In particular, single-component MHD is an approximation based on the smallness of the relative velocities of the components (in the two-component case, the current flow velocity is $u \equiv j/ne$) compared to the average mass velocity, $|v_\alpha - v^\beta| \ll v$. Generally speaking, when this inequality is in the opposite direction, EMH applies.

It is true that no universal criteria exist for the applicability of the hydrodynamic approach in plasma physics, or equivalently, for the applicability of MHD or EMH. Thus, the single fluid approximation may be valid even when the above inequality is violated. In most concrete problems, however, this question causes no special difficulty. As an example, we might note the applicability of MHD to large-scale motion with $a \gg$

c/ω_{pi} (this corresponds to the condition $u \ll v_A \equiv B/\sqrt{4\pi\rho}$, where v_A is the characteristic scale for the hydrodynamic velocity). This is a fairly widespread situation in astrophysical problems and, indeed, Alfvén, the founder of magnetohydrodynamics, started out with this case. For laboratory devices the spatial scales are generally much smaller and the density is much higher, so that the opposite inequality may be satisfied. Then $v_i \ll u \simeq v_e$ and the ion motion can be neglected.

The first problems in this approximation were solved in the 1960s by Morozov, Bryzgalov, and Shubin [6, 7]. They were essentially studying the Hall effect. In the steady state it obeys the equation

$$\operatorname{rot}[\mathbf{j},\ \mathbf{B}] = 0. \tag{1.1}$$

The case of short characteristic times was examined by Gordeev and Rudakov [8], who used the purely electron equation

$$\frac{4\pi e}{c}\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}\left(\frac{1}{n}[\mathbf{B},\ \operatorname{rot}\mathbf{B}] + \frac{4\pi}{n}\nabla P_e\right) \tag{1.2}$$

to describe nonpotential high-frequency instabilities.

Further development of the theory, however, was constrained by two circumstances. First, there is the multidimensionality of the problems. Unlike the case of single-fluid hydrodynamics, no meaningful example of one-dimensional EMH exists. All EMH effects are at least two-dimensional. Second, comparison with experiment is complicated. Plasmas that evolve in accordance with EMH are, as a rule, short-lived and small-scale objects, while the diagnostics for them leave much to be desired. Nevertheless, in recent years interest in EMH has grown considerably in connection with the need to describe high-energy plasmas (primarily inertial confinement fusion systems). Recently it has served as a basis for studies of the instabilities of high-current ion beams in plasma channels [9], the generation of magnetic fields and the filamentation of particles in laser flares [10], the dynamics of fast Z-pinches [11], field penetration in the skin effect [12, 13], and a number of other effects.

We now demonstrate the transition to the EMH approximation beginning with the system of equations for two-component MHD,

$$\left.\begin{aligned} \frac{d\mathbf{p}_e}{dt} &= -e\mathbf{E} - \frac{\nabla P_e}{n} - \frac{e}{c}[\mathbf{v}_e,\ \mathbf{B}] + \frac{e}{\sigma}\mathbf{j},\\ \frac{d\mathbf{p}_i}{dt} &= Ze\mathbf{E} - \frac{Z\nabla P_i}{n} + \frac{Ze}{c}[\mathbf{v}_i,\ \mathbf{B}] - \frac{Ze}{\sigma}\mathbf{j},\\ \frac{\partial \mathbf{B}}{\partial t} &= -c\operatorname{rot}\mathbf{E}. \end{aligned}\right\} \tag{1.3}$$

To simplify the later calculations, we set σ = const and $P_\alpha = P_\alpha(n_\alpha)$. Then Eqs. (1.3) reduce to a system of two equations,

$$\left.\begin{aligned} \frac{\partial}{\partial t}\,\mathrm{rot}\,\pi_e &= \mathrm{rot}\,[\mathbf{v}_e,\ \mathrm{rot}\,\pi_e] - \frac{e}{\sigma}\,\mathrm{rot}\,\mathbf{j};\\ \frac{\partial}{\partial t}\,\mathrm{rot}\,\pi_i &= \mathrm{rot}\,[\mathbf{v}_i,\ \mathrm{rot}\,\pi_i] + \frac{Ze}{\sigma}\,\mathrm{rot}\,\mathbf{j}, \end{aligned}\right\} \tag{1.4}$$

where $\boldsymbol{\pi}_\alpha = \mathbf{p}_\alpha + (e_\alpha/c)\mathbf{A}$ is the generalized momentum of a given component. In the limit of single-component MHD ($\mathbf{p}_e \to 0$, $\mathbf{v}_e \approx \mathbf{v}_i$) with a conductivity $\sigma \to \infty$, the system of equations (1.4) reduces to the well-known equation for a field frozen in a material. In the general case, ideal conductivity leads to freezing of each component of the curl of the generalized momentum of a given component. (This generalization for single-component MHD is given in the review by Braginskii [1].) This statement remains valid even in the relativistic case and when quasineutrality is violated.

The EMH equation is, strictly speaking, the electron equation (1.4), where one can set $\mathbf{v}_e = \mathbf{j}/(ne)$ when $a \ll c/\omega_{pi}$ and solve it independently of the ion equation. The ions merely create a motionless ($\mathbf{v}_i \ll u$) background for the fast electron flows. If, in addition,

$$\partial/\partial t \ll \omega_{Be}, \quad a \gg c/\omega_{pe} \tag{1.5}$$

(i.e., the field component in $\boldsymbol{\pi}_e$ predominates), then this equation can be simplified and reduced to

$$\frac{\partial \mathbf{B}}{\partial t} + \mathrm{rot}\left[\frac{\mathbf{j}}{ne},\ \mathbf{B}\right] = -\frac{c}{\sigma}\,\mathrm{rot}\,\mathbf{j}, \tag{1.6}$$

which describes the freezing of **B** in the electron current and the diffusion of the field. The second EMH equation (the continuity equation) can be written as $\mathrm{div}\,\mathbf{j} = 0$ when $\partial/\partial t \ll \omega_{pe}$ (or the displacement current is small and the electron flow is quasineutral against the ion background). It is identically satisfied because of the Maxwell equation

$$\mathrm{rot}\,\mathbf{B} = \frac{4\pi}{c}\,\mathbf{j}. \tag{1.7}$$

When $\sigma \to \infty$, Eqs. (1.4) [as well as Eq. (1.6)] can be written in a Hamiltonian form. The canonical variables λ and μ found by Zakharov [14] ($\mathrm{rot}\,\boldsymbol{\pi}_e = [\nabla\lambda, \nabla\mu]$) are related to the Clebsch variables of ordinary hydrodynamics. When the inertia of the electrons can be neglected, **B** =

$[\nabla\lambda, \nabla\mu]$. For configurations with linked field lines **B** ($\int \mathbf{B}\mathbf{A}d^3\mathbf{r} \neq 0$), unique Clebsch variables cannot be introduced, but this presents no obstacle to obtaining a Hamiltonian form [14]. In the EMH approximation the main term in the Hamiltonian (and often the only term) is the energy of the magnetic field.

Sometimes the assumption of an isentropic flow, i.e., P and n related by an adiabatic equation, is used to justify Eq. (1.6). This, in fact, is unnecessary. An arbitrary form for $P(n)$, as assumed in deriving Eqs. (1.4), is fully satisfactory. As in the case of an isentropic flow, this requires that the current flow velocity be fairly large compared to the drift velocity, i.e.,

$$u \gg v_{Te}\rho_{Be}/a. \tag{1.8}$$

If Eq. (1.8) is not satisfied, we cannot neglect the fact that electrons are arriving at each point at a given time from substantially different points. Using the estimate $B \sim (c/4\pi)ja$, we arrive at the condition

$$\beta = 8\pi n T_e/B^2 \ll 1. \tag{1.8a}$$

Condition (1.8a) limits the validity of Eq. (1.6), but the opposite inequality does not in general exclude the applicability of EMH. Frequently it is enough to "cut" the connection between P and n. Then terms of the form $[\nabla n, \nabla T]$, which are responsible for generating the magnetic field, appear on the right-hand side of Eq. (1.6). The natural limit for applicability of the hydrodynamic approach when $T \neq 0$, however, is the condition $\rho_{Be} \ll a$ [1–5].

The foundation of EMH is thus Eqs. (1.6) and (1.7) and the following discussion will be based on them. Depending on the nature of the problem, Eq. (1.6) can be supplemented by terms corresponding to the inertia of the electrons, to effects associated with the electron pressure (in particular, the mechanism for generating a magnetic field owing to nonparallel ∇n and ∇T), and so on.

The choice of specific examples and models is mainly related to the scientific interests of the authors who have, however, tried to discuss the most important aspects of EMH. Thus, Chapter 2 is devoted to a study of the main mechanism for the evolution of the magnetic field in EMH, namely current transport. Since Eq. (1.6) is written in curl form, EMH flows have a rotational character. [As can be seen from Eqs. (1.3) and (1.4), Maxwell's equations play a not insignificant role in this.] The general properties of such flows in the two-dimensional case and the difference from three-dimensional situations are examined in Chapter 3. Three-dimensional turbulence is discussed in Chapter 4. Also studied there is a specific EMH effect, namely EMH resistance. Chapter 4 is the most important in ideological terms. Problems which arise during practical applications of EMH to real situations are discussed in Chapter 5, using as an example the Z-pinch, one of the most widespread devices in plasma physics. One important application of EMH may be the modelling of kinetic effects by, for example, representing the electrons as two or more fluids with different temperatures and hydrodynamic velocities [15]. Chapter 6 is devoted to this approach. It is close to the multibeam model in the theory of collective phenomena, only in our case the main topic is the generation and transport of magnetic fields. Chapter 7 gives a review of experimental situations which fall in the framework of EMH, and the prospects for comparing theory and experiment are discussed.

2. CONVECTIVE SKIN PHENOMENA IN PLASMAS

2.1. Nonlinear Skin Effect

One of the most characteristic properties of EMH is the transport of a magnetic field by a current. This property shows up especially clearly in the skin effect.

Up to now the penetration of an external magnetic field into a plasma has often been examined on the basis of conventional concepts of diffusion. In reality, diffusion may be strongly supplemented or even exceeded by convective transport.

In terms of EMH the magnetic field dynamics obey the formulas [cf. Eq. (1.6)]

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot}\left[\frac{\mathbf{j}}{ne}, \mathbf{B}\right] + c \operatorname{rot} \frac{\mathbf{j}}{\sigma} = 0 \tag{2.1}$$

and

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{rot} \mathbf{B}, \quad \sigma = \frac{ne^2}{m} \tau_e. \tag{2.2}$$

In general, when the inequality $\omega_{Be}\tau_e \gg 1$ is satisfied, the second term of Eq. (2.1), which is responsible for transport of the field frozen in the electrons at the current flow velocity $j/(ne)$, will dominate the third, diffusion term. (Note that because of the fundamental multidimensionality of the problem, in specific situations a more careful estimate may be required. In this inequality unity must be replaced by a geometric factor equal to the ratio of the characteristic scale lengths in two mutually perpendicular directions, along **j** and perpendicular to **j**.) These qualitative discussions are fully confirmed by the exact solution of the problem, to which we now proceed.

In accordance with the features of the phenomenon under consideration, the most typical initial condition for Eqs. (2.1) and (2.2) is a jump in the field at the boundary between the plasma and the vacuum. Since the current flow velocity is much greater than the hydrodynamic flow velocity, the motion of this boundary can be neglected. Thus, let the plasma occupy the half space $z > 0$ and at time $t = 0$ let $B = 0$ inside it, while at all times t the field is a constant $\mathbf{B}_0$ at the boundary. We first consider the plane case with $\mathbf{B}_0 \| \mathbf{e}_y$ and $\partial/\partial y = 0$. In this simplest of geometries, Eq. (2.1) can degenerate into a conventional diffusion equation. Indeed,

$$[\mathbf{j}, \mathbf{B}] = -c \frac{\nabla B^2}{8\pi} + \frac{c}{4\pi} (\mathbf{B}\nabla) \mathbf{B}.$$

The second term on the right, which only could give a nonzero contribution to rot [**j**, **B**], is identically equal to zero because there is no curvature in the magnetic field lines. The convective term, however, does not go to zero if we include a possible gradient in the plasma density. Then (σ = const)

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{c}{8\pi e} \left[\nabla B^2, \nabla \frac{1}{n} \right] = \frac{c^2}{4\pi\sigma} \Delta \mathbf{B}. \tag{2.3}$$

Let the density gradient be parallel to the plasma boundary and perpendicular to B (Fig. 1). Then the nonlinear convective term causes transport of the field in the z direction. When the depth of penetration is sufficiently small, as determined by the inequality $\partial^2/\partial z^2 \gg \partial^2/\partial x^2$, Eq. (2.3) transforms to the Burgers equation

$$\frac{\partial B}{\partial t} + kB \frac{\partial B}{\partial z} = D \frac{\partial^2 B}{\partial z^2}, \tag{2.4}$$

where $k = (c/4\pi e)(\partial/\partial x)n^{-1}$ and $D = c^2/4\pi\sigma$.

It is clear that at early times (more precisely, for $t < D/(k^2 B_0^2)$, while the gradient in B is large, the penetration of the field into the plasma is determined by diffusion, as usual (here the geometric factor mentioned above also appears). After the B profile becomes fairly flat, however,

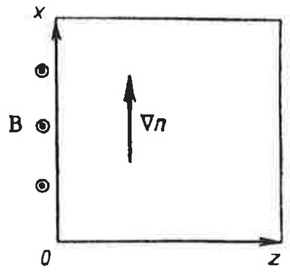

Fig. 1. The geometrical configuration for the problem of field penetration into a plasma owing to a density gradient.

transport of the field by the current-carrying electrons becomes predominant and the character of the solution becomes very different.

If the vectors **B** and ∇n and the propagation direction $\mathbf{e}_z$ form a left-handed triplet, as in Fig. 1, then $k < 0$ in Eq. (2.4) and nonlinear field transport begins to compete with diffusive transport. As $t \to \infty$ this leads to a steady state with

$$B = \frac{B_0}{1 - (kB_0/2D)\,z}. \tag{2.5}$$

It is not difficult to trace the evolution of this solution. The substitution $B = -(2D/k)(\partial/\partial z)\ln|\varphi|$ is known to convert the Burgers equation into a linear diffusion equation [16]. Its solution corresponding to the boundary conditions at $z = 0$ has the form [13]

$$\varphi = -\operatorname{erf}\left(\frac{z}{2\sqrt{Dt}}\right) + \exp\left[\frac{kB_0}{2D}\left(z + \frac{kB_0}{2}\,t\right)\right]\operatorname{erfc}\left(\frac{z + kB_0 t}{2\sqrt{Dt}}\right). \tag{2.6}$$

It is easy to see that as $t \to \infty$, we have

$$\varphi \to \frac{1}{\sqrt{\pi Dt}}\left(\frac{2D}{kB_0} - z\right),$$

which corresponds to Eq. (2.5).

When the vectors B_0 and ∇n are oriented oppositely and $k > 0$, the convective terms also cause transport of the magnetic field into the plasma, even with ideal conductivity $\sigma \to \infty$ ($D \to 0$). Field penetration takes place in the form of a travelling wave moving at a constant velocity $kB_0/2$, whose leading edge is controlled by the competition between the nonlinearity and diffusion:

$$B=\frac{B_0}{2}\left[1-\operatorname{th}\frac{kB_0}{4D}\left(z-\frac{kB_0}{2}t\right)\right]. \tag{2.7}$$

The formation of this wave can also be studied analytically.

If $1/n(x)$ has a minimum at some x, which can be set equal to zero, then in the upper half plane the field will penetrate into the plasma in the form of the wave (2.7) and in the lower half plane it will approach the steady-state profile (2.5) in accordance with Eq. (2.6).

A gradient mechanism for field penetration into a forbidden region $z > \delta_{sk}$ is known in solid state physics [17, 18]. Whereas it is related to ∇n in the present case, in a solid the temperature gradient plays an analogous role (Nernst–Ettingshausen effect). References 17 and 18 dealt with linear thermomagnetic waves produced by this effect.

In a more general geometry, where the magnetic field lines are not straight, the equation for the field evolution is still nonlinear,

$$\frac{\partial \mathbf{B}}{\partial t}+\frac{c}{4\pi ne}\operatorname{rot}(\mathbf{B}\nabla)\mathbf{B}=\frac{c^2}{4\pi\sigma}\Delta\mathbf{B}, \tag{2.8}$$

even in a plasma with a uniform density. For a cylindrical geometry $\mathbf{B}\parallel \mathbf{e}_\varphi \times \partial/\partial\varphi \equiv 0$, in the approximation of a small penetration depth $\partial/\partial z \gg 1/r$, it again reduces to a Burgers equation (2.4) with $k=-c/(2\pi ner)$, so that the convective effects mentioned above (trapping of the field at the boundary and its rapid penetration) also occur.

Note that although the Burgers equation is highly nonlinear, linear analogs of these phenomena are well known, both in solids and in plasmas: drift waves and helicons propagating against a large constant background magnetic field [4, 19]. It might be pointed out that in the nonlinear skin effect the finite value of B simultaneously serves as the amplitude of the wave and of the external field in which it propagates.

We should emphasize one important fact. The Burgers equation (2.4) is not only nonlinear, but is also exactly integrable. Its solutions are uniquely related to the solutions of the linear heat conduction problem. For this reason, the problem of magnetic field penetration into a plasma in terms of EMH allows an exact analytic solution in a rather standard statement of the problem, despite the strong nonlinearity of the equations.

2.2. The Skin Effect in the Presence of Charged-Particle Beams

The injection of energetic beams of charged particles into a plasma can, in principle, strongly change the situation examined above. The medium becomes a multicomponent system. Beams of this sort, however, can often be introduced into the system of EMH equations as an "external" current, which contributes to the Maxwell equation, but not to Ohm's law [20–22] [cf. Eqs. (2.1) and (2.2)]:

$$\mathrm{rot}\,\mathbf{B} = \frac{4\pi}{c}(\mathbf{j}_b + \mathbf{j}); \quad \mathbf{E} = -\frac{1}{nec}[\mathbf{j},\ \mathbf{B}] + \frac{\mathbf{j}}{\sigma}. \tag{2.9}$$

In this approximation the beams do not introduce qualitatively new effects in the convective field transport, but they make the problem much more diverse. We begin by considering the limits of applicability of this approximation.

First, representing a beam as an external current $\mathbf{j}_b$ which is independent of the magnetic field means that the distortion in the trajectories of the beam particles is small, i.e., the inequality $\rho_b \gg a$ is satisfied, where a is the characteristic scale length of the problem, or, equivalently, the mechanical component is dominant in the generalized momentum, $p_b \gg (e_b/c)A$. This condition can be written in a third form, namely that all currents in the problem be small (including the beam current) compared to the Alfvén current of the beam (for a geometric factor on the order of unity):

$$I \ll I_{Ab} \equiv \frac{m_b c^3}{e_b}\beta_b\gamma_b.$$

Second, this representation means that the Coulomb drag of the particles in the medium is neglected because of their rather high energy. This is possible when $j_b/(n_b e) \gg v_{Te}$. Otherwise, it is generally necessary to include the increase in the plasma electrons owing to the beam particles in the second of Eqs. (2.9).

The failure of these conditions is discussed in Chapter 6. If, however, they are satisfied, then instead of Eq. (2.8) we obtain

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{c}{4\pi ne}\,\mathrm{rot}\,(\mathbf{B}\nabla)\,\mathbf{B} - \frac{1}{ne}\,\mathrm{rot}\,[\mathbf{j}_b,\ \mathbf{B}] = \frac{c^2}{4\pi\sigma}\,\Delta\mathbf{B} + \frac{c}{\sigma}\,\mathrm{rot}\,\mathbf{j}_b. \tag{2.10}$$

In deriving Eq. (2.10) we have set $n = \mathrm{const}$ in order to focus attention on the effects related to $\mathbf{j}_b$. According to Eq. (2.10) these effects include volume field production by the external current (this is also exam-

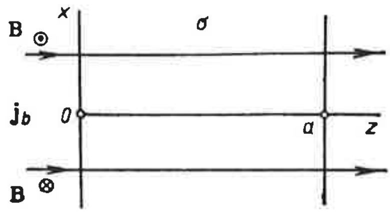

Fig. 2. The geometrical configuration for the problem of field transport into a plasma by a particle beam.

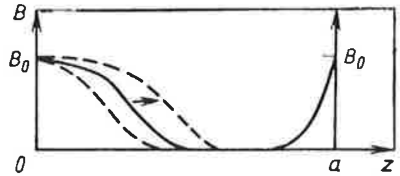

Fig. 3. Transport of a field by a beam (plane geometry).

ined in Chapter 6) and linear convection of the magnetic field by the reverse current of the beam, which obeys the last term on the left of Eq. (2.10) and is discussed below.

In order to illustrate this, we turn again to a plane model problem. Let a beam of "ribbon" or "knife" geometry penetrate a plane layer of plasma $0 < z < a$ with $\mathbf{j}_b \parallel \mathbf{e}_t$ and $\mathbf{B} \parallel \mathbf{e}_y$ (Fig. 2). At the initial time $t = 0$ the reverse current in the conducting medium is completely compensated by the beam current and $\mathbf{B} = 0$. Outside the conducting layer at any time, $\mathbf{B} = \mathbf{B}_0$, where $\mathbf{B}_0$ is the magnetic induction of the intrinsic field of the beam. Equation (2.10) degenerates to a linear equation in this case [cf. Eq. (2.3)] and when $\partial/\partial z \gg \partial/\partial x$ transforms to

$$\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial z} = D\frac{\partial^2 B}{\partial z^2}, \quad u = \frac{i_b}{ne}.$$

Qualitatively, its solution is analogous to the solution (2.4) examined above. On the left ($z = 0$) the field is carried into the conductor with a current-flow velocity u, so that the depth of the skin layer increases with t as

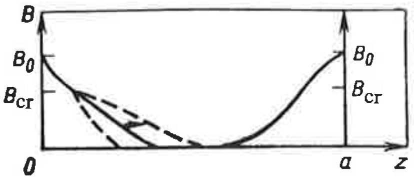

Fig. 4. Transport of a field by a beam (cylindrical geometry).

$$B = \frac{B_0}{2}\left[\exp\left(\frac{uz}{D}\right)\operatorname{erfc}\left(\frac{z+ut}{2\sqrt{Dt}}\right) + \operatorname{erfc}\left(\frac{z-ut}{2\sqrt{Dt}}\right)\right].$$

On the right ($z = a$) the steady-state profile

$$B = B_0 \exp\left[\frac{u}{D}(z-a)\right]$$

develops because of the competition between diffusive penetration and linear transport of the field, i.e., the effective penetration depth is $l \sim D/u$. The general form of $B(z)$ is illustrated in Fig. 3.

We note an interesting effect: if the external current $\mathbf{j}_b = -n_b e \mathbf{v}_b$ is produced by an electron beam, then the field penetrates in the direction counter to the beam. The possible transport of an external field that is considerably higher than B_0 is also of interest. It is only necessary that it also be parallel to the y-axis.

We now consider the more realistic problem of a cylindrically symmetric beam. The system is similar to that shown in Fig. 2, but the z-axis is the symmetry axis and $\partial/\partial\varphi \equiv 0$. Then $B = B_\varphi$ and when $\partial/\partial z \gg 1/r$, Eq. (2.10) becomes

$$\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial z} - kB\frac{\partial B}{\partial z} = D\frac{\partial^2 B}{\partial z^2}, \quad \text{where } k = \frac{c}{2\pi ner}. \qquad (2.11)$$

Now the effective velocity of transport of the magnetic field, as in Eq. (2.1), depends on B as $v_{ef} = u - kB$. If j_b is independent of r in some neighborhood of the symmetry axis, the linear transport predominates and only when $B = B_0 v_{ef}$ does it go to zero. If, on the other hand, as is more probable, j_b decreases with radius, then there is a critical field $B_{cr} \equiv u/k = 2\pi j_b r/c$ at which v_{ef} changes sign.

The dynamics of field penetration in a plasma after diffusion has smeared out the jump is fairly complicated in this case. On the left boundary ($z = 0$), linear transport of the field predominates when $B < B_{cr}$

and further smooths out the $B(z)$ profile, so that the diffusion term is small and Eq. (2.11) reduces to the equation for a simple wave,

$$\frac{\partial}{\partial t}(B - B_{cr}) - k(B - B_{cr})\frac{\partial}{\partial z}(B - B_{cr}) = 0. \tag{2.12}$$

The solution of this equation is well known to be

$$z = -k(B - B_{cr})t + f(B - B_{cr}), \tag{2.13}$$

where the function f is determined from the initial conditions. Smoothing of the profile (2.13) makes it possible to set $\partial B/\partial z = 0$ asymptotically when $B = B_{cr}$. For $B > B_{cr}$ the competition between nonlinear transport and diffusion of the field leads to the establishment of a steady-state profile for B [cf. Eq. (2.5)] of the form

$$(B - B_{cr})^{-1} = kz/(2D) + (B_0 - B_{cr})^{-1}. \tag{2.14}$$

The behavior of the field on the right-hand boundary of the plasma is of special interest in the cylindrical problem. If j_b depends weakly enough on r, so that $B_{cr} > B_0/2$, then, as in the plane case, $\partial/\partial t \to 0$ asymptotically, but the field profile now has an inflection point

$$\frac{B - B_{cr}}{B_{cr}} = \tanh\left[\frac{kB_{cr}}{2D}(z + z_0)\right], \tag{2.15}$$

where z_0 is chosen from the condition $B(a) = B_0$. If, on the other hand, $B_{cr} < B_0/2$, then the steady-state solution (2.15) is impossible and the magnetic field penetrates into the plasma as the travelling wave with a constant velocity $v = (kB_0/2 - u = k(B_0 - 2B)/2$ of the form

$$B = \frac{B_0}{2}\left[1 + \text{th}\,\frac{kB_0}{4D}(z + vt)\right] \tag{2.16}$$

[cf. Eq. (2.7)]. As an example, Fig. 4 shows the form of the solution (2.13)–(2.15) for $B_{cr} > B_0/2$.

We note that even here it is possible to obtain an exact solution to the problem using Eq. (2.11), which again reduces to Burgers equation when the substitution $b = u - kB = k(B_{cr} - B)$ is made. In particular, instead of Eqs. (2.13) and (2.14) it is possible to find an expression analogous to Eq. (2.6), but which is considerably more complicated [12]. Various effects can occur in an ion diode [22] or when a beam is injected into a plasma [12, 13, 21].

In conclusion, we note that although Eqs. (2.4) and (2.11) have a one-dimensional form, the problem is really significantly multidimensional. Thus, convection of a field in the z direction is caused not so much by the current j_z [which may be entirely absent in terms of Eq. (2.4)] as by transport owing to the currents j_x or j_r perpendicular to $\mathbf{e}_z$. It is easy to see that in a real geometry or in a plasma that is bounded in the direction orthogonal to $\mathbf{e}_z$, current transport of magnetic energy along the boundaries of the medium takes place from the region where the magnetic field is trapped into the region where rapid penetration is taking place. This is in complete agreement with the solutions obtained here.

3. STABLE TWO-DIMENSIONAL ELECTRON VORTICES

3.1. Vortices as Fundamental Objects

Particular solutions in the form of stable vortices are interesting in that they appear during the evolution of quite arbitrary initial perturbations. The traditional construction of the theory of strong turbulence along the lines of the Kolmogorov–Obukhov theory is based on forced averaging and does not include any particular or especially important solutions. At the present time it is clear that during the evolution of different systems, particular but comparatively universal solutions appear which are known as structures. A complete picture of turbulence which includes these structures is lacking. So far only the types of solutions which serve as structures and, in rare cases, the conditions under which they appear have been established.

It is fairly evident that during the decay of a spatially bounded perturbation, stable solutions appear (if they exist at all, and we shall show this for the case of two-dimensional vortices) in asymptotic form. It is not at all obvious that stable solutions can appear as the result of the evolution on the average of a uniform perturbation that occupies all space. For the case of solitons in nonintegrable systems that is true [23], but for two-dimensional vortical turbulence in the most important medium of all (an ideal fluid) the answer is not known at any level of rigor. There is no answer even for the case of electron hydrodynamics.

Later in this chapter we shall examine all types of stable vortices in relative detail, since they are described by equations which are the same for various media (Section 3.3). Attempts based on thermodynamics to narrow the class of asymptotically important vortices are unlikely to be correct. The review by Petviashvili and Yan'kov [23] contains a critique on this topic.

We note also that all the subsequent discussion in this chapter is in terms of integrals of motion, which are most suitable for describing stable vortices.

3.2. Vortices in Uniform Plasmas

The main types of stable vortices are monopole and dipole vortices, all of which realize a maximum energy for fixed values of the other integrals of motion [23–27]. Since there are an infinite number of integrals corresponding to freezing, there is an infinite number of forms for the stable vortices.

In a uniform plasma the two-dimensional equation for freezing of the curl of the generalized momentum takes the form

$$\frac{\partial}{\partial t}\,\mathrm{rot}\left(\frac{mc}{4\pi ne}\,\mathrm{rot}\,\mathbf{B}+\frac{e}{c}\,\mathbf{A}\right)=\mathrm{rot}\left[-\frac{c\,\mathrm{rot}\,\mathbf{B}}{\partial(x,y)},\ \mathrm{rot}\left(\frac{mc}{4\pi ne}\,\mathrm{rot}\,\mathbf{B}+\frac{e}{c}\,\mathbf{A}\right)\right], \tag{3.1}$$

or, in dimensionless form,

$$\frac{\partial\omega}{\partial t}=\frac{\partial(b,\omega)}{\partial(x,y)}\equiv\frac{\partial b}{\partial x}\,\frac{\partial\omega}{\partial y}-\frac{\partial b}{\partial y}\,\frac{\partial\omega}{\partial x}, \tag{3.2}$$

where $\omega = b - \Delta b$.

It can be seen immediately from Eq. (3.2) that all configurations for which the level contours of b and Δb coincide (such as circular vortices) are stationary.

The stability of vortices is conveniently explained in terms of the constants of motion. Equation (3.2) has the following constants of motion:

(a) energy

$$\mathscr{E}=\int[b^2+(\nabla b)^2]\,d^2\mathbf{r}, \tag{3.3}$$

(b) momentum

$$\mathbf{p}=\int\left(nm\mathbf{v}+\frac{ne}{c}\,\mathbf{A}\right)d^2\mathbf{r}=\mathrm{const}\int\omega\,[\mathbf{r},\ \zeta]\,d^2\mathbf{r},\ \ \zeta\parallel\mathbf{B}, \tag{3.4}$$

(c) angular momentum

$$M=\int\omega r^2 d^2\mathbf{r}, \tag{3.5}$$

(d) freezing

$$J_F = \int F(\omega)\, d^2\mathbf{r}, \tag{3.6}$$

where F is an arbitrary function.

In the form (3.3) it is clear that the energy is the sum of the magnetic and kinetic energies, although for stability studies it is most convenient to express the energy in terms of the quantity ω, which is conserved along the trajectories, using the relation $b = 1/2\pi \int \omega(\mathbf{r}_1) K_0(|\,\mathbf{r} \neq \mathbf{r}_1\,|) d^2\mathbf{r}_1$ which follows from the formula $\omega = b - \Delta b$:

$$\mathscr{E} = \int \left[b^2 + (\nabla b)^2 \right] d^2\mathbf{r} = \int b\omega d^2\mathbf{r} = \frac{1}{2\pi} \int\int \omega(\mathbf{r}_1)\, \omega(\mathbf{r}_2)\, K_0(\,|\,\mathbf{r}_1 - \mathbf{r}_2\,|\,)\, d^2\mathbf{r}_1 d^2\mathbf{r}_2, \tag{3.7}$$

where $K_0(r)$ is the MacDonald function.

An electrostatic analog is useful for the subsequent discussion: in electrostatics the energy can be written both as the energy of the field (single-volume integral) and as the energy of interaction of charges (a double-volume integral). In our case of conserved "charge," it is $\omega(\mathbf{r})$, while the "interaction potential" $K_0(r)$ falls off at small distances as $\ln r$ and at large distances exponentially. "Screening" of the interaction over distances greater than c/ω_{pe} occurs when the energy and momentum of the magnetic field are taken into account. If we assume that the interaction potential $K_0(r)$ falls off monotonically, while the "charge" $\omega(\mathbf{r})$ is frozen in an incompressible fluid, then it is evident that the maximum energy will be attained in the solutions where the "charges" of one sign are as close as possible. These include the circular solutions where $\omega(r)$ is an arbitrary function that decreases monotonically from the center [it may also be required that $\omega(r)$ decrease rapidly, in order for the integrals (3.3)–(3.6) to be finite]. It is interesting to note that no solutions for which the energy is a minimum have been found.

The stability of the above vortices becomes fully obvious if we examine the integral (3.5). They correspond to the minimum absolute value of this integral. It might seem that for large-sized vortices, the "binding energy" would be exponentially small and the vortex would be "fragile"; however, this is not so. Only neighboring parts actually interact, as in a liquid droplet. In this case the expression for the energy is simpler and can be written in the form

$$\mathscr{E}' = \text{const} \int (\nabla\omega)^2\, d^2\mathbf{r}. \tag{3.8}$$

This formula can be obtained as follows: we begin by partially expressing b in Eq. (3.3) in terms of ω to give

$$\mathscr{E} = \int [b^2 + (\nabla b)^2]\, d^2\mathbf{r} = \int b\omega d^2\mathbf{r} = \int (\omega + \Delta b)\, \omega d^2\mathbf{r}. \tag{3.9}$$

$\int \omega^2 d^2\mathbf{r}$ is a constant of motion, so it can be dropped, and for the remaining part we use $\Delta b \ll b$, i.e., $b \simeq \omega$, to give

$$\mathscr{E}' = \int \Delta b\omega d^2\mathbf{r} \simeq \int \Delta\omega\omega d^2\mathbf{r} = -\int (\Delta\omega)^2\, d^2\mathbf{r} \simeq -\int (\nabla b)^2\, d^2\mathbf{r}. \tag{3.10}$$

The stable solutions, therefore, minimize the kinetic energy of the electrons, while the total energy (including the magnetic field energy) reaches a maximum, since the variation in the magnetic field energy is greater than the variation in the kinetic energy and has the opposite sign.

Recalling that the equation of motion conserves the momentum (3.4), it is possible to prove the existence of stable noncircular "dipole" vortices. Consider two circular vortices with a monotonically decreasing function $|\omega|$ of size a, which differ only in the sign of the "charge." In isolation, such vortices are stable. Including the integral (3.4) means that the distance l between the centers of the vortices, i.e., the "dipole moment," is conserved. If we assume that $a \ll l$, then the interaction energy of the vortices is small compared to the energy of each vortex, and the maximum energy is realized for a pair of two almost circular vortices [24]. This sort of pair moves along a straight line perpendicular to the dipole moment, carrying electrons along with it and producing an enhanced thermal conductivity.

3.3. A New "Universal" Two-Dimensional Equation in a Weakly Inhomogeneous Medium

The equation obtained in this section describes two-dimensional hydrodynamical flows of the electron or ion component in a plasma, as well as of the oceans and atmospheres of planets. The ability to describe the motion of different media originates in the similar structure of the constants of motion for these media. This equation generalizes the Korteweg–de Vries and Kadomtsev–Petviashvili equations.

The similarity of drift waves in plasmas to Rossby waves in the atmosphere has long been known. First of all, with appropriate rendering of the equations in dimensionless form, the dispersion relations for linear waves,

$$\omega = k_x/(1 + k^2)\,, \qquad (3.11)$$

are the same (here the y-axis is in the direction of the inhomogeneity). It has been found [23] that under certain simplifying assumptions the nonlinear parts of the equations, which are fairly complicated, also are the same. The similarity of these equations has been attributed to freezing of the curl of the momentum in a continuous medium [24–26] (in the case of a plasma, the curl of the generalized momentum of each component). For two-dimensional motions, the curl has only one component. We denote it by the letter z, and the freezing equation takes the form

$$\partial z/\partial t + \operatorname{div} \mathbf{v} z = 0. \qquad (3.12)$$

The motion of plasmas, liquids, and gases can often be regarded as incompressible, so that the two-dimensional velocity $\mathbf{v}$ can be expressed in terms of the scalar b as

$$\mathbf{v} = [\mathbf{e}_z,\ \nabla b]. \qquad (3.13)$$

In order to close the system of Eqs. (3.12) and (3.13), we must introduce a coupling between b and z. One of the simplest relationships, which takes the nonlinearity, nonuniformity, and nonlocalization into account in the first approximation, is

$$z = b + b^2/2 + y - r_0^2 \Delta b. \qquad (3.14)$$

Substituting Eqs. (3.13) and (3.14) in Eq. (3.12), we find the "universal" equation which we have been seeking:

$$\frac{\partial}{\partial t}\left(b + \frac{b^2}{2} + y - r_0^2 \Delta b\right) + \frac{\partial\left(b,\ b + b^2/2 + y - r_0^2 \Delta b\right)}{\partial (x,\ y)} = 0. \qquad (3.15)$$

Written in this form, which is an extended version of Eq. (3.2), Eq. (3.15) emphasizes the freezing of the curl in the material. This equation is often written in another form [27] which takes the smallness of several of the terms in Eq. (3.14) into account. For example, let the terms $b + y$ dominate in Eq. (3.14); then Eq. (3.15) takes the form

$$\partial b/\partial t + \partial b/\partial x = 0,$$

or, in this approximation, $\partial/\partial t = -\partial/\partial x$.

Now adding the terms $-r_0^2\Delta b + b^2/2$ and replacing $\partial/\partial t$ by $-\partial/\partial x$ in them, we obtain

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}\left(-b + \frac{b^2}{2} - r_0^2\Delta b\right) - \frac{\partial (b, r_0^2\Delta b)}{\partial (x, y)}. \tag{3.16}$$

This notation, in which the time derivative appears in the simplest form, was first used by Korteweg and de Vries. It is convenient for evaluating the constants of motion. For example, by direct differentiation with respect to time and eliminating $\partial b/\partial t$ with the aid of Eq. (3.16) it is possible to confirm that the momentum and energy are conserved [27]:

$$p = \int \frac{b^2}{2} d^2\mathbf{r} \tag{3.17}$$

and

$$\mathscr{E} = \int \left[\frac{1}{3} b^3 + r_0^2 (\nabla b)^2\right] d^2\mathbf{r}. \tag{3.18}$$

It is not as simple to confirm the existence of analogous integrals for the more exact Eq. (3.15), and the analogs to Eqs. (3.17) and (3.18) are considerably more complicated.

In place of Eq. (3.14) it is possible to take

$$z = b - r_0^2\Delta b + y - by - y^2/2 \tag{3.19}$$

and again obtain Eq. (3.16) as a result of some simplifications. We note that the Korteweg–de Vries equation can also be obtained by taking the nonlinearity into account in either the continuity equation or the velocity equation. It is also possible to assume low compressibility in Eq. (3.13) and again arrive at Eqs. (3.15) or (3.16). Up to now Eq. (3.15) has been derived using general considerations. For specific cases, such as the EMH equations (1.4), it can be obtained by simplifying the equation for the frozen quantity rot $\boldsymbol{\pi}_e$:

$$-\frac{c}{e}\,\mathrm{rot}\,\pi_e = \mathbf{B} + \mathrm{rot}\frac{c^2}{\omega_{pe}^2}\,\mathrm{rot}\,\mathbf{B}. \tag{3.20}$$

For a nonuniform Z-pinch configuration, in which we shall assume that the perturbations are small in scale and independent of angle, $B = B_0(1 + b)$ with $b \ll 1$. We rewrite Eq. (3.20) in plane orthogonal coordinates (x, y), where the x-axis coincides with the current streamlines of the initial state, $nr^2 = \mathrm{const}$, as

$$\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x}\right)(b - \lambda_0^2 \Delta b - \beta_0 b) - g_0 \frac{\partial (b, \lambda_0^2 \Delta b)}{\partial (x, y)} = 0, \quad (3.21)$$

where $v_0 = (c/4\pi ne)(1/r)(\partial/\partial y)(rB_0)$ is the current flow velocity, $\lambda_0 = c/\omega_{pe}$, $\beta_0 = (1/nr^2)(\partial/\partial y)(nr^2) - (v_0/g_0)$, and $g_0 = cB_0/4\pi ne$.

In deriving the coefficients v_0 and β_0, we have taken into account the plasma nonuniformity, the curvature of the field lines, and the terms which arise upon "straightening" the coordinate axis in the Laplacian.

We conclude with a discussion of why the term with a vortical nonlinearity was retained in Eq. (3.16) along with the usual nonlinear term that leads to reversal of the wave. The reason is that we are considering drift motions in which the material is displaced primarily in a direction perpendicular to the wave vector, and this displacement may be considerably larger than the wavelength. Naturally, the transport of the "frozen" perturbation must be included with such displacements. Originally the Korteweg–de Vries and Kadomtsev–Petviashvili equations were derived for waves with longitudinal material motion, so that it was correct to drop the term with the vortical nonlinearity.

3.4. Stable Vortices and Solitons in Nonuniform Plasmas

Monopole vortices, which are stable in a uniform plasma, are also stable in a nonuniform plasma if the corrections for the nonuniformity are small. A similar situation holds for dipole vortices. The soliton solution found by Petviashvili also appears along with the vortices.

We shall base the discussion on Eq. (3.16) derived in the preceding section,

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}\left(-b + \frac{b^2}{2} - r_0^2 \Delta b\right) + \frac{\partial (b, r_0^2 \Delta b)}{\partial (x, y)}. \quad (3.22)$$

We first note that $\partial(b, \Delta b)/\partial(x, y) = 0$ for those solutions which depend only on the radius and that the remainder of the equation includes dispersion and nonlinear terms which are typical for equations with soliton solutions. Seeking a solution in the form $b = b(x - (1 + v)t, y)$, we arrive at the equation

$$vb + b^2/2 - r_0^2 \Delta b = 0. \quad (3.23)$$

In the centrally symmetric case, when it is permissible to drop the Jacobian, Eq. (3.23) has the form

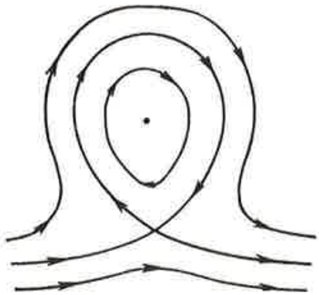

Fig. 5. Current paths in a moving vortex.

$$vb + \frac{b^2}{2} - r_0^2 \frac{1}{r} \frac{d}{dr} r \frac{db}{dr} = 0. \qquad (3.24)$$

This equation has been solved numerically [25] and has a soliton solution which decays monotonically to zero at infinity. This solution is stable and it maximizes the energy (3.18) for a fixed momentum (3.17). Its stability has been demonstrated by the Zakharov–Kuznetzov method [27]. The scale length R of a soliton is related to the amplitude by $b \simeq r_0^2 R^2$. The resulting solution appears to be a typical soliton, but this is not certain. Indeed, the velocity of the soliton in the chosen dimensionless system of units is unity when the corrections are neglected, while the drift velocity of the plasma is $L\Delta b$, where L is the characteristic scale length of the nonuniformity. If the drift velocity is much greater than the displacement velocity, i.e.,

$$Lr_0^2/R^3 \gg 1, \qquad (3.25)$$

then the current streamlines are closed inside some region (Fig. 5).

Since Eq. (3.22) is valid only when $R \ll \sqrt[3]{r_0^2 L}$ (otherwise the position dependence of the velocity of the linear waves must be taken into account [27]), our solution always contains a trapped plasma. This is a property of vortices. We note that in experiments with rotating fluids, all formations which have any pretense of being solitons will have trapped some of the fluid [29, 30]. It seems natural that if we slightly vary the curl of the generalized momentum on the closed flow lines passing inside the separatrix (Fig. 5), then, by analogy with vortices in a uniform plasma, the solution will not lose stability. Thus, it has been stated [23] that Petviashvili's solution is only the center of an "island of stability" and that the family of stable solutions is infinitely parametric. Subsequently, a more rigorous meaning [26] has been given to these arguments. It turns out that vortices that are stable in a uniform plasma (Section 3.1) will remain stable in a nonuniform plasma as well, provided that the corrections owing to the nonuniformity are small.

A large number of papers deal with comparatively special solutions that do, however, have the undoubted advantage of being written down in analytic form. They all contain matched solutions, but the resulting discontinuities in the higher derivatives are not a major shortcoming. The most widely known is the dipole solution of Larichev and Reznik [31], which is a generalization of the vortex solution for a uniform liquid [32].

The solutions mentioned above all contain closed contours of constant vorticity and they transport material, since they are essentially vortices of a nonuniversal form. The soliton solutions of Petviashvili are the only ones which do not necessarily contain a region with trapped material (when $R \ll \sqrt[3]{r_0^2 L}$ there is no trapping) and the validity of using Eq. (3.22) when this inequality holds can be ensured by choosing parameters such that the phase velocity of the drift waves depends only weakly on position, i.e., $|\nabla v_{ph}| \ll v_{ph}/L$.

In conclusion, we note that vortex equilibrium equations which contain an arbitrary function were obtained long ago for electron hydrodynamics [32]. For ion hydrodynamics this type of equation has been obtained by Stupakov [33]. The possibility of functional arbitrariness in such problems was first pointed out previously by Gordeev [34].

3.5. Pseudo-Two-Dimensional Vortices

EMH effects depend strongly on the geometry of the problem. For this reason it is appropriate to seek models which, on the one hand, describe (even if qualitatively) the three-dimensional effects and, on the other, can be solved analytically as well as the two-dimensional (degenerate) situations examined above. One such model involves a flow of electrons in a thin layer (sheet) of plasma located in a vacuum.

Thus, let us assume that a plasma sheet of thickness $\delta \to 0$, but with a finite "surface" density $N = n\delta$, conductivity $\Sigma = \sigma\delta$, and current density $\mathbf{J} = \mathbf{j}\delta$, lies in the $z = 0$ plane. The EMH equation inside the layer is, as before, written in the form (1.6). In the two-dimensional equation for the z-component of the magnetic field b ($B_z(x, y, 0) \equiv b(x, y)$) frozen in J (for $\Sigma = \infty$), which follows from this equation, however, one can replace the volume characteristics by surface characteristics everywhere, i.e.,

$$\mathbf{e}_z \frac{\partial b}{\partial t} = -\operatorname{rot} \frac{[\mathbf{J}, \mathbf{e}_z]\, b}{Ne} - \operatorname{rot} \frac{c\mathbf{J}}{\Sigma}. \qquad (3.26)$$

(Because of the small thickness of the sheet, $\delta \ll \delta_{sk}$ or $\delta \ll c/\omega_{pe}$, the tangential components of **B** are not frozen in the flux.) The only condition for this is that n, σ, and **j** not have gradients with respect to z inside the layer (otherwise mutual conversion of the potential and normal components of **B** would begin).

All the information on the implicit three-dimensionality of the problem is contained in the relationship between b and **J** (Biot–Savart law),

$$b\mathbf{e}_z = \frac{1}{c}\int \frac{[\mathbf{J}(\mathbf{r}'),\ \mathbf{r}-\mathbf{r}']}{|\mathbf{r}-\mathbf{r}'|^3}\, d^2\mathbf{r}'. \tag{3.27}$$

(Here the integral at the point $\mathbf{r} = \mathbf{r}'$ is taken to denote the principal value.) We actually need the inverse of Eq. (3.27), which is easily obtained with the aid of the Fourier transform,

$$\mathbf{J} = -\frac{c}{4\pi^2}\int \frac{b(\mathbf{r}')\,[\mathbf{e}_z,\ \mathbf{r}-\mathbf{r}']}{|\mathbf{r}-\mathbf{r}'|^3}\, d^2\mathbf{r}' = \frac{c}{4\pi^2}\,\mathrm{rot}\int \frac{b(\mathbf{r}')\,\mathbf{e}_z}{|\mathbf{r}-\mathbf{r}'|}\, d^2\mathbf{r}'. \tag{3.28}$$

Thus, the three-dimensionality of the problem manifests itself through the nonlocal relationship between b and **J**.

We first examine the case of a streaming flow of electrons with $\mathbf{J} = J(x)\mathbf{e}_y$ and $b = b(x)$. It obeys the equation

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial y}\frac{1}{N}\frac{Jb}{e} - \frac{c}{\Sigma}\frac{\partial J}{\partial x}, \tag{3.29}$$

and b and $2\pi J/c$ are related through the Hilbert transform

$$b(x) = \frac{1}{\pi}\int \frac{2\pi J(x')}{c(x'-x)}\, dx', \quad \frac{2\pi J(x)}{c} = -\frac{1}{\pi}\int \frac{b(x')}{x'-x}\, dx'. \tag{3.30}$$

(here the integrals are taken in the sense of the principal value). Some examples of such relationships include the functions $(1 + x^2/a^2)^{-1}$ and $x/a(1 + x^2/a^2)^{-1}$ used below or $\cos(x/a)$ and $\sin(x/a)$.

Equations (3.29) and (3.30) correspond to Eq. (2.4) for the conventional plane situation, but the nonlocalization of the nonlinearity and dissipation introduce new effects which change the hierarchy of the corresponding terms in the equation. Thus, on the one hand, this sort of nonlinearity has self-stabilizing dispersive properties, as is already evident from the following solitonlike particular solution of Eq. (3.29) (for $\Sigma = \infty$):

$$b = \frac{G}{1 + (x - paGt)^2/a^2}, \quad p = \frac{c}{2\pi e}\frac{\partial}{\partial y}\frac{1}{N}.$$

[here and in the following we neglect the variation of $p(y)$, which is valid when $\partial/\partial y \ll \partial/\partial y$ and clearly correct when $N \propto (y - y_0)^{-1}$]. On the other hand, if the nonlinearity causes steepening of the profile, then this process is not stopped by dissipation either. An example is explosive current contraction (soliton collisions),

$$J = \frac{c}{2\pi} G \frac{1 + t\,(v + pG)}{x^2/a^2 + [1 + t\,(v + pG)]^2}; \quad v = \frac{c^2}{2\pi\Sigma}$$

when $G < -v/p$ [compare with contraction in the framework of Eq. (2.7) and Section 4.2]. One remarkable property of Eq. (3.29) is that, despite its integrodifferential form, it does allow an exact solution, as does the Burgers equation (2.4), although this solution is obtained by a completely different approach involving entering the complex plane and treating x as a complex variable. In fact, according to Eq. (3.30), b and $2\pi J/c$ are related in a way analogous to the real and imaginary parts of the generalized susceptibility (the Kramers–Kronig relations). This makes it possible to introduce the function $\omega = b + \mathrm{i}\cdot 2\pi J/c$ that is analytic in the upper half x plane. Equation (3.29) and its conjugate equation for J, which follows from Eq. (3.29) through the Hilbert transform, combine into a single equation for w,

$$\mathrm{i}\,\frac{\partial w}{\partial t} = pw^2 - v\,\frac{\partial w}{\partial x}, \qquad (3.31)$$

which is integrated trivially over its characteristics.

Now let $N = \text{const}$ and $\Sigma = \infty$, but with an arbitrary flow of electrons over the plane, i.e., in general a vortical flow. In the problem of a fully uniform flow along the z axis, for this case the equation analogous to Eq. (3.26) degenerates into the identity $\partial B_z/\partial t = 0$ [see Eq. (3.2)], and only if the inertia of the electrons is included will a nontrivial evolution of the flow result. In this geometry, because of the other coupling between the incompressible flux $\mathbf{J}$ and the quantity b frozen into it, this is not so: $(\mathbf{J}\nabla)b \equiv 0$, and only flows which satisfy the condition [cf. Eq. (3.2)]

$$\partial\left(\int \frac{b\,(\mathbf{r}')}{|\,\mathbf{r} - \mathbf{r}'\,|}\, d^2\mathbf{r}',\ b\right) \Big/ \partial\,(x,\ y) = 0$$

will be stationary. This corresponds, in particular, to circular vortices. Yet another characteristic difference (besides the nontrivial evolution in the massless approximation) of the pseudo-two-dimensional vortices is their nonlocalization. The magnetic field from a current J_φ bounded in r will decrease toward infinity on penetrating through the vacuum only according to a power law, even when $m \to 0$. Including the electron inertia leads to freezing in **J** of the quantity $\omega = b + (mc/Ne^2)\mathbf{e}_z \,\mathrm{rot}\, \mathbf{J}$. The constants of motion for such vortices are completely analogous to Eqs. (3.4)–(3.7); only the interaction potential of the "charges" in Eq. (3.7) transforms to $\mathbf{H}_0(r) - N_0(r)$, where $\mathbf{H}_0$ is the Struve function and N_0 is the Neumann function. That is, it behaves as $-\ln r$ for $r \ll 1$ and as $1/r$ for $r \gg 1$ (here the unit of measurement of r is not c/ω_{pe}, but c^2/Ω_{pe}^2, where $\Omega_{pe}^2 = 4\pi Ne^2/m$). Thus, our conclusions about the stability of certain classes of vortices remain valid. If the characteristic size of the flow is $a \ll 1$, then the relationship between the flux **J** and the quantity rot **J** frozen into it degenerates into a local coupling, the system ceases to "feel" its boundedness in z, and this case reduces to the two-dimensional flow of an ideal fluid examined in Section 3.2.

4. TURBULENCE AND EMH RESISTANCE

4.1. Stable Three-Dimensional Vortices and Three-Dimensional Turbulence

We shall examine a model of turbulence in which the breakup of scale lengths takes place only in a few relatively small regions, as does reconnection in MHD. At the same time, stable configurations with toroidal magnetic surfaces exist.

In the case of an extremely idealized model, the ion density can be regarded as constant, while the ion motion and the electron inertia can be neglected. Then the equation for the evolution of the magnetic field, when rendered dimensionless such that $\mathbf{v} = \mathrm{rot}\,\mathbf{B}$, takes the form

$$\partial \mathbf{B}/\partial t = \mathrm{rot}\,[\mathrm{rot}\,\mathbf{B},\ \mathbf{B}]. \tag{4.1}$$

This equation conserves the energy

$$\mathscr{E} = (1/2) \int B^2 d^3\mathbf{r} \tag{4.2}$$

and can be written in Hamiltonian form [14]. It is extremely close to the equation for the motion of a nonviscous, incompressible fluid, but its properties are still different. The first distinctive feature is the existence of stable localized solutions of Eq. (4.1), which minimize the energy when the freezing condition is met [23, 35]. The latter statement means that the variation has the form

$$\delta \mathbf{B} = \mathrm{rot}\,[\mathrm{rot}\,\mathbf{q},\ \mathbf{B}_0], \tag{4.3}$$

where $\mathbf{q}$ is an arbitrary small vector. We shall show that a minimum of the energy (4.2) is attained in the stationary solutions. The condition $\int \mathbf{B}_0 \delta \mathbf{B} d^3\mathbf{r} = 0$, together with Eq. (4.3), implies that

$$0 = \int \mathbf{B}_0\, \mathrm{rot}\,[\mathrm{rot}\,q,\ \mathbf{B}_0]\, d^3\mathbf{r} = \int \mathbf{q}\, \mathrm{rot}\,[\mathrm{rot}\,\mathbf{B}_0,\ \mathbf{B}_0]\, d^3\mathbf{r},$$

which, since $\mathbf{q}$ is arbitrary, means that

$$\mathrm{rot}\,[\mathrm{rot}\,\mathbf{B}_0,\ \mathbf{B}_0] = 0, \tag{4.4}$$

i.e., our assertion is proved. If the field lines are linked, then a variation of the form (4.3) cannot drive Eq. (4.2) to zero, but would seem to mean that a nontrivial minimum field energy exists for given frozen fluxes. We shall consider the example of an initial field where every field line forms a ring and any two field lines are linked N times. When the energy is minimized, the result is a stable configuration in terms of Eq. (4.1) with field lines which are wrapped around the toroidal surfaces an integral number of times.

This sort of analysis is, unfortunately, not universal. Arnol'd [36] was apparently the first to note that, in general, the field lines fill three-dimensional regions densely and when the energy is minimized they cannot be stacked on the surfaces, as Eq. (4.4) would require. In actual physical problems, however, the field topology is usually simple enough to avoid such difficulties.

Let us consider the possible dynamics of the turbulence in terms of Eq. (4.1). We shall assume, as usual, that dissipation is activated at small scale lengths and does not affect the course of the large-scale processes. We specify an initial configuration $\mathbf{B}(r)$ of a general form and allow it to evolve according to Eq. (4.1). The excess energy will immediately begin to be lost in bending oscillations of the field lines [in the case of quasi-classical linearization of Eq. (4.1), these are conventional helicons]. Generally speaking, however, it is possible to approach a minimum energy state only through a modification of the topology, i.e., by activating reconnection (as in conventional MHD [37]). As a result, stable configurations should be obtained with toroidal (in general, irrational) windings. The excess energy is partially dissipated during reconnection and partially transformed into helicons and subsequently into small-scale oscillations through three-wave interactions on the helicon branch.

A completely different model has been constructed by Vainshtein [38] on the pattern of an ideal fluid. In this model the frequency of breakup of the scale length is $\nu_\lambda \propto B_\lambda/\lambda^2$, while the constancy of the energy flux over the scale lengths implies that $B_\lambda \propto \lambda^{2/3}$. Thus, for sufficiently small spatial scales, the frequency $\nu_\lambda \propto \lambda^{-4/3}$ is much lower than the helicon frequency $\Omega_h \propto B_0/\lambda^2$, which contradicts our assumptions: the turbulence is weak rather than strong. Noting also that in terms of Eq. (4.1) steady-state stable configurations exist on scale lengths much greater than the dissipative scale length, we see that the feasibility of this model [38] appears very questionable.

4.2. EMH Resistance

The concept of EMH resistance is extremely important in a whole range of practical problems. As yet there are no unified ways of calculating it.

First we examine the need for and usefulness of introducing a new terminology. It is well known that on a microscopic level the development of resistance to a flow of particles corresponds to scattering of individual particles on some objects (other particles or waves) in which the former lose their momentum and energy of directed motion. In general, as mentioned repeatedly above, in EMH the energy and momentum of an electron flow contain both mechanical and field components:

$$\mathscr{E} = \frac{nmu^2}{2} + \frac{B^2}{8\pi}; \quad \pi = nm\mathbf{u} - \frac{ne}{c}\,\mathbf{A}. \tag{4.5}$$

The standard mechanisms for electron scattering (such as Coulomb collisions with ions [1] or the quasilinear interaction with ion acoustic noise [4, 39, 40]) only cause changes in the first terms of Eqs. (4.5) without affecting the second. The reason for this limitation is fairly simple: because of the smallness of the scattering objects, only individual particles interact with them, while the field components of the momentum and energy of a single electron depend on the motion of all the electrons as a whole. The magnetic field is an integral characteristic of the current.

Consequently, in order to change **B** and **A** in Eq. (4.5), substantially macroscopic obstacles to the electron flux must be present on which many particles scatter simultaneously. It is easy to see that the characteristic size of the obstacle, a, must satisfy the inequality $a > c/\omega_{pe}$. In this case, because the field components dominate the mechanical components in Eq. (4.5), the electron current will experience a very large resistance, namely EMH resistance.

Let us consider a simple example. The stationary flow of a current in an ideally conducting plasma is described in terms of massless EMH by the equation

$$\text{rot}\,[\mathbf{j}/n,\ \mathbf{B}] = 0. \tag{4.6}$$

In two-dimensional degenerate situations, when $\mathbf{j} \perp \mathbf{B}$ and $(\mathbf{B}, \nabla)n, B \equiv 0$, it generally corresponds to an electron flow along the contours of some function $F(\zeta, \eta)$, where (ζ, η) are the coordinates on surfaces perpendicular to $\mathbf{B}$. For example, $F = nr^2$ in the case $\mathbf{B} \parallel \mathbf{e}_\varphi$ and $F = n$ in the case $\mathbf{B} \parallel \mathbf{e}_z$. [We have already met this property of EMH, even in the more general case of $m \neq 0$, in the previous chapter. See the Jacobians in Eqs. (3.2) and (3.15).] Thus, sharp changes in F (owing, for example, to inhomogeneity of the plasma) can serve as macroscopic obstacles of this sort. When the current flow lines (4.6) encounter such barriers in their path, they cannot overcome them.

Finite σ changes this situation, which begins to depend on the shapes of the obstacles. If an obstacle is absolutely impenetrable (a boundary with the vacuum), then the lines of $\mathbf{j}$ are curved near the obstacle and the current begins to flow along it [7]. A "boundary layer" of thickness $\delta \sim a/(\omega_{Be}\tau_e) \ll a$ is formed on the surface of the obstacle. A power $Q = (j^2/\sigma)v_{\text{layer}} \simeq (cB/4\pi\sigma)^2(\delta a^2/\sigma) \sim (B^2/8\pi)ua^2$ (in a cylindrical geometry with $\mathbf{B} \parallel \mathbf{e}_\varphi$) dissipates within this layer, where $u \sim cB/(nea)$ is the current flow velocity far from the boundary layer; i.e., the entire magnetic energy carried along by the current incident on the obstacle is dissipated (for simplicity in these estimates, all the geometric factors have been assumed to be on the order of unity). The value of Q is completely independent of σ and is determined purely by the hydrodynamic characteristics of the flow. For this reason alone, the effect merits the name EMH resistance. In addition, Q remains unchanged as $\sigma \to \infty$, when the inertia of the electrons becomes of major importance [11]. Then $\delta \sim c/\omega_{pe}$ and the energy of the flow is converted, not into heat, but into vortices produced by the obstacle. The subsequent independent existence of these vortices is described in Chapter 3. This fact makes it possible to define the EMH resistance as $R = Q/I^2$. For this example $(I \sim cBa^2 \sim neua^2)$, $R \sim u/c^2$ or 30 u/c ohms. An analogous effect can occur in other cases. For example, during injection of an electron beam into a plasma, because of trapping of its magnetic field near the boundary [see Eq. (2.15)], the plasma current flowing along this boundary experiences the same sort of resistance: $R \sim u/c^2 = (u_b/c^2)n_b/n$. If, however, this obstacle is just some fluctuations in the plasma density, then the current may flow through it, but with significant changes in its properties. Let us examine this effect for the example of a plane geometry [Eq. (2.3)], i.e., with $\mathbf{B} \parallel \mathbf{e}_z$. Let $\mathbf{j} = j(x)\mathbf{e}_y$ and $(\partial/\partial y)n \neq 0$. Then in a plasma with a monotonic $n(y)$ profile ($\partial/\partial y \ll \partial/\partial x$), the electron current caused by the motion of the electrons toward higher densities will undergo a contraction [cf. Eq. (2.7)],

$$B = -B_0 \operatorname{th} \frac{p\sigma B_0}{2} x, \quad p = \frac{1}{ec} \frac{\partial}{\partial y} \frac{1}{n} > 0,$$

and experience a resistance. This behavior differs only in the geometrical factors from the previous example. For an opposed current direction at $x = \pm L$ [cf. Eq. (2.5)],

$$\frac{B}{B_0} = G \operatorname{tg} \frac{p\sigma B_0}{2} Gx,$$

where the constant G is determined from the condition $B(\pm L) = \pm B_0$; that is, in the steady state the current flows mainly along the boundaries and experiences the same (in terms of order of magnitude) resistance.

Thus, it is as if each individual obstacle to the electron current formed an EMH "shock" wave (dissipative or collisionless), within which energy is removed from the magnetic field, independently of the specific mechanism. When there is a multiplicity of such obstacles [nonmonotonic $n(y)$], it is possible to use spatial averaging and to examine the complete evolution of even the time-independent problem using the same equation (2.3). Indeed, let $n = n_0/(1 + \alpha \cos(ky))$ and, as before, let the spatially averaged current be $\langle \mathbf{j} \rangle \parallel \mathbf{e}_y$. Then, writing B_z as $B(x, t) + \tilde{B}(x, t) \sin(ky)$ ($\tilde{B} \ll B$, $\partial/\partial x \ll k$), we obtain the following relationship between B and $\tilde{B}$,

$$\left.\begin{aligned} \tilde{B} &= \frac{\alpha\sigma}{2n_0eck} \frac{\partial}{\partial x} B^2, \\ \frac{\partial B}{\partial t} &= \frac{c^2}{4\pi\sigma} \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\frac{\alpha\sigma B}{n_0ec} \right)^2 + 1 \right] \frac{\partial B}{\partial x}, \end{aligned}\right\} \tag{4.7}$$

which is valid for $c^2k^2/(4\pi\sigma) \gg 1/t$. Clearly, "headlong" passage of a current through an obstacle leads to an enhancement of the resistance (and the rate of diffusion of the magnetic field) by a factor of $(\omega_{Be}\tau_e)^2$.

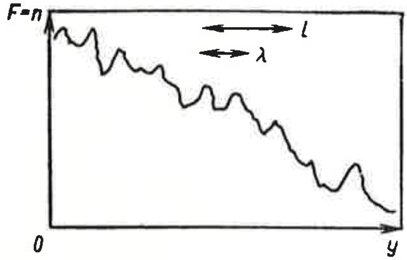

Fig. 6. The continental slope of country F.

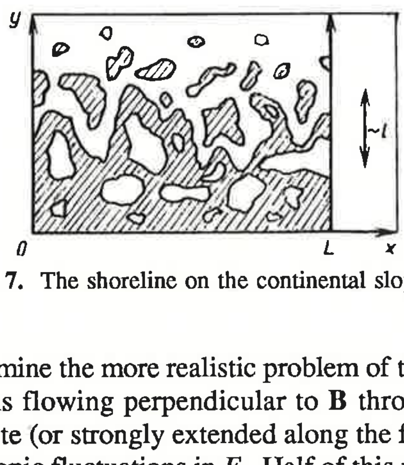

Fig. 7. The shoreline on the continental slope.

Let us now examine the more realistic problem of the resistance experienced by electrons flowing perpendicular to **B** through a nonuniform plasma in which flute (or strongly extended along the field) perturbations in n create macroscopic fluctuations in F. Half of this problem has essentially already been solved: if there are no contours of F connecting the electrodes (viewed here only as a source and sink of electrons, rather than as equipotentials), then the situation can be described by a slight modification of the system of equations (4.7). The presence of such contours, however, can change the situation fundamentally, since the resistance along such a path is lower than on any others.

This problem of the EMH resistance is close to the "flow" or "flood" problem of Shklovskii and Éfros [41]. In particular, the analogy of a mountainous country F flooded with water may be useful in studying it: the behavior of the contours can be traced by studying the coalescence of isolated "lakes" and the appearance of "shore" lines that connect the electrodes. For concreteness, as before we shall examine the plane case with $F = n(x, y)$ and electrodes located at $x = 0$ and L.

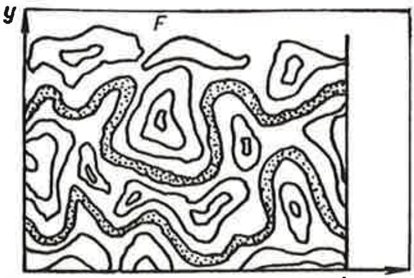

Fig. 8. F = const contours joining two electrodes.

Let $n = n_0(y) + \delta n(x, y)$, where n_0 is a monotonic function and δn is random, with characteristic spatial scale lengths of l and λ, respectively, where $l \gg \lambda$. These scale lengths are not completely standard. Here we mean that when the argument of n_0 changes by l and that of δn changes by λ, both functions change by the same amount. If $l \neq \infty$, i.e., $\nabla n_0 \neq 0$ [in a cylindrical geometry with $F = nr^2$ this also corresponds to n_0 = const], then there are many paths connecting the electrodes. In this case the country F is located on a continental shelf (Fig. 6) and for any level of flooding there will be a shore line connecting $x = 0$ and $x = L$. Their contribution to the conductivity is determined by the average length $(l/\lambda)^{\alpha}L$. The linearity in L is a consequence of the fact that when $L \gg l$ the shore line on the slope "wanders" in the y direction by no more than l (Fig. 7). The width of these paths is $S(\lambda/l)^{\beta}$, where S is the size of the electrodes in the y direction (Fig. 8). Since the area of these paths in the XY plane is naturally less than the total area of the plasma, we have $\alpha \leq \beta$, where the inequality holds for the case of Fig. 8. Unfortunately, only one of the two characteristic indices, α and β, can be found. Let us consider a divergence-free flow with a flux $\mathbf{q} = -[\mathbf{e}_z, \nabla n]$ moving along the contours. It flows from the cathode and the amount arriving at the cathode is given by

$$J = -\int_{y_1}^{y_2} \mathbf{e}_x[\mathbf{e}_z, \nabla n]\, dy = n_0(y_2) - n_0(y_1) \sim (y_2 - y_1)/l,$$

according to the definition of l ($y_2 - y_1 \gg l$). On the other hand, in the volume between the electrodes typically $q \sim 1/\lambda$, so that J is transported through an area $(y_2 - y_1)\lambda/l$, i.e., $\beta = 1$ [42]. If, however, $l = \infty$ (i.e., n_0 = const), then the contours are basically closed. Nevertheless, some link between the cathode and anode exists, even for arbitrarily large L. That corresponds to the shore line at the moment half the country F is flooded, when the paths joining the electrodes along the continent disappear and lie in the water [41].

The length of the connecting paths in this case is of order $L(l/\lambda)^{\gamma}$, while their width approaches 0 as $L \to \infty$. This means that the resistance is much greater than in the previous case.

To summarize, the concept of EMH resistance makes it possible to explain many nontrivial effects, even with the simplest equation (4.6) and its time-dependent analog (2.3). Unfortunately, the limits of applicability of this description have not been clarified. For example, including the slow motion of the ions may make it necessary, first, to determine the λ spectrum of δn self-consistently (although the effect of the density fluctuation spectrum on the resistance may be quite large, as can be seen from the simple example with $(\partial/\partial y)\delta n = 0$ and $\alpha = \beta$) and, second, to introduce additional dissipation into the problem owing to "reversal" of the field in accordance with Eq. (2.3) when n "moves" slowly. The possible existence of EMH resistance in a nondegenerate three-dimensional geometry is also of great interest. Indeed, in this case the two required conditions (quasineutrality and frozen **B**) do not place such strong limits on the electron motion, so that the number of degrees of freedom is greater by unity. For example, in this case if we allow density fluctuations that are limited along **B** ($(\mathbf{B}\nabla)n \neq 0$), then the electrons acquire the ability to move between them while simultaneously expanding along **B**. Then, however, the longitudinal current cannot turn sideways and each small obstacle will develop long current "whiskers." This kind of flow may begin to emit helicons, at which point another mechanism for EMH resistance begins to appear. In any case, this situation must be investigated further.

The question of EMH resistance, therefore, requires deep and detailed study. Such studies might not yield solutions of the model problems but lead to the creation of a unique "ordering" which will allow us to make very simple estimates by analogy with the system developed for the anomalous resistance produced by ion acoustic instabilities [4, 39, 40].

5. THE Z-PINCH

5.1. EMH Effects in the Z-Pinch

The EMH description can be applied successfully to such popular plasma devices as the Z-pinch. As a result a number of nontrivial effects will be revealed.

The condition $a < c/\omega_{pi}$ for applicability of EMH to the Z-pinch ($a \sim r$) is written in terms of the linear ion density [4] as

$$\Pi_i \equiv ZNe^2/(AMc^2) < 1, \quad N = \int_0^\infty n \cdot 2\pi r dr. \tag{5.1}$$

The inequality (5.1) is rather typical of a whole range of experiments with short-pulse, high-current discharges (high-current diodes [43], plasma focus [44], vacuum spark [45], etc.). This means that various EMH effects described above can come into play for them (in particular, convective transport of the field by an electron current). These effects can change the dynamics of the Z-pinch considerably from the behavior implied by a single fluid model.

The situation is by no means trivial, and the condition (5.1) alone does not guarantee the significance (or even the existence) of these effects. Indeed, in the case of the Bennet equilibrium [4], which is extremely characteristic of the Z-pinch, [**j**, **B**] is compensated fully by the plasma pressure. If the ion pressure is low ($P_i \ll P_e$), then this compensation also appears in the electron equation, i.e., convective EMH effects are absent. For small deviations from the Bennet equilibrium, these effects do occur but because they are strongly suppressed, the ion motion cannot be neglected and cooperative electron effects are predominant. Some dramatic examples of these effects (such as the appearance of sausage-type instabilities owing to overlapping of helicon and Alfvén modes) can be found elsewhere [4, 46].

Strong compensation of the convective term in the equation for the magnetic field does not occur if $P_i > P_e$ or if a true equilibrium is absent and the magnetic force is balanced by the inertia of the ions. Precisely the latter situation is considered in Section 6. However, even then, the electron pressure makes some contribution to the dynamics of the field.

Joule heating of the plasma can also be important in the dynamics of a Z-pinch, especially if we note that in an equilibrium pinch $c_s \simeq v_A$, while the condition (5.1) means that the current flow velocity exceeds c_s, i.e., the threshold for the ion acoustic instability of the current has been surpassed. The resulting anomalous resistivity of the plasma [4, 39, 40] substantially increases the role of resistive effects [47, 48]. A criterion for the existence of these effects is that the temperature increment owing to dissipative plasma heating, ΔT, exceed its initial temperature T, i.e.,

$$T < \Delta T \simeq \frac{mj^2}{ne^2} \nu_{\rm ef} \frac{L}{u}, \quad j \sim \frac{cB}{R}$$

where L is the length of the pinch, R is its radius, and u is the current flow velocity. This criterion can be rewritten in the form [11]

$$\nu_{ef} > \omega_{pe} \frac{R}{L} \sqrt{\frac{T}{mc^2}} \sqrt{\frac{8\pi nT}{B^2}}. \tag{5.2}$$

When Eq. (5.2) is satisfied, the impedance of a high-current Z-pinch (for example, a plasma diode) can be determined by ohmic, rather than hydrodynamic effects. Section 5.3 is devoted to a description of this situation. Unfortunately, up to now both effects (convective field transport and Joule heating of the plasma) have only been examined separately and in terms of idealized models.

5.2. Electron Flows in Low-Density Pinches

Convective transport of the field by the current leads to stabilization of the sausage instability in Z-pinches.

Effects associated with convective field transport in a geometry simulating the constriction of a Z-pinch were first examined by Morozov et al. [6, 7]. Subsequently, however, these effects were not usually taken into account until much later [11] (except in the kinetic model of Imshennik et al. [49, 50] where they are automatically included in the analysis although not isolated in a pure form).

It is easiest to include the transport effect in the approximation of ideal freezing-in of the field in the electrons ($P_e \to 0$, $\nu_{ef} \to 0$, $a \gg c/\omega_{pe}$). For axially symmetric systems in this case, the equation for the magnetic field [see Eqs. (2.1) and (2.8)] is conveniently rewritten in terms of a function of the current $I \equiv (cB/2)r$ ($\mathbf{B} \parallel e_\varphi$),

$$\frac{\partial I}{\partial t} + \frac{\mathbf{e}_\varphi r}{4\pi e}\left[\nabla I^2,\ \nabla \frac{1}{nr^2}\right] = 0 \tag{5.3}$$

[compare with Eqs. (2.3) and (2.8)]. In the steady state ($\partial/\partial t \equiv 0$) this leads to an effect, mentioned several times above, in which a current flows along the nr^2 = const curves: $I = I(nr^2)$. This equation, in turn, yields a nontrivial focussing of the electron current in a high-current diode as it flows into the dense plasma at the anode foil. A complete picture must include the processes that inhibit focussing. The most natural of these processes in this problem is finite electron pressure. The combined dynamics of the electrons and magnetic field are described in this case by the equation ($\partial/\partial t \equiv 0$)

$$\mathbf{E} - \left[\frac{\mathbf{j}}{nec}, \mathbf{B}\right] + \frac{\nabla P}{ne} = 0, \tag{5.4}$$

where $\mathbf{E} = -\nabla\varphi$ and $j = (c/4\pi)\,\mathrm{rot}\,\mathbf{B}$.

For simplicity we shall assume that $I(r)$ is monotonic, while the pressure (or temperature) varies adiabatically along the current flow lines:

$$\frac{\mathbf{j}}{ne}\nabla P - \gamma P \frac{\mathbf{j}}{ne} \nabla \ln n = 0,$$

i.e., $T = T_0(I)\,(n/n_0)^{\gamma-1}$. We shall also assume that the plasma density is a given function $n = n(z)$ (neglecting the ion dynamics). Taking the curl of the generalized Ohm's law and making the substitution $z \to n(z)$, we obtain

$$2n\frac{\partial I^2}{\partial n} - \frac{\partial I^2}{\partial r}\left[r + 2\pi c^2 \frac{dT_0}{dI^2} r^3 \left(\frac{n}{n_0}\right)^{\gamma-1} n\right] = 0 \tag{5.5}$$

instead of Eq. (5.3). The characteristics of this equation are the curves [11]

$$\frac{r(I)}{r_0} = \left(\frac{n}{n_0}\right)^{-1/2}\left[C(I) - A(I)\left(\frac{n}{n_0}\right)^{\gamma-1}\right]^{-1/2}. \tag{5.6}$$

Here $A(I) = -[2\pi/(\gamma-1)]c^2 r_0^2 n_0 (dT/dI^2)$, and $C(I)$ is determined from the boundary conditions. It is clear that as $n(z)$ increases the current flow lines are initially pressed toward the axis as $r \propto n^{-1/2}$ and then, if $A > 0$, rapidly move away from the axis, going to infinity when $n/n_0 = (C/A)^{1/(\gamma-1)}$. In the region of maximum compression, the electron pressure is on the order of the magnetic pressure.

We shall try to examine the constriction of a Z-pinch starting with these effects. As it rises, the magnetic pressure exceeds the plasma pressure, so that for preliminary estimates we can use the model of ideal freezing of the field in the electrons with a current flowing along the $nr^2 =$ const curves. It seems quite natural that in the neighborhood of a neck, nr^2 passes through a minimum as a function of z. Let the radial boundary of the neck be diffusive. Then, in the steady-state solution the current flow lines move away from the axis in the region $(\partial/\partial r)\,(nr^2) > 0$ and, if this solution falls off more rapidly than r^{-2} when $r \to \infty\; n(r)$, part of the current flow lines will turn back on passing through the curve $(\partial/\partial r) \times (nr^2) = 0$, so that the total current through the pinch will be lower than

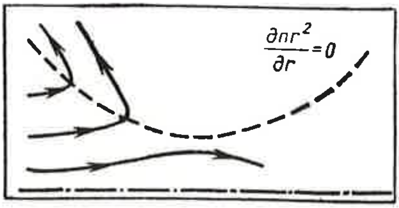

Fig. 9. Current paths in the constriction of a Z-pinch with a diffuse boundary.

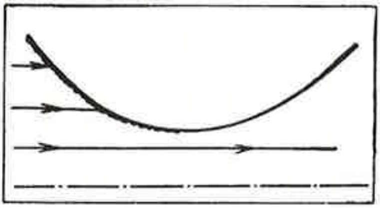

Fig. 10. Current paths in the constriction of a Z-pinch with a sharp boundary.

without a constriction (Fig. 9). If $n(r)$ falls more slowly, then the total current is not reduced, while the expansion of the current flow lines leads to a drop in the magnetic pressure. Including the electron pressure in the framework of Eq. (5.5) leads to still more rapid loss of current flow lines from the neck. That is, in the framework of EMH the constriction does not capture the current.

Finally, a formal analysis of a sharp neck boundary, on which the density drops suddenly to zero, shows that the current flow lines resting against this boundary will (in accordance with Chapter 4) form a boundary layer with EMH resistance and rapid dissipation of the magnetic energy (Fig. 10). Ultimately, this allows us to assume that effects related to EMH will lead to stabilization of the sausage instability when $\Pi_i \sim 1$.

The degeneracy of the axially symmetric geometry in a Z-pinch should foster the formation of discontinuities. In fact, Eq. (5.3) is nothing other than the equation for a simple wave. After reversal of the I profile, two possibilities arise. Either dissipation sets in and a stationary travelling wave with a steep front develops (Chapter 2) or the inertia of the electrons begins to show up and the wave begins to break up into separate vortices (Chapter 3). In the latter case, the energy of the magnetic field is transformed into kinetic energy of the electrons.

5.3. The Resistive Pinch

Including the motion of the ions has a significant effect on the EMH characteristics and impedance of a resistive pinch.

Electron flows in a resistive pinch with electrodes or, equivalently, in a high-current resistive diode have been examined in the approximation of a plane (x, y) geometry [51]. The anomalous resistance owing to the ion-acoustic instability of the current was modeled by $\sigma = (\omega_{pe}/4\pi)nev_{Te}/j$. The electrons were assumed to have no inertia. The diode was assumed to operate in a quasistationary regime since the duration of the current pulse usually exceeds both the electron and the ion times of flight. The latter fact made it necessary to include the ion motion; that is, a transition from EMH to two-component hydrodynamics was required. The electron flow was assumed to be magnetized ($\omega_{Be}\tau_e \gg 1$), which justified the neglect of the thermal force and electron thermal conductivity. Thus, the following system of equations was solved:

$$\left.\begin{aligned}
&\frac{nAM}{Z}(\mathbf{v}\nabla)\mathbf{v} = -\nabla\left(\frac{B^2}{8\pi}+P\right), \quad \mathbf{v}\equiv\mathbf{v}_i, \quad P\equiv P_e,\\
&\operatorname{div}(n\mathbf{v}) = 0,\\
&\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \frac{1}{ne}\nabla\left(\frac{B^2}{8\pi}+P\right),\\
&\mathbf{u}\nabla P - \gamma P\mathbf{u}\nabla\ln n = (\gamma-1)\,j^2/\sigma,\\
&\mathbf{j} = -ne\mathbf{u} = (c/4\pi)\operatorname{rot}\mathbf{B},\\
&\mathbf{E} = -\nabla\varphi.
\end{aligned}\right\} \qquad (5.7)$$

In principle, even when $\omega_{Be}\tau_e \gg 1$ the thermal force and electron heat fluxes can make a significant contribution to the system of Eqs. (5.7) (see [1]), although their role is not evident when the conductivity is anomalous [52]. A solution was also obtained for this case, as well as for a rather general form of the conductivity, $\sigma = f(B, P)n^{\alpha}j^{\beta}$.

Here the electrode geometry is generally arbitrary. Only their potentials are specified. In such a general statement of the problem it is possible to proceed quite far in solving the problem because of the plane geometry, specifically by transforming to orthogonal curvilinear coordinates (φ, B). The general solution is, nevertheless, extremely complicated and is expressed only in quadrature form; hence, we shall not derive it here, but refer the reader to the details elsewhere [51] and only point out some of its most interesting properties:

a) from the natural assumption that **v** and **u** are parallel near the electrodes, it necessarily follows in the general case that $v \equiv 0$ throughout the entire volume; i.e., the plasma in a diode is in Bennet equilibrium, while the system of Eqs. (5.7) degenerates to a system of EMH equations with a compensated convective term;

b) the ratio of the longitudinal and transverse scale lengths of a pinch is $L_{\|}/L_{\perp} \sim \omega_{Be}\tau_e \gg 1$; i.e., when the electron flow is magnetized the interelectrode gap is considerably larger than the thickness of the plasma cloud [cf. Eq. (5.2)];

c) the impedance of plasma diodes has been calculated for several simple geometries and a method pointed out for calculating it in the general case [at least in an implicit form $F(U, I) = 0$], but most importantly, it has been proven that a stationary solution exists for the problem of a resistive pinch with electrodes in the approximation of multispecies hydrodynamics.

6. GENERATION OF MAGNETIC FIELDS

The freezing of the magnetic field into the electrons, a property which makes the analysis of the dynamics of a field in a plasma so much easier, fails as the field pressure is raised. This situation modifies the separation of a cold component (which "maintains" freezing) from a hot electron fluid. This approach also makes it possible to take qualitative account of kinetic effects.

Magnetic field generation can be defined briefly as the enhancement of the magnetic energy of a plasma through an internal source within the medium itself. In the classical problem it is assumed that the energy capacity of the source is considerably greater than the energy of the magnetic field; that is, there are no "coarse" prohibitions on increases in $B^2/(8\pi)$. In the framework of EMH these sources can be either the thermal energy of the electrons or their kinetic energy.

The first, most often encountered case, occurs when the inequalities $\beta \gg 1$ (thermal energy predominant) and $a \gg c/\omega_{pe}$ (negligible kinetic energy, where a is the characteristic dimension) are satisfied and corresponds to field generation by a thermal emf. It can be described qualitatively by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{c}{e}\left[\nabla \frac{1}{n},\ \nabla p\right] + \frac{c^2}{4\pi\sigma}\Delta \mathbf{B}, \tag{6.1}$$

in which convective terms which are small in the parameter β^{-1} are completely absent, by comparison with Eq. (5.4) and all the preceding discussion. In fact, however, the intensity of field generation and the level which can be attained are extremely sensitive to different kinetic effects which have not been included in the purely hydrodynamic approach of Eq. (6.1): heat conduction, thermal force, and others which determine the degree of noncollinearity of ∇n and ∇P [53, 54]. In this situation it is useful, while keeping within the framework of the simpler hydrodynamic description, to modify it by separating the electrons into two components (hot and cold) [15]. This two-component approach, as will be shown below, makes it possible to include effects which follow from "infinite-component hydrodynamics" or, in other words, from the kinetics.

The nontriviality of the two-component description was first pointed out by Aliev et al. [55]. Isolating a cold component also restores the freezing effect for the magnetic field which was lost during the transition from Eq. (5.4) to Eq. (6.1) and aids in the qualitative analysis of this situation.

Let us consider in detail the case where the density of the hot component is low. It is interesting because it occurs in a pure form in two important practical cases: in laser flares [56] and during the interaction of an electron beam with a plasma when $\rho_b \ll a$ [see Eq. (2.2)].

In both cases, knowledge of the restrictions imposed on magnetic field generation is very important. As we also intend to apply the results to collisionless plasmas, we shall neglect the effect of electron scattering on their dynamics, assuming that the inequality $\omega_{Be}\tau_e \gg 1$ is satisfied. Thus, the initial system of equations has the form

$$\left.\begin{aligned}
&0 = -n_\alpha e\mathbf{E} - \frac{n_\alpha e}{c}\,[\mathbf{v}_\alpha,\ \mathbf{B}] - \nabla n_\alpha T_\alpha,\\
&\alpha = c,\ h,\quad T_h \gg T_c,\quad n_h \ll n_c,\quad n_c + n_h = Zn_i,\\
&\partial n_h/\partial t + \operatorname{div}(n_h\mathbf{v}_h) = 0,\\
&\operatorname{rot}\mathbf{B} = -\frac{4\pi e}{c}\,(n_c\mathbf{v}_c + n_h\mathbf{v}_h),\\
&\partial B/\partial t = -c\operatorname{rot}\mathbf{E}.
\end{aligned}\right\}\qquad (6.2)$$

With the aid of the small parameters written out above, this system of equations can be reduced to two equations for n_h and $\mathbf{B}$ ($n_c \simeq Zn_i(\mathbf{r})$),

$$\partial n_h/\partial t + \operatorname{div}(n_h\mathbf{v}_h) = 0 \qquad (6.3)$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{c}{e}\left[\nabla \frac{1}{n_c},\ \nabla P_h\right]. \tag{6.4}$$

(here it is assumed that $[\nabla n_\alpha, \nabla P_\alpha] = 0$ for each component in isolation). The first equation is just the continuity equation, while the second describes the generation of magnetic fields [cf. Eq. (6.1)].

It is clear that since $v_h \sim c\nabla P_h/(n_h eB)$, according to Eq. (6.2), n_h changes n_c/n_h times faster than B. This is a consequence of the freezing of the magnetic field in the cold component and the equation $\nabla_c \simeq -(n_h/n_c)\mathbf{v}_h$ ($\beta \gg 1$!). Naturally, such rapid evolution in a practically static magnetic field should lead to the establishment of some sort of equilibrium state that will drive the right-hand side of Eq. (6.3) to zero. In other words, a functional relationship exists between n_h and B and this means that the slow (compared to the relaxation of n_h) magnetic field generation described by Eq. (6.4) is greatly modified and might better be referred to as nonlinear dynamics.

Let us demonstrate this for a simple example in which $\mathbf{B} \parallel \mathbf{e}_z$, $\partial/\partial z = 0$, the electron flow takes place in the r, φ plane, T_h = const, and Eq. (6.3) is rewritten in the form [$\mathbf{v}_h$ is found from Eqs. (6.2)]

$$\frac{\partial n_h}{\partial t} = \frac{cT_h}{er}\,\frac{\partial\,(n_h,\ 1/B)}{\partial\,(r,\ \varphi)}.$$

Let $n_h = n_h(\varphi)$, while $b = B(r)$, when $t = 0$; i.e., n_h and B are not related in any way. Then $n_h = n_h(\varphi - f(r)t)$, where $f(r) = (cT_h/er)(d(1/B)/dr)$, and, as noted before in solving Eq. (6.3), B can be assumed to be time independent. In general, $f(r) \neq$ const and the profile evolves with "twisting" of the contours (Fig. 11). Already after a few rotations, a strong $n_h(r)$ dependence appears; i.e., $n_h(B)$ (the specific form of this function is determined by the initial conditions). It is easy to see that in this simplest of cases, this result can also be obtained through the pure kinetics: the drift of hot electrons in a nonuniform field along the B = const curves at different velocities on different curves leads to establishment of a $P_h(B)$ dependence. Substituting this expression in Eq. (6.4), we obtain

$$\frac{\partial B}{\partial t} = \frac{cT_h}{er}\,\frac{dn_h}{dB}\,\frac{\partial\,(1/n_c,\ B)}{\partial\,(r,\ \varphi)};$$

that is, the nonlinear dynamics of B in a plane geometry is purely a matter of transport and the amplitude of B does not increase. This dynamics, however, generally leads to "reversal" of the B profile (i.e., to an unusual

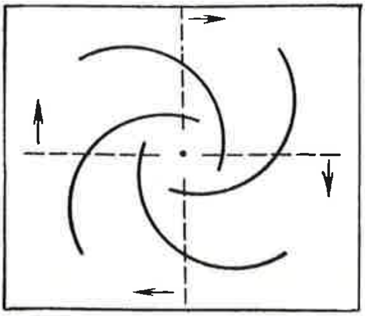

Fig. 11. The evolution of the function $n_h(B)$.

filamentation of the current which has been observed in numerical calculations [57]), after which either collisions of the cold electrons with ions or the inertia of the hot electrons (finite gyroradius ρ_{Bh}) set in.

Thus, Eqs. (6.3) and (6.4) also allow the following interpretation: they describe the rapid establishment of an equilibrium distribution of the hot particles in a quasistatic magnetic mirror and its slow evolution through transport of the field by cold particles (in a highly collisional plasma the mirror can evolve even when n_c = const because of magnetic field diffusion). This process has been examined for the one-dimensional case by Gordeev et al. [58]. This analogy allows us to conclude immediately that in general rapid relaxation results in the establishment of the well-known equilibrium relation for particles in a mirror, $P_h = P_h(U)$, where $U = \int ds/B$ and ds is the element of length along a field line [4] (it should, however, be recalled that unlike in ordinary mirrors, here the particles are ultimately confined as a result of the ions' inertia, rather than by the magnetic field). In principle, the way this relationship is established can be followed analytically by analogy with the plane case, by writing Eq. (6.3) in a curvilinear coordinate system with one of the axes directed along **B**. Nevertheless, in reality this coordinate system is by no means as simple as the cylindrical system used above. For instance, it cannot be orthogonal when $\mathrm{rot}\,\mathbf{B}\cdot\mathbf{B} \neq 0$. When the dependence of P_h on B is so indirect, the nonlinear magnetic field dynamics is no longer purely due to drift and B may increase (or decrease) somewhat as a result of it, although even here the reversal effect sets in rapidly.

If the electron motion perpendicular and parallel to **B** does not undergo a redistribution in the system (it might be aided by, for example, a low rate of collisions with the ions [5]), then this picture can be generalized by introducing two pressures, $P_{\|}$ and $P_{\perp}$. The dependence of n_h on B is then naturally more complicated [15], but the picture does not change qualitatively.

It is easy to see that the above discussion referred to a regular flow of hot electrons along a system of several surfaces (In the isotropic case the surfaces are $\int ds/B$ = const) with P_h balanced. (Here we are speaking of an averaged hydrodynamic velocity. The drift trajectories of individual particles may differ strongly.) When this condition is violated, the random motion of the electrons in a certain region makes P_h = const inside that region because of mixing and, therefore, brings the evolution of B to an end, i.e., makes $\partial B/\partial t \equiv 0$. Therefore, in a two-component electron plasma with $n_h \ll n_c$, magnetic field generation is already strongly suppressed in the stage when $\beta \gg 1$.

When $n_h \sim n_c$ the equations for the magnetic field and the electron density do not separate in time and must be solved jointly. Nevertheless, separation of a cold component (the "maintainer" of field freezing) is always formally possible if the curl part of the electric field is orthogonal to **B** (for example, if $\nabla P \perp \mathbf{B}$ or $\nabla n \perp \mathbf{B}$), i.e., $\mathbf{E}_{\mathrm{rot}} = (1/c)[\mathbf{v}, \mathbf{B}]$, and may be useful even if it appears that $n_h \ll n_c$. For example, with this approach it is easy to show that the topology of the magnetic field is conserved (the Hopf invariant $\int \mathbf{BA}d^3\mathbf{r}$ [4]) as the latter evolves (although, it is true, only when there are no problems with convergence of the integrals at infinity).

The second case, where a field is generated because of the kinetic energy of the electrons, can, in the simplest variant of a one-component electron fluid, be referred to as an electron dynamo by analogy with conventional MHD. Here the condition that the energy of the source should predominate reduces to a simple limit on the characteristic scale length of the electrons' motion, $a \ll c/\omega_{pe}$ ($\beta \to 0$), and the condition that a field be generated reduces to the presence of an energy flux at large scale lengths. This flux exists [23] in the simplest two-dimensional problem ($\mathbf{B} \parallel \mathbf{e}_z$, $\mathbf{B}\cdot\mathbf{j} = 0$) but is apparently absent in the more general three-dimensional cases. This is exactly the opposite of the situation in single-fluid MHD [59].

The more interesting generation of a large-scale magnetic field ($a \gg c/\omega_{pe}$) in a cold plasma is possible only when the medium includes some other components which have a kinetic energy, such as an electron or ion beam. This corresponds precisely to the term $(c/\sigma)\,\mathrm{rot}\,\mathbf{j}_b$ in Eq. (2.10), which leads in the first stage to a linear growth in B with time [20]. For the problem with uniformity along j_b, this equation simplifies to

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{c}{\sigma}\,\mathrm{rot}\,\mathbf{j}_b + \frac{c^2}{4\pi\sigma}\,\Delta\mathbf{B}. \qquad (6.5)$$

Here energy is collected from the beam as it is slowed down in the induced electric field. [For $u_b < v_{Te}$ and a Spitzer conductivity, friction of the beam on the plasma electrons and removal of the latter set in, while a factor $(1 - z_b^2/z^2)$ appears in the term responsible for field generation.] From a physical standpoint, when a field is generated by this mechanism, the currents produced by the plasma and beam electrons are separated in space. This separation can also occur over distances shorter than the beam radius ($a \ll r_b$). This happens not only because of the finite conductivity of the plasma, as in the example considered above, but also because of the helicon instability in a multicomponent medium (transport of B by an electron fluid) [9, 60]. Then field generation is accompanied by filamentation of the beam and in order to describe it we must supplement Eq. (6.5) [when uniformity along $\mathbf{j}_b$ fails, Eq. (2.10)] with the equation of continuity and an equation for the dynamics of the beam particles [9, 10, 20, 60]. The resulting system of equations is rather complicated and all that can be obtained from it is the growth rate for the filamentation instability. It can be found for the dissipative case using Eq. (6.5) to be $\gamma \sim (1/B)cj_b/\sigma a$, while the characteristic scale length and B are determined from the condition that the field should have a significant effect on the motion of the beam particles: $a \sim r_b\sqrt{I_b/I_{Ab}}$, $B = j_b a/c$, and $I_{Ab} = m_b c^2 v_b/e_b$. Thus, $\gamma \sim c^2/(\sigma a^2) \sim c^2/(\sigma v_b^2) I_b/I_{Ab}$. For development of the helicon instability it is enough that the generated field should be sufficiently greater than the already existing field and $a \sim r_b\sqrt{I_b/I}$ (where I is the total current). Here the growth rate is simply $\gamma \sim u_\perp/a$, where $u_\perp$ is the displacement velocity of the plasma electrons perpendicular to j_b, which is n_b/n times smaller than the transverse displacement velocity of the beam, which in turn is on the order of the transverse velocity of the particle beams, $v_\perp \sim v_b\sqrt{I/I_b}$; i.e., $\gamma \sim (n_b/n)(v_b/r_b)$ [60].

In many cases, therefore, a description of magnetic field generation in terms of EMH requires that the multicomponent character of the plasma be taken into account. This leads to some interesting and nontrivial effects.

7. EMH EFFECTS IN EXPERIMENTS

The simplest example of transport of a magnetic field by a current can be observed with the aid of a conventional radio receiver: the static known as atmospheric whistlers or helicons. These waves are created in the ionosphere and obey the equation

$$\partial \mathbf{B}/\partial t = \mathrm{rot}\,[\mathbf{u},\ \mathbf{B}], \tag{7.1}$$

which must be linearized against a background of B_0 = const and n_0 = const (where n_0 is the electron density) to yield

$$\partial \mathbf{B}_1/\partial t = \mathrm{rot}\,[\mathrm{rot}\,\mathbf{B}_1,\ \mathbf{B}_0] = (\mathbf{B}_0\nabla)\,\mathrm{rot}\,\mathbf{B}_1. \tag{7.2}$$

For a harmonic wave ($\mathbf{B}_1 = \mathbf{B}e^{i\mathbf{kr}}$ and $\mathbf{B}_0 \parallel \mathbf{e}_z$)

$$\partial \mathbf{B}/\partial t = k_z \mathbf{B}_0\,[\mathbf{k},\ \mathbf{B}]. \tag{7.3}$$

The helicity of the waves is clear in this formulation. The role of helicons in EMH is analogous to the role of Alfvén waves in single-fluid hydrodynamics. When $k_z = 0$ the frequency of the helicons goes to zero. A nonzero frequency can be obtained by including the position dependence of B_0 or n_0. These waves are referred to as gradient waves and they can be detected experimentally. For example, transport of a magnetic field by a current owing to a temperature gradient (the Nernst–Ettingshausen effect) has been observed in semiconductor plasmas [17]. One concrete manifestation of this phenomenon was the penetration of a field into a region where it was excluded by the classical skin effect.

Generally speaking, solid state plasmas represent, on the one hand, an important and large area for the development of EMH and, on the other, a good prospect for experimental studies owing to more extensive diagnostic development. This is primarily related to the absence of restrictions from above on the characteristic scale lengths and times for processes. Thus, if the charge of the current carriers (electrons or holes) is compensated by the lattice, then the motion of a second component can always be neglected (i.e., $c/\omega_{pi} \to \infty$). Unfortunately, although solid state EMH effects have already been observed experimentally (see the references cited above), the theoretical analysis is proceeding rather slowly. This is all the more unfortunate since the first papers on EMH [6,7] were concerned with semiconductor plasmas.

The theory of all linear waves is the same, so that the helicons mentioned above are hardly a specific feature of EMH. A more specific experimental manifestation of EMH is the disruption of the symmetries of single-fluid hydrodynamics. Single-fluid hydrodynamics is symmetric with respect to the substitution $\mathbf{B} \to -\mathbf{B}$ and is insensitive to the direction of the current. For example, in studies of the reconnection of magnetic field lines, devices have been used where initially the current flows only along a single axis, $j_z(x, y)$, and the magnetic field lies in the $B_x(x, y)$, $B_y(x, y)$ plane [62, 63]. (In the UN–Feniks machine the Z axis was bent into a ring [63].) In single-fluid hydrodynamics this symmetry is preserved during the evolution of the field, but including transport of the field by the current leads to the appearance of a third component of the magnetic field, $B_z(x, y)$. Indeed, electrons moving in the Z direction carry a field line at a velocity which is generally different at different points, so that the line moves out of the (x, y) plane. In some experiments the bending of magnetic field lines has actually been observed.

Another example of the disruption of symmetry has been observed in the constrictions of Z-pinches. In single-fluid hydrodynamics the directions toward the cathode and anode are indistinguishable, but in the two-fluid case the direction of motion of the electrons is distinct, so that constrictions in which the current flow velocity is on the order of the Alfvén velocity must be asymmetric. Such an asymmetry has been observed in many experiments with microscopic pinches and exploding wires.

The evolution of the magnetic field in the constrictions is described by an equation like the Burgers equation for the discontinuities in the magnetic field (current layers). In a study of exploding wires by currents, Aivazov et al. [61] provide a direct link between observations of annular formations which emit in the x-ray region and this phenomenon. The ratio of the current flow velocity to the Alfvén velocity in these experiments was $u/v_A \geq 3$. A similar situation apparently held in the corona of a radiatively cooled wire [64].

It might appear that in devices with a small current flow velocity ($u \ll v_A$), the magnetic field configuration should be strongly distorted over the time $\tau = L/u$ required for the electrons to move a distance equal to the size of the device. This, however, is not so. The equation for this evolution has the form

$$\partial \mathbf{B}/\partial t = \mathrm{rot}\,[\mathbf{j}/(ne),\ \mathbf{B}]\,.$$

There is no evolution if the vector $[\mathbf{j}/(ne), \mathbf{B}]$ has a potential. In equilibrium configurations, $[\mathbf{j}, \mathbf{B}] = \nabla P$ is a vector with a potential and, if we assume additionally that the density $n(\mathbf{r})$ is constant on the magnetic surfaces, as is usually the case, then the vector $[\mathbf{j}/(ne), \mathbf{B}]$ also has a potential (see Chapter 5 on this topic). For this reason, transport of the magnetic field is important primarily for inertial CTR, where equilibrium does not necessarily hold. Without affecting the equilibrium conditions, the transport effect does influence stability. The simplest examples are provided by the cylindrical Z-pinch [11, 46].

We mention the phenomenon of anomalous electron heat conduction in tokamaks because of its great importance. Ordinary electron hydrodynamics offers a poor description of it, although we believe the prospects for EMH have not been exhausted in this area. Since there are no reviews on this topic, we can only recommend original papers [65–67].

The theory of plasma circuit breakers, used to steepen megampere current pulses with switching times of 10^{-5}–10^{-8} sec, promises to become an important area of application for EMH [68, 69]. The increased resistance of a circuit breaker is associated with a drop in the plasma density so that the range of densities at which EMH works is always reached:

$$M_ic^2/Ze^2 \gg na^2 \gg mc^2/e^2. \tag{7.4}$$

Here a is the characteristic size and n is the density. Nevertheless, this has not been noted by most authors and they "match" the region $na^2 \gg M_ic^2/Ze^2$ with the region $na^2 \ll mc^2/e^2$. In the meantime, including transport of the magnetic field by the current leads to the appearance of such specific phenomena as EMH resistance (Chapter 4). The experimentally observed [69] evolution of the magnetic field in such systems is extremely similar to transport of **B** along the contours with nr^2 = const (Chapter 5).

The observed focussing of the current in high-current diodes [70] can be explained in terms of EMH [11]. At least the range of parameters required for applicability of the theory clearly shows up along the path from the cathode to the anode.

There are also a whole range of plasmas for which the EMH approximation may be satisfied, but diagnostic difficulties and the as yet insufficient popularity of EMH make it impossible to test the agreement of theory with experiment. These might include collisionless shock waves, laser flares, and the plasmas in many relatively small devices. Thus, filamentation, which is well known in laser flares [56], has also been observed in high-current diodes [71]. One possible explanation is based on the existence of fast electrons (of which there is never a lack during pulsed plasma heating [72]) and makes considerable use of EMH effects [10, 73]. We also note a method proposed by Petviashvili [25] for modelling plasma vortices in bowls of water. The experimental results obtained in this way [29, 30, 74] are basically in agreement with the theory (Chapter 3).

The Hall effect has been used to explain the sliding of the discharge along the anode observed in the plasma focus [75].

CONCLUSION

It can hardly have slipped by the attentive reader that the substantial nonlinearity in the phenomena discussed here is related to the anomalously low frequency of the linear oscillations, rather than to the large amplitude of the perturbations. These frequencies were low since we were examining perturbations that are constant along the magnetic field. In many plasma devices the magnetic field has shear, and the perturbations can only be approximately constant along the field. The nonlinear theory of such perturbations has not yet been constructed but will obviously be developed over the next few years.

Another important area of research is the application of EMH theory to the interpretation of experiments. Up to now this interpretation has been based either on ordinary single-fluid hydrodynamics or on electron hydrodynamics with the inertia of the electrons included but the intrinsic magnetic field of the current left out. Between these two cases lies the region of applicability of EMH, which occupies a large range in the electron density [on the order of $M_i/(Zm)$].

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