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Nonlinear growth of the tearing mode

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The resistive tearing mode is analyzed in the nonlinear regime; nonlinearity is important principally in the singular layer around $\mathbf{k}\cdot\mathbf{B}=0$. In the case where the resistive skin time τ_R is much longer than the hydromagnetic time τ_H, exponential growth of the field perturbation is replaced by algebraic growth like t^2 at an amplitude of order $(\tau_H/\tau_S)^{4/5}$. Application of the theory to the unstable tearing modes of a tokamak with a shrinking current channel yields good agreement with the observed amplitudes of the $m\geq 2$ oscillations. The analysis excludes the very long wavelength mode, and $m=1$ in the tokamak, for which the "constant-ψ" approximation is invalid.

I. INTRODUCTION

The nonlinear phase of the tearing mode[1] is of interest in many connections,[2] not the least in the interpretation of the magnetic perturbations observed in tokamaks.[3]

Here, we consider those nonlinear effects that are important near the singular layer of the tearing mode. In accord with an earlier suggestion,[4] we find that sizeable nonlinear eddy currents arise, producing forces which oppose the flow pattern and which quickly assume the role played in the linear theory by the inertia. At this point the exponential growth in time is replaced by algebraic growth on a much slower time scale.

The simplest configuration subject to tearing-mode instability is the sheet pinch. For a sheared equilibrium field $B_y \simeq B_y' x$ the structure of tearing mode perturbations near the singularity at $x=0$ is shown in Fig. 1. If the plasma were perfectly conducting and incompressible, the area within a surface of constant flux $\psi = B_y' x^2/2$ would be invariant. With the addition of merely a perturbation $B_x = B_{x1}\sin ky$ so that $\psi = B_y' x^2/2 + (B_{x1}/k)\cos ky$, the flux outside a ψ surface of fixed area changes by an amount $\delta\psi = -B_{x1}^2/8k^2\psi$, at least far from the separatrix. This flux change is forbidden, and must be removed by the appearance of y-independent eddy currents δj_z, which will be sharply peaked for small x. This picture is not entirely correct, however, since the resistive singular layer of the tearing mode is narrower than the skin depth for the time scales of interest, so that the eddy currents are, in fact, resistively relaxed: nonetheless, they provide the dominant nonlinear effect near the singular layer.

The perturbation $B_x = B_{x1}\sin ky$ growing with growth rate γ induces a current $j_{z1} = (\gamma B_{x1}/k\eta)\cos ky$ which provides the x-direction linear forces $-j_z B_y' x$ indicated on Fig. 1. These drive the flow pattern of narrow vortices which is shown. Moving away from the resistive singular layer the induced electric field produces a flow $v_x = -E_z/B_y = -(\gamma B_{x1}/kB_y' x)\cos ky$. For incompressible flow, as would be implied for instance by a strong equilibrium field B_z, this requires a strongly sheared flow $v_y(x)$ over the layer $x \sim x_T$ i.e., the narrow vortex pattern shown, with $v_y \sim v_x/kx_T \sim \gamma B_{x1}/k^2 B_y' x_T^2$. That this shear flow be driven against inertia by the torque produced by the linear forces requires $\gamma\rho v_y/x_T \sim k j_{z1} B_y' x_T$, which gives $x_T \sim (\gamma\rho\eta)^{1/4}/(kB_y')^{1/2}$, thus determining the width of the singular layer. The mode will grow if the perturbed currents produce magnetic perturbations $B_y = B_{y1}\cos k$ which correctly match with those in the outer regions i.e., if the outer perturbations have a positive jump in B_{y1}. This is the standard linear tearing-mode picture.

Next consider the effect of the nonlinear forces. It is clear from Fig. 1 that the second order forces $j_z B$ do not contribute to the net torque driving the vortex flow. Assuming that the resistive decay of y-independent eddy currents is rapid on the time scale of interest (the linear tearing-mode growth time is indeed long compared with the skin time appropriate to the singular layer), then the vortex flow will induce second-order y-independent eddy currents $\delta j_z = -v_y B_x/\eta$ of the form shown in Fig. 2, and of magnitude $\delta j_z \sim \gamma B_{x1}^2/k^2 B_y' \eta x_T^2$. The y-direction third-order nonlinear forces $\delta j_z B_x$, indicated on Fig. 1, then provide a torque opposing the vortex flow. As the amplitude grows these forces will replace the inertia as the dominant mechanism opposing the growth of the mode. That the torque produced by the linear forces balances the opposing torque of the third order forces requires $kj_{z1}B_y' x_T \sim \delta j_z B_{x1}/x_T \sim \gamma B_{x1}^3/k^2 B_y' \eta x_T^3$, which with $j_{z1} \sim \gamma B_{x1}/k\eta$ gives $x_T \sim (B_{x1}/kB_y')^{1/2}$; thus, in the nonlinear phase, the singular-layer width is comparable to the width of the separatrix or "magnetic island." The detailed analysis which follows shows, however, that growth of the mode is not stopped, but merely drastically slowed, in the nonlinear regime.

II. ANALYSIS

We introduce a flux function ψ such that $B_x = -\partial\psi/\partial y$; $B_y = \partial\psi/\partial x$. In terms of ψ the field diffusion equation $\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B} - \eta\mathbf{j})$ may be written

$$\frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\psi = \eta j_z \qquad (1)$$

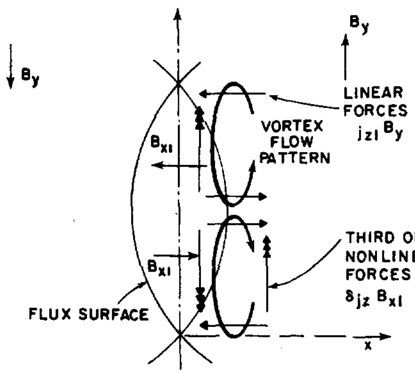

FIG. 1. Tearing mode structure in the singular layer. A perturbation $B_x=B_{x1}\sin ky$ of the equilibrium field $B_y=B_y'x$ produces flux surfaces which include the magnetic island shown. The vortex flow carrying plasma into the magnetic island is, in linear theory, driven against inertia by the linear forces $j_{z1}B_y$ indicated by single-headed arrows. In the nonlinear theory, second-order y-independent eddy currents δj_z arise; these produce third-order forces $\delta j_z B_{x1}$, indicated by triple-headed arrows, which oppose the vortex flow, and which replace the effect of the inertia as the mode grows.

together with

$$\nabla^2\psi=4\pi j_z. \quad (2)$$

The strong uniform field B_z implies incompressibility of the velocity $\mathbf{v}=(E_y/B_z,\ -E_x/B_z,\ 0)$: thus, we may introduce a stream function ϕ such that $v_x=-\partial\phi/\partial y$; $v_y=\partial\phi/\partial x$. There remains the equation of motion:

$$\rho(\partial\mathbf{v}/\partial t+\mathbf{v}\cdot\nabla\mathbf{v})=-\nabla p+\mathbf{j}\times\mathbf{B}. \quad (3)$$

In the limit of strong uniform B_z, it is appropriate to eliminate the unwanted components j_x and j_y by operating on Eq. (3) with $\mathbf{e}_z\cdot\nabla\times$ and using $\nabla\cdot\mathbf{j}=0$. Assuming the density ρ to be uniform and unperturbed (a reasonable assumption within the narrow singular layer), we obtain

$$\rho\left(\frac{\partial w_z}{\partial t}+\mathbf{v}\cdot\nabla w_z\right)=\mathbf{B}\cdot\nabla j_z, \quad (4)$$

where w_z is the vorticity

$$w_z=(\nabla\times\mathbf{v})_z=\nabla^2\phi. \quad (5)$$

We take a fundamental linear mode with $\psi=\psi_1\cos ky$, and we expand the unperturbed field $B_y(x)\simeq B_y'x$ around the singular surface taken to be at $x=0$. Within the singular layer, we may extend the familiar "constant-ψ" approximation[1] to all the nonlinear harmonics except the zeroth harmonic:

$$\psi(x,y,t)=\psi_0(x)+\tilde{\psi}(y,t), \quad (6)$$

$$\tilde{\psi}(y,t)=\sum_{n=1}^{\infty}\tilde{\psi}_n(t)\cos nky. \quad (7)$$

We recall that linear tearing modes are slow growing with respect to the resistive skin time appropriate to the singular layer of width x_T

$$\partial/\partial t\ll\eta/4\pi x_T^2. \quad (8)$$

We may assume that this continues to hold in the nonlinear regime. It follows from this assumption that in Eq. (6) we may use for $\psi_0(x)$ its unperturbed value: $\psi_0(x)=B_y'x^2/2$. Consider the corrections $\delta\psi_0$ to the zeroth harmonic; from the above assumption together with Eqs. (1) and (2), we have $\eta\delta\psi_0/x_T^2\sim v_{y1}k\tilde{\psi}_1\sim \phi k\tilde{\psi}_1/x_T\sim\psi_1(\partial\tilde{\psi}_1/\partial t)/B_y'x_T^2$ implying that $\delta\psi_0/B_y'x_T^2\ll (\tilde{\psi}_1/B_y'x_T^2)^2$, shows that $\delta\psi_0$ may indeed be neglected in Eq. (6).

Substituting Eq. (6) into Eq. (1), we obtain

$$\frac{\partial\tilde{\psi}}{\partial t}-\left(\frac{\partial\psi}{\partial y}\right)_\psi B_y'x=\eta j_z-\eta_0 j_{z0}, \quad (9)$$

where we have used $\partial\psi_0/\partial t=\eta_0 j_{z0}$. For simplicity, we first suppose that the unperturbed quantities η_0 and j_{z0} are essentially uniform within the singular layer, so that $\eta=\eta_0$. We also restrict ourselves, first, to the case where the inertia may be neglected. Equation (4) then becomes $\mathbf{B}\cdot\nabla j_z=0$, implying

$$j_z=j_z(\psi). \quad (10)$$

We may eliminate ϕ from Eq. (8) by dividing by x and averaging over y at constant ψ; we obtain

$$j_z(\psi)=j_{z0}+\eta^{-1}\left\langle\frac{\partial\tilde{\psi}(y,t)/\partial t}{[\psi-\tilde{\psi}(y,t)]^{1/2}}\right\rangle_y \Big/ \langle[\psi-\tilde{\psi}(y,t)]^{-1/2}\rangle_y, \quad (11)$$

where $\langle f\rangle_y\equiv\int_0^{2\pi/k}fk\,dy/2\pi$.

As in the standard tearing-mode treatment, the perturbed fields in the regions outside the singular layer would, in the plane case, be solutions of the linearized versions of Eqs. (2) and (10), i.e., for the nth harmonic ψ_n

$$\frac{\partial^2\psi_n}{\partial x^2}-n^2k^2\psi_n=4\pi j_{zn}=\frac{4\pi\psi_n}{B_y}\frac{\delta j_{z0}}{\partial x}. \quad (12)$$

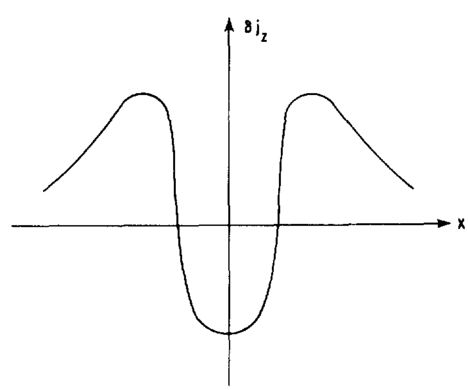

FIG. 2. Form of the second-order y-independent eddy currents.

The outer solutions are, of course, strongly affected by the geometry, and the plane case is not representative of, say, cylindrical geometry.[5,6] For our stability analysis, however, all that we require of the outer solutions ψ_n are the discontinuities in their logarithmic derivatives across the singularity, i.e., the quantities

$$\Delta_n' = [\partial \ln\psi_n/\partial x]_{0-}^{0+}. \quad (13)$$

We must match these logarithmic-derivative discontinuities in the outer solution to those arising asymptotically from the solutions within the singular layer. We substitute Eq. (11) into Eq. (2), and perform the matching for each harmonic (approximating $\nabla^2\psi_n \simeq \partial^2\psi_n/\partial x^2$ within the singular layer):

$$\Delta_n'\tilde{\psi}_n = 8\pi \left\langle \cos nky \int_{-\infty}^{\infty} j_z\, dx \right\rangle_y$$

$$= \frac{16\pi}{\eta(2B_y')^{1/2}} \int_{\psi_{\min}}^{\infty} d\psi \left\langle \frac{\partial\tilde{\psi}/\partial t}{(\psi-\tilde{\psi})^{1/2}} \right\rangle_y \times \left\langle \frac{\cos nky}{(\psi-\tilde{\psi})^{1/2}} \right\rangle_y \bigg/ \langle(\psi-\tilde{\psi})^{-1/2}\rangle_y. \quad (14)$$

A similarity solution of Eq. (14) exists in which $\tilde{\psi}=t^2\tilde{\Psi}$; writing $\tilde{\psi}_n=t^2\tilde{\Psi}_n$ and $\psi=t^2\Psi$, we have

$$\Delta_n'\tilde{\Psi}_n = \frac{32\pi}{\eta(2B_y')^{1/2}} \int_{\Psi_{\min}}^{\infty} d\Psi \left\langle \frac{\tilde{\Psi}}{(\Psi-\tilde{\Psi})^{1/2}} \right\rangle_y \times \left\langle \frac{\cos nky}{(\Psi-\tilde{\Psi})^{1/2}} \right\rangle_y \bigg/ \langle(\Psi-\tilde{\Psi})^{-1/2}\rangle_y \quad (15)$$

and multiplying by $\tilde{\Psi}_n$ and summing over n, we obtain the quadratic form

$$\sum \Delta_n'\tilde{\Psi}_n^2 = \frac{32\pi}{\eta(2B_y')^{1/2}} \int_{\Psi_{\min}}^{\infty} d\Psi \times \left\langle \frac{\tilde{\Psi}}{(\Psi-\tilde{\Psi})^{1/2}} \right\rangle_y^2 \bigg/ \langle(\Psi-\tilde{\Psi})^{-1/2}\rangle_y. \quad (16)$$

In the case where the fundamental is linearly unstable ($\Delta_1'>0$), but all harmonics are relatively strongly stable ($\Delta_n'/\Delta_1' \ll -1$, $n\geq 2$), Eq. (16) shows that a solution exists in which the fundamental dominates ($|\tilde{\psi}_1| \gg |\tilde{\psi}_n|$, $n\geq 2$). Returning to Eq. (14), setting $\tilde{\psi}=\tilde{\psi}_1 \cos ky$, and scaling the integration variable ψ to $\tilde{\psi}_1$, we obtain

$$\Delta_1'\tilde{\psi}_1^{1/2} = \frac{16\pi A}{\eta(2B_y')^{1/2}} \frac{\partial\tilde{\psi}_1}{\partial t}, \quad (17)$$

where

$$A = \int_{-1}^{\infty} dW \left\langle \frac{\cos ky}{(W-\cos ky)^{1/2}} \right\rangle_y^2 \bigg/ \langle(W-\cos ky)^{-1/2}\rangle_y.$$
$$\simeq 0.7.$$

Integrating Eq. (17), we obtain

$$\tilde{\psi}_1^{1/2} = 0.25\int^t (\eta\Delta_1' B_y'^{1/2}/4\pi)\, dt$$
$$\simeq 0.25(\eta\Delta_1' B_y'^{1/2} t/4\pi), \quad (18)$$

which shows that, in the nonlinear phase, the exponential growth of ψ_1 is replaced by growth like t^2, approaching finite amplitude only on the skin time. The singular-layer width x_T, which is now the "magnetic-island" width $x_T = 2(\tilde{\psi}_1/B_y')^{1/2}$, grows like t. It is easy to verify that the assumption (8) remains valid until the mode reaches finite amplitude.

We may generalize these results to include the case of a nonuniform resistivity $\eta_0(x)$, and a nonuniform current density $j_{z0}(x)$ within the singular layer. We will assume, however, an equilibrium on the resistive time scale: $\eta_0(x)j_{0z}(x)=\text{const}$. We also assume the classical dependence of the resistivity on electron temperature: $\eta \propto T^{-3/2}$. Within the singular layer, the temperature will be determined by parallel and perpendicular thermal conduction:

$$0 = \nabla_\perp \cdot \kappa_\perp \nabla_\perp T + \mathbf{B}\cdot\nabla[\kappa_{\|}(\mathbf{B}\cdot\nabla T)/B^2]. \quad (19)$$

In the long mean-free-path regime, the parallel thermal conductivity is very large, implying $\mathbf{B}\cdot\nabla T=0$, i.e.,

$$T = T(\psi).$$

Equation (19) imposes a constraint on the function $T(\psi)$, which may be obtained by noting that $\mathbf{B}\cdot\nabla = B_y' x(\partial/\partial y)_\psi$, dividing Eq. (19) by x, and averaging over y at constant ψ:

$$\langle x^{-1}\nabla_\perp \cdot \kappa_\perp \nabla T\rangle_y = 0.$$

Within the singular layer $\kappa_\perp$ is essentially constant, and the $\partial/\partial x$ term dominates in ∇; using $\partial/\partial x = B_y' x\partial/\partial\psi$, and expressing x in terms of ψ and y from Eq. (6), we have

$$\frac{\partial}{\partial\psi}\left\{ \langle[\psi-\tilde{\psi}(y,t)]^{1/2}\rangle_y \frac{\partial T}{\partial\psi} \right\} = 0.$$

This has the solution $\partial T/\partial\psi = \text{const}/\langle[\psi-\tilde{\psi}(y,t)]^{1/2}\rangle_y$; the constant may be determined by assuming that the appropriate boundary condition far from the singular layer is that the heat flow be held fixed [i.e., $T'(x)$ fixed]. We then obtain

$$T(\psi) = \begin{cases} T(0) + \dfrac{T_0'(0)}{(2B_y')^{1/2}} \displaystyle\int_{\psi_s}^{\psi} \frac{d\psi}{\langle[\psi-\tilde{\psi}(y,t)]^{1/2}\rangle_y}, & (\psi>\psi_s) \\ T(0), & (\psi<\psi_s), \end{cases} \quad (20)$$

where ψ_s is the separatrix. The resistivity follows:

$$\eta(\psi) = \begin{cases} \eta(0) + \dfrac{\eta_0'(0)}{(2B_y')^{1/2}} \displaystyle\int_{\psi_s}^{\psi} \frac{d\psi}{\langle[\psi-\tilde{\psi}(y,t)]^{1/2}\rangle_y}, & (\psi>\psi_s) \\ \eta(0), & (\psi<\psi_s) \end{cases}$$

Including these corrections on the right-hand side of Eq. (19), and using $\eta_0'/\eta_0=-j_{z0}'/j_{z0}$, we find that Eq. (11) must be modified in that the first term on the right becomes

$$\begin{cases} j_{z0}(0)+\dfrac{j_{z0}'(0)}{(2B_y')^{1/2}}\displaystyle\int_{\psi_s}^{\psi}\frac{d\psi}{\langle[\psi-\tilde{\psi}(y,t)]^{1/2}\rangle_y}, & (\psi>\psi_s) \\ j_{z0}(0), & (\psi<\psi_s). \end{cases} \tag{21}$$

The extra terms in $j_z(\psi)$ that are sinusoidal in y are odd on opposite sides of the separatrix and, thus, make no contribution to Eq. (14); it follows that our results are unaltered by these new terms.

It is important to note that our nonlinear treatment is a little different from standard perturbation theory. Even if the fundamental $\tilde{\psi}_1$ dominates in $\tilde{\psi}$, so that the rather simple result expressed in Eq. (17) holds, inspection of Eq. (11) reveals that the perturbed current density in the singular layer contains *all* harmonics. Standard pertrubation theory does, nonetheless, give a qualitatively correct solution, in which the plasma inertia may be carried along. With a perturbation expansion $j_z=j_{z0}+\tilde{j}_{z1}\cos ky+\delta j_{z0}$, the second-order zeroth-harmonic term δj_{z0} will be given by the second-order zeroth-harmonic component of Eq. (1), in which $\partial\delta\psi_0/\partial t$ may be neglected because of (8). We obtain $\delta j_{z0}=-(k\tilde{\psi}_1/2\eta)\partial\tilde{\phi}_1/\partial x$, where $\phi=\tilde{\phi}_1\sin kx$ is the first-order stream function. Through third order, the terms like $\sin ky$ in $\mathbf{B}\cdot\nabla j_z$ are $-kB_y'x\tilde{j}_{z1}+k\tilde{\psi}_1\partial\delta j_{z0}/\partial x$. Substituting into Eq. (4), keeping only the linear terms from the inertia, and approximating $\nabla^2\tilde{\phi}_1\simeq\partial^2\tilde{\phi}_1\partial x^2$, we obtain

$$\left(\rho\frac{\partial}{\partial t}+\frac{k^2\tilde{\psi}_1^2}{2\eta}\right)\frac{\partial^2\phi_1}{\partial x^2}=-kB_y'xj_{z1}$$
$$=-\frac{kB_y'x}{\eta}\left(\frac{\partial\tilde{\psi}_1}{\partial t}-kB_y'x\phi_1\right), \tag{22}$$

the latter after using the first-order equation (1). Equation (22) explicitly reveals, on the left, that the nonlinearity replaces the inertia as the mode grows: without the nonlinear term, Eq. (22) reduces to the standard linear-theory treatment of the singular layer.[1] The critical amplitude at which the (exponentiating) linear solution ceases to be valid can be obtained by comparing the two terms on the left in Eq. (22):

$$\tilde{B}_{x1}/B_y'a\sim(2\eta\rho\gamma)^{1/2}/B_y'a\sim(\Delta'a)^{2/5}(ka)^{1/5}(\tau_H/\tau_S)^{4/5} \tag{23}$$

where $\tau_S=4\pi a^2/\eta$, $\tau_H=(4\pi\rho)^{1/2}/B_y'$, $\gamma=0.5(\Delta'a)^{4/5}\times(ka)^{2/5}\tau_S^{-3/5}\tau_H^{-2/5}$ (the linear growth rate), and a is some scale length. Since $\tau_H/\tau_S\ll1$, this critical amplitude is very small.

III. APPLICATION TO THE MAGNETIC PERTURBATIONS IN TOKAMAKS

In the constant-current phase of tokamak discharges, small amplitude magnetic perturbations are observed on pickup loops placed outside the plasma column.[3] The observed modes are helical: they have azimuthal mode numbers m of 2–4 (although $m=4$ is only rarely seen); the toroidal mode number n is always unity. During a discharge, the sequence of onset is $m=4$, 3, and then 2, with higher amplitudes at lower m. The modes are slow-growing and long-lived; typically, the $m=3$ oscillations persist until the $m=2$ oscillations begin. An interpretation of these perturbations as resistive tearing modes has been supported by calculations of the linear stability criteria[6] (i.e., calculations of Δ') for the current profiles typical of present tokamaks. In this section we show that the perturbation amplitudes, and their slow growth, may be explained by the nonlinear tearing-mode theory presented above.

To define the amplitude of the mode we use the half-width of the magnetic island: $\xi=2(\tilde{\psi}_1/B_y')^{1/2}$. Allowing the "equilibrium" to vary slowly in time, Eq. (18) then gives

$$\xi=(0.5/B_y'^{1/2})\int^t(\eta\Delta'B_y'^{1/2}/4\pi)\,dt. \tag{24}$$

We can change to a cylindrical geometry for the equilibrium simply by substituting

$$B_y'\rightarrow r\frac{\partial}{\partial r}\left(\frac{B_\theta}{r}\right).$$

Values of Δ' have been calculated for a variety of equilibrium current profiles $j_z(r)$ typical of tokamak experiments.[6] We employ the profile which was designated "rounded model" in Ref. 6:

$$j_z(r)=Ir_0^4/\pi(r_0^4+r^4)^{3/2};\quad B_\theta(r)=2Ir/(r_0^4+r^4)^{1/2}. \tag{25}$$

Here, I is the total current, and r_0 is a measure of the radius of the current channel. For this profile, the "safety factor" $q(r)=rB_z/RB_\theta$ is given by

$$q(r)=q(0)(1+r^4/r_0^4)^{1/2};\qquad q(0)=2IB_zr_0^2/R. \tag{26}$$

For a perturbation with azimuthal and toroidal mode numbers m and n, respectively, the singular surface r_s is where

$$m/n=q(r_s)=q(0)(1+r_s^4/r_0^4)^{1/2}. \tag{27}$$

Assuming that the equilibrium has $\eta(r)j_z(r)=\text{const}$, we have

$$\eta(r)=\eta(0)(1+r^4/r_0^4)^{3/2}. \tag{28}$$

We suppose, as is typical of tokamak discharges in their constant-current phase, that the current channel is shrinking due, perhaps, to a thermal instability,[7] or to neutral gas and impurity effects.[8] The time scale of the shrinking is the resistive skin time; specifically, we assume that the quantity r_0 in Eq. (25) decreases according to

$$\frac{d(r_0^2)}{dt}=-\frac{C\eta(r_0)}{4\pi}, \tag{29}$$

where C is some numerical constant of order unity, which can be chosen to fit Eq. (29) to the known rate of shrinking of the current profile. We may apply our results, expressed by Eq. (18), to determine the mode amplitudes in this case, in which onset of a given (m, n) mode occurs when the singular surface for the mode emerges from the axis, i.e., when $q(0)$ falls below m/n. As the current channel shrinks further, the singular surface moves outward in r, and the actual plasma elements involved in the tearing-mode vortex-flow change. The required flow can arise, however, since plasma inertia is negligible on the time scales of interest. Strictly, if the flux is still to be measured so that $\psi=0$ is the singular surface, Eq. (1) should be corrected for the relative velocity between the singular surface and the plasma by the addition to $\mathbf{v}$ of a term $v_r\sim(\eta/4\pi r_0)$; such a correction has a negligible effect on Eq. (9), however, in view of the constant-ψ approximation within the singular layer (estimating $\partial/\partial t\sim\eta/4\pi r_0 x_T$).

For a given (m, n) mode, it is convenient to transform the integration variable t in Eq. (24) to the position of the singular surface r_s: from Eqs. (26), (27), and (29) we have

$$-\frac{C\eta(r_0)}{4\pi r_0^2}=\frac{d\ln(r_0^2)}{dt}=\frac{d\ln q(0)}{dt}=-\frac{2x_s^3}{1+x_s^4}\frac{dx_s}{dt},$$
$$x_s=r_s/r_0. \tag{30}$$

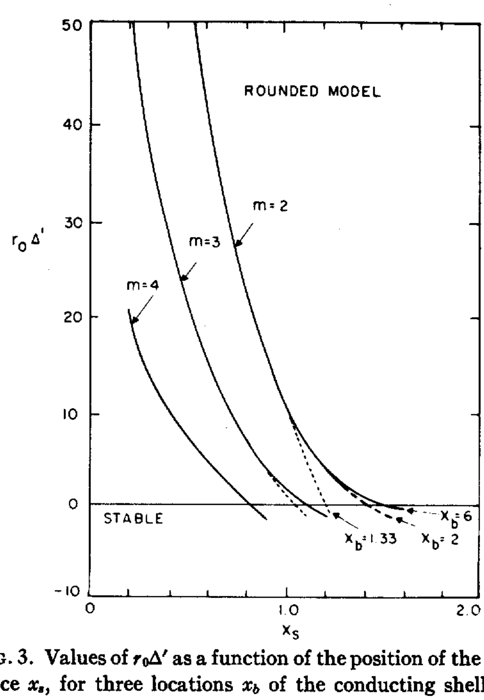

FIG. 3. Values of $r_0\Delta'$ as a function of the position of the singular surface x_s, for three locations x_b of the conducting shell (reproduced from Ref. 6).

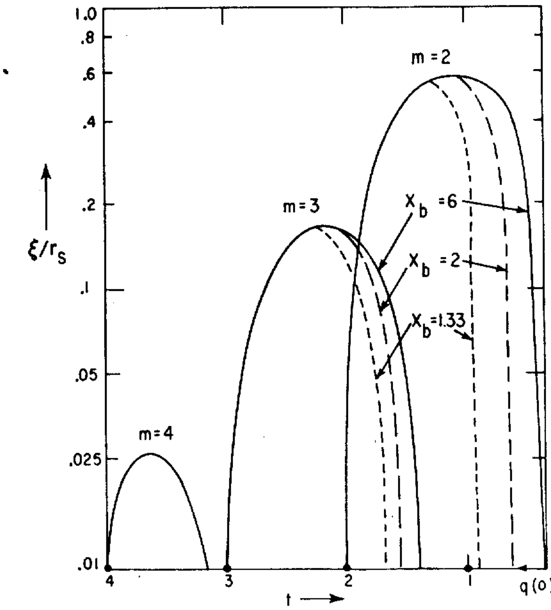

FIG. 4. Mode amplitudes as functions of time for a constant-current tokamak with shrinking current channel, and three locations x_b of the conducting shell. Amplitudes are expressed by the half-widths ξ of the magnetic islands in terms of their radii r_s. Time is expressed by the "safety factor" on axis $q(0)$. Amplitudes are computed taking $C=4$ in Eq. (29); otherwise scale ξ like C^{-1}.

Substituting Eqs. (28) and (30) into Eq. (24):

$$\frac{\xi}{r_s}=\frac{0.35(1+x_s^4)^{3/4}}{Cx_s^3}\int_0^{x_s}\frac{(\Delta' r_0)x_s^5}{(1+x_s^4)^{1/4}}dx_s. \tag{31}$$

Values of $\Delta' r_0$ as a function of x_s may be found in Ref. 6, and are reproduced in Fig. 3. We insert these into Eq. (31), evaulate the integral numerically, and thus obtain values of ξ/r_s as functions of the position x_s of the singular surface for the mode being considered. The results are made more transparent by transforming the independent variable x_s to $q(0)$, using $m=q(0)(1+x_s^4)^{1/2}$, since $q(0)$ is a rough (reverse) measure of time. When this has been done, the results[9] are as shown in Fig. 4, which has been calculated for $C=4$; for other values of C the amplitudes ξ/r_s simply scale like C^{-1}.

The relative amplitudes of the $m=4$, 3, and 2 modes shown in Fig. 4 are in good agreement with the observations. The absolute amplitudes of the modes are a little larger than the observations would suggest, but the absolute levels in our theory depend upon our choice of $C=4$; this choice was made by comparing Eq. (29) with the typical 60-kA ST-tokamak discharge, in which the current channel shrinks from $r_0\sim10$ cm to $r_0\sim6$ cm in about 10 msec with $T_e(r_0)\sim750$ eV and $\eta/\eta_{\mathrm{Sp}}\sim4$ where η_{Sp} denotes the Spitzer resistivity. Figure 4 reveals one further interesting consequence of our theory: there is a small interval of time during which both the $m=3$ and $m=2$ modes are present; this occurs despite the fact that, according to the linear stability criteria,[6] the $m=3$ mode becomes stable before the onset of the $m=2$ mode. The reason is clear from Eq. (24): nonlinearly, the $m=3$ mode takes a certain time to relax after Δ' has become negative. Figure 4 reveals that this relaxation of the $m=2$ and 3 modes after linear stability is achieved is, typically, aided by a close conducting shell. (Our results with a close conducting shell at r_b are offered only as a rough indication of the effect since, in our model, $x_b=r_b/r_0$ remains fixed and the position of the shell moves with r_0; thus, the case $x_b=1.33$ contemplates a shell unreasonably close to the shrunken current channel.)

IV. DISCUSSION

It is obvious that our nonlinear analysis is restricted to that class of tearing modes for which, in the linear theory, the "constant-ψ" approximation is valid for the singular layer. At least in the limit of large magnetic Reynolds number τ_S/τ_H, this class includes all modes with $ka\sim 1$, where a is some typical scale length in our x direction, and, in particular, it includes $m\geq 2$ modes in tokamaks. However, as was pointed out in Appendix D of Ref. 1, if arbitrarily long wavelengths in our y direction are permitted, the fastest growing linear mode has $ka\sim(\tau_H/\tau_S)^{1/4}\ll 1$. The large growth rate arises due to the fact that such a mode would be only weakly stable (indeed marginal in the limit $ka\rightarrow 0$) in perfect-conductivity theory. The "constant-ψ" approximation fails for this long wavelength mode,[1] as does our assumption (8). That this mode might be able to exponentiate to large amplitudes, before nonlinear effects restrain it, is demonstrated by a recent computational treatment[2] in which, by appropriate choice of boundary conditions, this is the principal mode excited; (note, however, that the choice[2] $\tau_S/\tau_H\sim 10^3$ does not permit a very clear distinction between the modes with $ka\sim 1$ and the fastest growing mode). This long-wavelength mode in the slab geometry is analogous to the $m=1$ mode in the tokamak. Because of the cylindrical geometry, the theoretical distinction between $m=1$ and $m\geq 2$ is more pronounced, however, since $m=1$ is not merely marginal, but actually unstable, in perfect-conductivity theory. Although the nonlinear theory of this $m=1$ instability, assuming perfect conductivity, has been worked out,[10] the theory with resistivity remains to be done.

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