Physics 218A Fall 2008

Quasilinear Theory

→ how instabilities / turbulence modify the profiles which drive them

→ mean field theory → evolves $\langle f \rangle$

→ useful as device for calculating turbulent transport coefficients

i.e. anomalous resistivity.

→ special application: turbulent / anomalous resistivity

→ see also: posted supplementary notes
→ Chapter 3 of book manuscript.

→ read Kulsrud: 11.1, 11.2, 14.1, 14.2, 14.7

Quasilinear Theory - Vlasov Plasma

i.) Motivation and Overview

⇒ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

$\Rightarrow$ growth/damping rate $\gamma_k = \gamma_k[\langle f\rangle]$

but $\langle f\rangle$ evolves... If $\langle f\rangle$ evolves slowly:

i.e "slowly" $\Rightarrow$ $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t)\rangle]$ → $\left\{ \begin{array}{l}\text{evolution driven}\\ \text{by instabilities}\end{array}\right.$

⇒ Physics: mean distribution evolution .... driven by relaxation.

⇒ quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f\rangle$...

③ quasilinear theory as "mindless mean field theory", i.e.

$\langle f\rangle = \langle f(v,t)\rangle$ where → $\langle\ \rangle$ eliminates spatial dependence
→ t understood "slow"

so if:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f\rangle}{\partial t} + \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f}\right\rangle = 0$$

i.e. generic mean field equation (for $\langle f\rangle$) for mean of conserved order parameter ($\langle f\rangle$)

$$\frac{\partial\langle f\rangle}{\partial t} + \frac{\partial}{\partial v}J_v = 0$$ → phase space continuity equation

$$J_v = \Gamma_v = \left\langle \frac{q}{m}Ef\right\rangle$$
$$= \frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

for: $E = \tilde{E}$

$f = \langle f\rangle + \tilde{f}$

elementary closure problem
i.e. relate $\langle f\rangle$ to $\langle\tilde{E}\tilde{f}\rangle$ → hierarchy!

How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(if) $\tilde{f} \to \tilde{f}_{Linear}$ (i.e. linear response of $\tilde{f}$)
plug in linear response

i.e. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

so $$J_V = -\frac{q^2}{m^2}\sum_{k,\omega}|\tilde{E}_{k,\omega}|^2 \frac{i}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only) i.e. contrast approach to criticality in usual ~~phase transition~~ (2nd order)
→ here growth of ~~order~~ parameter in broken symmetric phase ... not noise driven

Q.L. equation is:

$$\left.\begin{aligned}\frac{\partial\langle f\rangle}{\partial t} &= \frac{\partial}{\partial v}D\frac{\partial\langle f\rangle}{\partial v}\\ D &= \frac{q^2}{m^2}\sum_k |\tilde{E}_k|^2\frac{i}{\omega - kv}\end{aligned}\right\}$$ "Q.L. equation"

i.e. mindless mean field theory...

with $$\left.\begin{aligned}\epsilon(k,\omega) &= 0\\ \partial_t|E_k|^2 &= 2\gamma_k|E_k|^2\end{aligned}\right\} \to \text{advance fields.}$$

But,
Surprisingly : Q.L.T. works quite well!

Key issue : why ?

{ N.B.: In contrast to critical phenomena, external noise ignured → instability driven ... }

④ Some questions to keep in mind : deterministic

i.) why is QL equation a diffusion equation ? When is this valid ?

↔ nature of "irreversibility" ....

ii.) can Q.L. equation be derived from Fokker-Planck theory ?

↔ also "irreversibility" related ...

iii.) how does Q.L. equation balance the energy-momentum budgets ?

iv.) when } how } does Q.L. theory ~~fail~~ ?

↔ related i.) .... What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated ?

v.) what is dynamics of quasilinear relaxation ?

i.e. physics ?

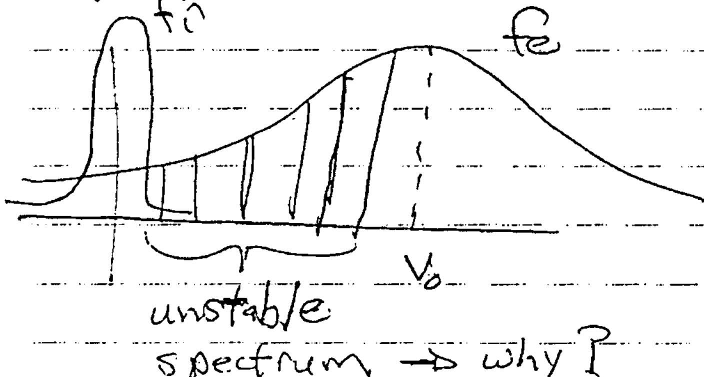
iii) <u>Basic Scales</u> / <u>Regime Definition</u>

① → Generally, Q.L.T. concerned with

i.) 'broad' spectrum of:

ii.) unstable waves

i.e for current-driven ion-acoustic (C.D.I-A.) turbulence:



②
→ In <u>finite</u> system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$$\omega_m / k_m = \omega(k_m)/k_m \equiv V_{ph\,m}$$

- wave-particle resonance occurs when

$$V = V_{ph,m}$$

then Sir Isaac ⇒

$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t)$$

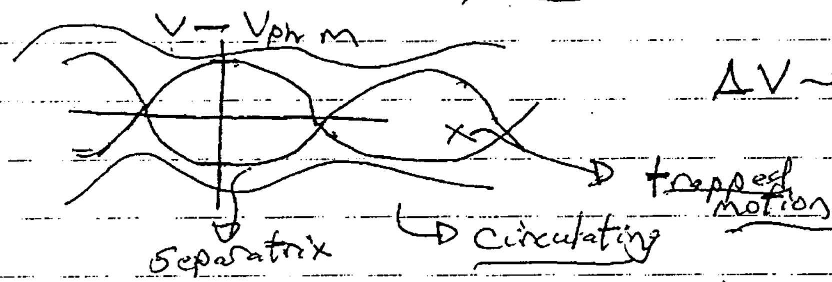
n.b. { deterministic, no RPA

and 1 resonance dominant ⇒

$$m\ddot{x} \cong q E_{m_0} \cos\left(k_m x_0 + (k_m v - \omega_m) t\right)$$

⇒ each resonant velocity defines a phase space island

i.e.

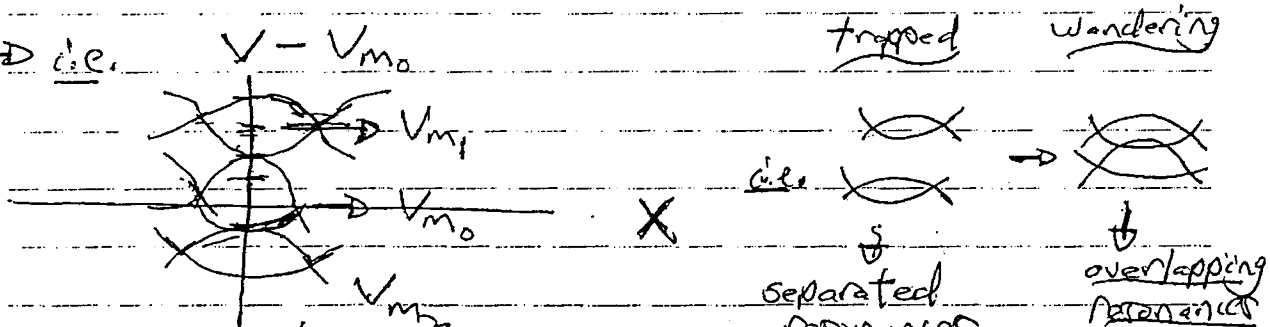


$$\Delta v \sim \left(\frac{q}{m} \phi_0\right)^{1/2}$$

QLT is concerned with the case of:

→ multiple, overlapping resonances → { separatrix proximity ⇒ destruction

→ i.e.



if overlap, hop

∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v, $D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$

Δv ~ resonance width

τ_{ac} → pattern time

↳ what is it?

Overlap condition (B.V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \gtrsim V_{ph_{m+1}} - V_{ph_m}.}$$

∴ ⇒ particle motion stochastic

→ fundamental irreversibility ⇒ orbit stochasticity (not dissipation, Landau damping → ⇔ contrast critical phenomena).

→ underpinning of diffusion equation.

③ → But, a swindle !! → use of un-perturbed orbit in estimate!

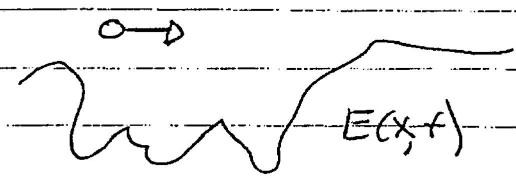
i.e. is $x \to x_0 + vt$ valid ?

Consider: linear, un-perturbed orbit ?!

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

∴ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern.

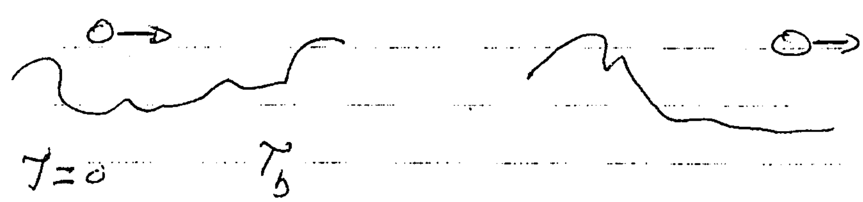
obviously, ① $\tau_L \ll \tau_b \Rightarrow$ unperturbed orbit is satisfactory approximation

(pattern changes prior $\longleftrightarrow$ bouncing)

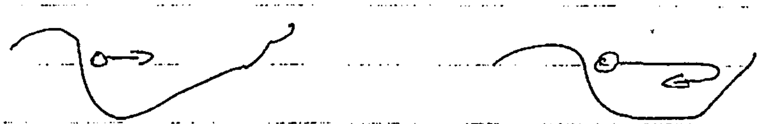
② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes,

so must consider orbit perturbation...

i.e.



vs.



∴ quasilinear theory relevant to evolution when:

① $\rightarrow$ orbits stochastic (Chirikov condition satisfied)

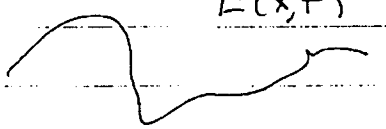
② $\rightarrow$ $\tau_{Life} < \tau_{bounce}$ $\rightarrow$ unperturbed orbits ~ valid.

3)

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

E(x,t)



$$\leftrightarrow \sum_k E_k e^{i(kx-\omega_k t)}$$

$$= \sum_k E_k \exp\left[i\left(k\left[x-\omega_k/k\right)t\right]\right)\right]$$

$v_{ph}(k)$

$\therefore \; \Delta(\omega_k/k) \equiv$ spread in phase velocities

sets dispersal rate

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength

$$1/T_{life} \equiv k\,\Delta(\omega_k/k)$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)$$

$$= \underbrace{\left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)}_{(v_g \,-\, v_{ph})}\Delta k = \left(v_g(k) - v_{ph}(k)\right)\Delta k$$

n.b. $\left\{\begin{array}{l} T_{life} \to \infty \text{ for non-dispersive waves!} \\ \text{Generally; QLT / weak turbulence encounters trouble for } \left\{\begin{array}{l}\text{non dispersive} \\ \text{weakly dispersive}\end{array}\right. \text{waves.}\end{array}\right.$

How systematize?

Consider: $\langle E(x_1, t_1)\, E(x_2, t_2) \rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, \tau)$, for $\left\{ \begin{matrix} \text{homogeneous} \\ \text{stationary} \end{matrix} \right\}$ fluctuations

$x_1 = x_+ + x_-$ $\qquad t_1 = t_+ + t_-$
$x_2 = x_+ - x_-$ $\qquad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \langle\ \rangle_{x_+, t_+}$

so

$$C(x, \tau) = \Big\langle \sum_{k,k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k + \omega_{k'})t_+} \, e^{i(k-k')x_-} e^{-i(\omega_k - \omega_{k'})t_-} \Big\rangle_{x_+, t_+}$$

x_+, t_+ average $\Rightarrow k = -k'$ $\quad \omega_k = -\omega_{k'}$

so

$$C(x, \tau) = \sum_k |E_k|^2 e^{ikx_-} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \left[\left(\frac{k-k_0}{\Delta k}\right)^2 + 1\right]$$

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

$$\therefore \langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2} + 1\right]} e^{ikx_{0-}} e^{i(kV - \omega_k)\tau}$$

integrating:

phase info. - irrelevant

$$\sim E_0^2 e^{ik_0 x_-} e^{-|\Delta k\, x_{0-}|} *$$

$$e^{i(k_0 V - \omega_{k_0})\tau} e^{-|\Delta(kV - \omega_k)|\tau}$$

oscillation (→ 0 on resonance)

correlation decay due dispersal ~~and its interplay with resonance.~~

n.b.: note that spread in doppler-shifted ω is critical now $\Delta(kV-\omega_k) = V\Delta k - V_{gr}\Delta k$

$= |(V-V_{gr})\Delta k|$

$V_{gr} = \frac{d\omega}{dk}$

so
$\langle E^2 \rangle = C(x_-, \tau)$

$= E_0^2 \, e^{ik_0 x_-} e^{i(k_0 V - \omega_{k_0})\tau} e^{-|\Delta k \, x_0|}$

$* \; \exp - \left[|(V-V_{gr})\Delta k| \, \tau \right]$

sets lifetime

$1/\tau_L = |(V-V_{gr}(k))\Delta k| \equiv (\text{Autocorrelation Time})^{-1}$

$\equiv 1/\tau_{ac}.$

Note:

- for resonant particles, $V = \omega_k/k$

$1/\tau_L = |(V_{ph}-V_{gr})\Delta k| \rightarrow$ recovers earlier!

- can think: $|V\Delta k| \rightarrow 1/\tilde{\tau}_{ac}^{\text{wave-particle}}$

$|V_{gr}\Delta k| \rightarrow 1/\tilde{\tau}_{ac}^{\text{wave}}.$

generally, shorter time dominates,
except for non-dispersive waves.

So, can enumerate key time scales

$\tau_{ac} = |\Delta k (v_{ph} - v_{gr})|^{-1}$

$\equiv$ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles

γ^{-1} = growth/damping time

$\tau_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1} \equiv$ trapping time

$\tau_{relax} = \left(\frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial t}\right)^{-1} \equiv$ avg. distribution relaxation time

So

$\tau_{ac} < \tau_{Tr} \rightarrow$ u.p.o. valid

$\tau_{ac}, \gamma^{-1} < \tau_{relax} \rightarrow \langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tau_{relax} \rightarrow$ Q.L.T. valid.

iii.) Energy - Momentum Budgets

→ key Point: There are two ways of implementing the book-keeping and accounting

i.e $\begin{cases} \text{resonant particles} & \text{vs.} & \text{'waves'} \\ \text{or} & & \\ \text{particles} & \underline{\text{vs.}} & \text{fields} \end{cases}$

keep in mind: Wave = Field + Non-resonant Particles

i.e for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field   non-resonant particle.

(show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$\frac{\partial}{\partial t} \int dv \, \frac{m v^2}{2} \langle f \rangle = -\int dv \, \frac{m v^2}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$= \int dv \, \cancel{m} v \frac{q}{\cancel{m}} \langle \tilde{E} \tilde{f} \rangle$$

Plugging in $\tilde{f}_k^{\text{Linear}}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t} \mathcal{E}_{kin} = -i \int dv \, \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{P}{\omega - kv} - i\pi \delta(\omega - kv) \right) \frac{\partial \langle f \rangle}{\partial v}$$

$$\frac{\partial}{\partial t} \mathcal{E}_{kin}^{res} = -\int dv \, \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \delta(\omega/k - v) \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \left. \frac{\partial \langle f \rangle}{\partial v} \right|_{\omega/k} |E_k|^2$$

As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

→ wave energy evolution :

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$

$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$

$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial \epsilon^r / \partial \omega}$; $\cancel{i}\gamma_k = -\frac{\cancel{i}\epsilon_{IM}}{\partial \epsilon^r / \partial \omega}$

$= -\epsilon_{IM} / \partial \epsilon^r / \partial \omega$

Now, $W \equiv$ Wave Energy Density

$$W = \sum_k \frac{\partial}{\partial \omega}(\omega \epsilon) \frac{|E_k|^2}{8\pi}$$

$$= \sum_k \omega_k \left.\frac{\partial \epsilon^r}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k \, \omega_k \left.\frac{\partial \epsilon^r}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$ $|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$

$$= \sum_k 2\left(\frac{-\epsilon_{IM}}{\cancel{\partial \epsilon^r / \partial \omega}}\right) \omega_k \frac{\partial \epsilon^r}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon_{IM}(k, \omega_k) \, \omega_k \left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\,\epsilon_{IM} = \frac{\omega_p^2}{k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_o = 1)$

$$\therefore\ \frac{\partial W}{\partial t} = \sum_k \pi\frac{q^2}{m}\frac{\omega_k\cancel{4\pi}}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial \cancel{k}}\right|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k\frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

$$\Rightarrow\ \boxed{\partial_t\,\mathcal{E}^{\text{resonant}}_{\text{kinetic}} + \partial_t W = 0}$$

Note:

- this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\underbrace{\frac{\partial}{\partial t}W}_{\text{wave energy}} + \underbrace{\underline{\nabla}\cdot\underline{S}}_{\text{divergence of wave energy density flux}} + Q = 0$$

$Q \rightarrow \langle\tilde{\underline{E}}\cdot\tilde{\underline{J}}\rangle$ coupling

For homogeneous system: $\underline{\nabla} \cdot \underline{S} = 0$

so $\frac{\partial W}{\partial t} + Q = 0$

Q ~ $\langle E \cdot J \rangle$ mediated by resonant particles (DC field)

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(RPKED) = 0}$$

RPKED ~ resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$$W = NRPKED + FED$$

NRPKED ~ non-resonant particle kinetic energy density

FED ~ field energy density

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}\underbrace{(RPKED + NRPKED)}_{PKED} = 0$$

So $\boxed{\dfrac{\partial}{\partial t}\,FED + \dfrac{\partial}{\partial t}\left(\rho\,KED\right) = 0}$ Energy Thm.

i.e. fields and particles conserve energy.

What is the physics of all this ? !

$$D = \mathrm{Re}\sum_k \frac{q^2}{m^2}|E_k|^2\left(i/\omega - kv\right)$$

QL diffusion

for general, weakly non-stationary state ---

$$= \sum_k \frac{q^2}{m^2}|E_k|^2\left(|\gamma_k| \Big/ \left((\omega-kv)^2+|\gamma_k|^2\right)\right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\simeq \sum_k \frac{q^2}{m^2}|E_k|^2\left\{\underbrace{\pi\delta(\omega-kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}}\right\}$$

Resonant Diffusion → irreversible - resonance overlap is underpinning

→ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy ....?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{\gamma_k}{\omega_k^2}$$

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \quad \text{where } |V_k|^2 = \underbrace{\frac{q^2}{m^2} \frac{|E_k|^2}{\omega_k^2}}_{\text{ponderomotive energy}}$$

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka' "fake diffusion"

→ vanishes in stationary state

Point is that can count non-resonant diffusion as:

$$D_{\text{non-resonant}} \left\langle \underline{\underline{\text{or}}} \; \frac{\begin{array}{c}\text{particle sloshing energy} \\ \text{part of } \underline{\text{wave}} \text{ energy density}\end{array}}{\begin{array}{c}\text{part of total particle} \\ \text{kinetic energy density}\end{array}} \right.$$

so <u>two</u> forms of energy conservation!

Note: Physically, the picture of plasma as gas $\begin{cases} \text{– resonant particles} \\ \text{– waves} \end{cases}$ or <u>equivalently</u>

<u>resonant particles</u> + <u>quasi-particles</u>

$$\Downarrow$$

$$\underline{\text{waves}} \left\langle \begin{array}{l} N(k, x, t) \\ WKE,\ \text{etc.} \end{array} \right.$$

is appealing and will pervade this course.

N.B.: Direct Proof of $\partial_t\,(PKED + FED) = 0$

From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv \, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} (\underbrace{kv - \omega}_{} + \omega) \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$\rightarrow$ {cancels denom; residual odd in k

$$= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

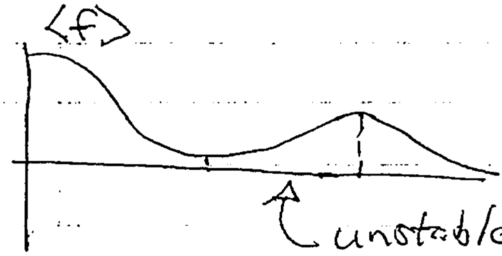
$$\omega_k = \omega_k^r + i\gamma_k$$

$$= -\sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= -\partial_t (FED) \quad \checkmark$$

iv.) Applications of Quasilinear Theory

→ Bump on Tail



unstable phase velocities (bump on tail)

$$\omega_k = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right)^{1/2}$$

Quasi-linear Equations:

$$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k), \gamma(k) \text{ from } \langle f \rangle$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi \delta(\omega_k - kv) + \frac{\gamma_k}{\omega_k^2} \right\}$$

$$\frac{\partial}{\partial t}\left(|E_k|^2/8\pi\right) = 2\gamma_k |E_k|^2/8\pi$$

Observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening

with ↕

- adjustment of core/bulk profile (i.e. effective "temperature")

i.e. first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v} \qquad * \langle f \rangle \text{ and } \int \Rightarrow$$

$\Rightarrow$

$$\frac{\partial}{\partial t} \int_{res} \frac{\langle f \rangle^2}{2} = - \int_{res} dv \; D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2$$

{ generalization ⇒ Zeldovich Thm.

stationarity ⇒

$$D^R \left(\partial \langle f \rangle / \partial v \right)^2 = 0$$

Now "res" → some finite interval of phase velocities

So

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay and damp

or

$\partial\langle f\rangle/\partial v = 0$; plateau forms, removing growth

N.B.: — In 1D $\rightarrow$ plateau
— can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2}\sum_k \frac{|E_k|^2}{8\pi}\delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2}\int dk\, \mathcal{E}_F(k)\,\delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}_F(\omega_{pe}/v)$$

so

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v}\left(2\gamma_{\omega_{pe}/v}\right)\mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} \Big/ \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k}$

$$\gamma_k = \gamma_{\frac{\omega_{pe}}{v}} = \pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}$$

so $$\frac{\partial D^R}{\partial t} = \frac{16\pi^2 q^2}{m^2 v}\left(2\pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}\right)\mathcal{E}(\omega_p/v)$$

$$= \left(\pi \omega_p v^2 \frac{\partial \langle f \rangle}{\partial v}\right) D^R$$, using D^R defn.

so

$$D^R(v,t) = D^R(v,0)\exp\left[\pi \omega_{pe} v^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial v}\right]$$

and:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial v}\left[\frac{D^R}{\pi \omega_{pe} v^2}\right]$$ $\Big\{$ using γ_k, D definitions

so

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]$$

$\therefore$ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_p v^2\int_0^t dt'\,\frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right] \end{aligned}}$$

Now, recall seek to know if:

i.) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\Big|_{t\to\infty} < 0$ (fluctuations damp)

ii.) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p v^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

but $\frac{16\pi^2 q^2}{m^2 V} \frac{\mathcal{E}_k(0)}{\pi \omega_p V^2} = 2 E_F(0) / (n m v_0^2/2)$

$\ll 1$, as $n \gg n_b$

$\therefore \quad \langle f(v,t)\rangle \cong \langle f(v,0)\rangle$, to good approx.

but, for resonant velocities,

$\rightarrow$ linear instability $\Rightarrow$ $\partial\langle f\rangle/\partial v > 0$

$\rightarrow$ $D^R \underset{t\to\infty}{\rightarrow} 0 \quad \Rightarrow \quad \partial\langle f\rangle/\partial v < 0$

but have (for $D^R \to 0$) $\quad \langle f(t)\rangle = \langle f(0)\rangle$!

$\therefore$ contradiction follows from assumption of $D^R(v,t) \to 0$

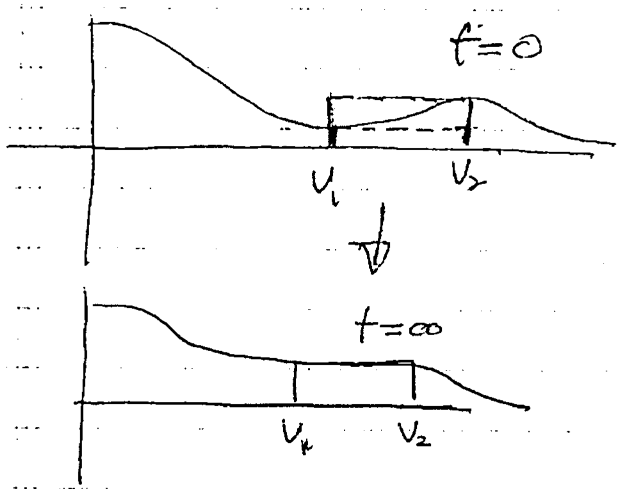
$\therefore$ have established that

$\partial\langle f\rangle/\partial v \big|_{res} \to 0 \Rightarrow$ plateau forms!

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(\text{RPKED}) + \frac{\partial}{\partial t}(\text{WED}) = 0$$

i.e.



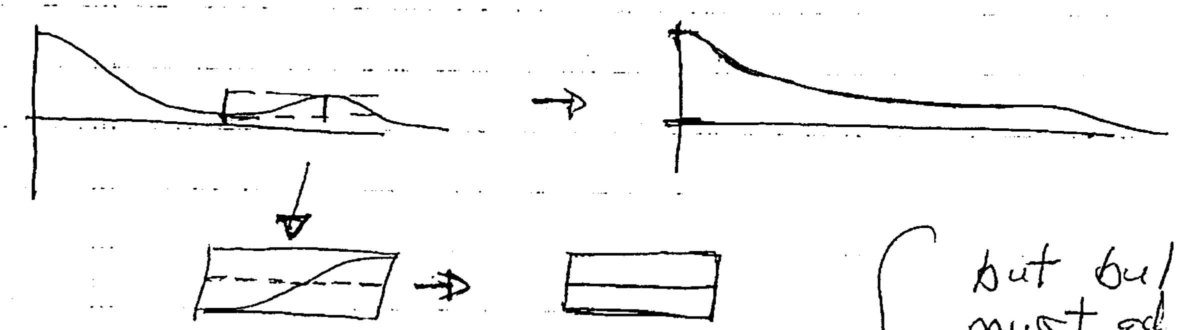
$k = \omega_p/v$

$$\therefore \quad \Delta\left(\int_{v_1}^{v_2} \frac{mv^2}{2}\langle f\rangle\right) = -\Delta\int_{k_1}^{k_2} W_k \, dk$$

but $W_k = 2\,\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv\, \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

→ can estimate A (RPKED) analytically, via construction



i.e. beam slows down

but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial \langle f \rangle}{\partial v}$$

$$\cong \frac{8\pi q^2}{m^2} \int dk\, \mathcal{E}(k) \frac{\gamma_k}{\omega_{p0}^2} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial \langle f \rangle}{\partial t} = \left(\frac{1}{nm} \frac{\partial}{\partial t} \int dk\, \mathcal{E}(k) \right) \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e} \int dk\, \mathcal{E}(k,t)$

so
$\Rightarrow$

$$\frac{\partial \langle f \rangle}{\partial T} = \frac{1}{2m} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle f \rangle = \left[m/2\pi \left[T + T(t) - T(0) \right] \right]^{1/2} \exp\left[\frac{-mv^2/2}{\left[T + T(t) - T(0) \right]} \right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \to T/2 + \frac{1}{n} \int dk \left[\mathcal{E}(k,\infty) - \mathcal{E}(k,0) \right]$$

i.e. electrons 'heated' by net increase in field energy ...

– can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta(RPKED) = -2\,\Delta(FED)$

but

$$\Delta(PKED) = -\Delta(FED)$$

so $\Delta(RPKED) = +2(\Delta(PKED))$

$$\Rightarrow \quad 0 = \Delta(RPKED) + 2\Delta(NRPKED)$$

and,

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (-2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above

→ heating is one-sided, to conserve momentum.