Physics 4E — Final Solutions — Spring 2015

Problem 1

$$P = \sigma T^4 \ , \quad \sigma = 5.6703 \times 10^{-8} \frac{W}{m^2 K^4} \quad \text{per unit area}$$

$$A = \text{area} = 2\pi r l = 2\pi \cdot 10^{-3} \cdot 2 \cdot 10^{-2} m^2 = 4\pi \cdot 10^{-5} m^2$$

$$P_{tot} = \sigma T^4 \cdot A = 40\,W \Rightarrow$$

$$T = \left(\frac{40\,W}{\sigma A}\right)^{1/4} = \left(\frac{40}{5.6703 \cdot 10^{-8} \cdot 4\pi \cdot 10^{-5}}\right)^{1/4} K$$

$$\Rightarrow \boxed{T = 2737\,K}$$

(b) $$\lambda_m = \frac{2.898 \times 10^{-3} m}{2737} = \boxed{10{,}588\,\text{Å}}$$

(c) $$R(\lambda) = \frac{C}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \quad ; \quad \frac{hc}{\lambda_m kT} = 4.96$$

$$\frac{R(\lambda_m)}{R(\lambda_m/2)} = \frac{1}{2^5} \frac{e^{2hc/\lambda_m kT} - 1}{e^{hc/\lambda_m kT} - 1} = \frac{1}{2^5} \frac{e^{9.92} - 1}{e^{4.96} - 1} = 4.5$$

$\boxed{\text{4.5 times more power}}$

Problem 2

de Broglie wavelength: $\lambda = \frac{h}{p}$

Compton wavelength $\lambda_c = \frac{h}{mc}$

$$\lambda = \lambda_c \Rightarrow \frac{h}{p} = \frac{h}{mc} \Rightarrow p = mc$$

Classically:

$$K = \frac{p^2}{2m} = \frac{mc^2}{2} = 255,500\ \text{eV}$$

Relativistically:

$$K = \sqrt{p^2c^2 + m^2c^4} - m^2c^2 = \sqrt{m^2c^4 + m^2c^4} - mc^2 \Rightarrow$$

$$\Rightarrow K = (\sqrt{2} - 1)\, mc^2 = 0.414\, mc^2 = 211,663\ \text{eV}$$

(c) The speed is close to the speed of light, so the classical answer is not correct, the relativistic one is

Speed: $\begin{matrix} E = \gamma mc^2 \\ p = \gamma m v \end{matrix} \Rightarrow v = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{p\cancel{c^2}}{\sqrt{2}\, m\cancel{c^2}} = \frac{c}{\sqrt{2}} \Rightarrow$

$$\boxed{v = 0.707c}$$

Problem 3

$$L = m v r_n = n\hbar \Rightarrow m v a_0 n^2 = n\hbar \Rightarrow$$

$$\Rightarrow n = \frac{\hbar}{m v a_0} = \frac{\hbar c}{m c^2 v a_0} = \frac{1973 \times 300{,}000}{511{,}000 \times 219 \times 0.529} = 10$$

$$\Rightarrow \boxed{n = 10}$$

(b) $\boxed{r_n = a_0 n^2 = 52.9\ \mathring{A}}$

(c) $$f = \frac{v}{2\pi r} = \frac{219 \times 10^{13}}{2\pi \times 52.9}\ Hz = \boxed{6.589 \times 10^{12}\ Hz}$$

(d) $$hf = 13.6\,eV\left(\frac{1}{9^2} - \frac{1}{10^2}\right) = 0.0319\,eV$$

$h = 4.136 \times 10^{-15}\ eV.s \Rightarrow \boxed{f = 7.71 \times 10^{12}\ Hz}$

(e) The answers in (d) and (e) would be the same for very large n (Bohr's correspondence principle).

Since $n = 10$ is not all that large, the answers are close but not the same.

Problem 4

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\,\Psi = E\,\Psi$$

$$\Psi(x) = C(1-x^2)^2 = C(1-2x^2+x^4)$$

$$\frac{\partial\Psi}{\partial x} = C(-4x+4x^3), \quad \frac{\partial^2\Psi}{\partial x^2} = C(-4+12x^2) \Rightarrow$$

$$\frac{\hbar^2}{2m}(4-12x^2) + V(x)(1-x^2)^2 = E(1-x^2)^2 \Rightarrow$$

$$\Rightarrow \boxed{V(x) = E + \frac{2\hbar^2}{m}\frac{(3x^2-1)}{(1-x^2)^2}}$$

$$V(x=0) = 0 = E - \frac{2\hbar^2}{m} \Rightarrow \boxed{E = \frac{2\hbar^2}{m} = 15.24\,eV}$$

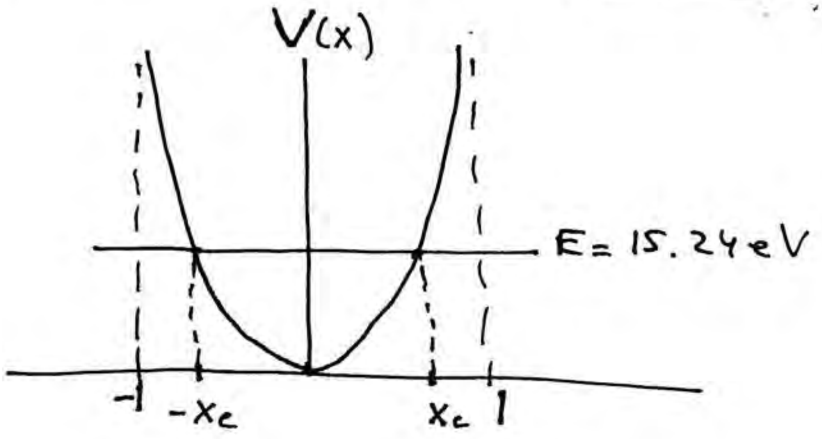
(b) $$V(x=0.5) = \frac{2\hbar^2}{m}\left(1 + \frac{\frac{3}{4}-1}{\left(\frac{3}{4}\right)^2}\right) = \frac{2\hbar^2}{m}\cdot\frac{5}{9} = \frac{10}{9}\frac{\hbar^2}{m} = 8.47\,eV$$

$$\boxed{V(x=0.5) = 8.47\,eV}$$

(c) Classical turning point: $E = V(x_c) \Rightarrow$

$$\Rightarrow 3x_c^2 - 1 = 0 \Rightarrow x_c^2 = \frac{1}{3} \Rightarrow \boxed{x_c = \pm\frac{1}{\sqrt{3}} = 0.577}$$

(d)



Problem 5

transmission probability is approximately

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(V-E)}\,d}$$

energy is $E = \frac{\hbar^2\pi^2}{2mL^2} = \frac{\hbar^2\pi^2}{2m(6\text{Å})^2} = 1.04\,\text{eV}$

$$T_l = 2T_r \Rightarrow \frac{T_l}{T_r} = 2$$

$$\frac{T_l}{T_r} = e^{-2\sqrt{\frac{2m}{\hbar^2}(V_l-E)}\,d_1 + 2\sqrt{\frac{2m}{\hbar^2}(V_r-E)}\,d_3}$$

$$\sqrt{\frac{2m}{\hbar^2}(V_l-E)} = 5.10\,\text{Å}^{-1} \; ; \; \sqrt{\frac{2m}{\hbar^2}(V_r-E)} = 3.58\,\text{Å}^{-1} \Rightarrow$$

$$2(3.58\,d_3 - 5.10\,d_1) = \ln 2 \Rightarrow$$

$$\Rightarrow d_3 = \frac{\frac{\ln 2}{2} + 5.10\,d_1}{3.58} \Rightarrow \boxed{d_3 = 0.24\,\text{Å}}$$

Problem 6

$$E_{n_1 n_2} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}\right) = \frac{\hbar^2\pi^2}{2mL_1^2}\left(n_1^2 + \frac{n_2^2}{4}\right)$$

$L_1 = 4\,\text{Å}$, $L_2 = 8\,\text{Å} = 2L_1$

To put in 5 electrons we need the three lowest energy states:

n_1	n_2	$n_1^2 + n_2^2/4$
1	1	$1 + 1/4 = 5/4$
1	2	$1 + 1 = 2$
2	1	$4 + 1/4 = 17/4$
1	3	$1 + 9/4 = 13/4$

$\Rightarrow$ 3 lowest states are $(1,1)$, $(1,2)$, $(1,3)$

Let $E_0 = \frac{\hbar^2\pi^2}{2mL_1^2} = \frac{3.81\pi^2}{4^2}\,\text{eV} = 2.35\,\text{eV}$

The groundstate is: 2 electrons in $(1,1)$, 2 electrons in $(1,2)$, 1 electron in $(1,3)$

$\Rightarrow$ energy is

$$E = E_0 \cdot \left[2 \times \frac{5}{4} + 2 \times 2 + \frac{13}{4}\right] = \frac{E_0}{4}\left[10 + 16 + 13\right] = \frac{39}{4} E_0$$

$$\Rightarrow \quad \boxed{E = \frac{39}{4} E_0 = 22.91\,\text{eV}}$$

Problem 7

$R(r) = Cr\left(1 - \frac{r}{a_0}\right)e^{-r/a_0}$; $P(r) = r^2R^2(r) \Rightarrow$

$$P(r) = C^2r^4\left(1 - \frac{2r}{a_0} + \frac{r^2}{a_0^2}\right)e^{-2r/a_0}$$

$\int_0^\infty dr\, P(r) = 1$ determines $C \Rightarrow$

$$1 = C^2\int_0^\infty dr\, r^4\left(1 - \frac{2r}{a_0} + \frac{r^2}{a_0^2}\right)e^{-2r/a_0} =$$

$$= C^2\left[\frac{4!\,a_0^5}{2^5} - \frac{2\cdot 5!}{2^6}a_0^5 + \frac{6!}{2^7}a_0^5\right] =$$

$$= C^2\left[\frac{3}{2^2} - \frac{15}{2^2} + \frac{45}{2^3}\right]a_0^5 = \frac{C^2a_0^5}{2^3}\cdot\left[6 - 30 + 45\right] = \frac{21}{8}C^2a_0^5$$

$$\Rightarrow \boxed{C = \left(\frac{8}{21}\right)^{1/2}\frac{1}{a_0^{5/2}}}$$

(b) The exponential part of $R(r)$ is of the form $e^{-Zr/na_0} \Rightarrow$ $\boxed{Z = n}$. The degree of the polynomial multiplying the exponential is $n-1$ $\Rightarrow n-1 = 2 \Rightarrow \boxed{n = 3} \Rightarrow \boxed{Z = 3}$

For small r the polynomial goes as $r^l \Rightarrow \boxed{l = 1}$

(c) $\langle \frac{1}{r} \rangle = C^2 \int_0^\infty dr\, r^3 \left(1 - \frac{2r}{a_0} + \frac{r^2}{a_0^2}\right) e^{-2r/a_0} =$

$= C^2 \left[\frac{3!}{2^4} a_0^4 - \frac{2 \cdot 4!}{2^5} a_0^4 + \frac{5!}{2^6} a_0^4 \right] =$

$= C^2 a_0^4 \left[\frac{3}{2^3} - \frac{3}{2} + \frac{15}{2^3} \right] = C^2 a_0^4 \frac{3}{4} = \frac{8}{21} \cdot \frac{3}{4} \frac{a_0^4}{a_0^5} = \frac{2}{7a_0}$

$\Rightarrow \boxed{\langle \frac{1}{r} \rangle = \frac{2}{7a_0}}$

(d) $V = -\frac{Z k e^2}{r} \Rightarrow \langle V \rangle = -Z k e^2 \cdot \frac{2}{7a_0} \Rightarrow$

$\langle V \rangle = -Z \cdot 14.4 \cdot \frac{2}{7 \cdot 0.529} \, eV = \boxed{-7.78\, Z eV} = \boxed{-23.34 eV}$

(e) $\langle \frac{p_{op}^2}{2m} \rangle = E - \langle V \rangle = -\frac{Z^2}{n^2} \times 13.6 eV + 7.78 Z eV \Rightarrow$

$\langle \frac{p_{op}^2}{2m} \rangle = Z\left(7.78 - 1.51 Z\right) = \boxed{9.75 eV}$

Problem 8

Vibrational state changes from $v = 0$ to $v = 1$, rotational from $l = 0$ to $l = 1$. So $l(l+1)$ changes from 0 to 2

$$hf = \hbar\omega + \frac{\hbar^2}{2I}\cdot 2$$

$I = \frac{1}{2} m R^2$, with $m = m_p =$ mass of proton

$$hf = \hbar\omega + \frac{2\hbar^2}{mR^2} \qquad . \quad \frac{\hbar^2}{2m_e} = 3.81\,\text{eV}\,\text{Å}^2, \quad \frac{m_e}{m_p} = \frac{1}{1836}$$

For H_2^+, $R = 1.06\,\text{Å}$. $\frac{2\hbar^2}{mR^2} = \frac{4 \times 3.81}{1836 \cdot 1.06^2}\,\text{eV} = 0.0074\,\text{eV}$

For H_2, $R = 0.74\,\text{Å} \Rightarrow \frac{2\hbar^2}{mR^2} = 0.015\,\text{eV}$

$\Rightarrow$ For H_2^+, $hf = \frac{hc}{\lambda} = 0.2074\,\text{eV} \Rightarrow \boxed{\lambda = 59{,}788\,\text{Å}}$

For H_2, $hf = \frac{hc}{\lambda} = 0.215\,\text{eV} \Rightarrow \boxed{\lambda = 57{,}674\,\text{Å}}$

(b) The characteristic temperature for rotation is $T_R = \frac{\hbar^2}{2Ik}$

Since I is larger for H_2^+, $T_R(H_2^+) < T_r(H_2)$

$\Rightarrow$ it becomes 'classical' earlier $\Rightarrow$ $\boxed{\text{molar heat capacity of } H_2^+ > 2R}$ at the temperature where it is $2R$ for H_2

(c) $T_R(H_2^+) = T_R(H_2)\left(\frac{R(H_2)}{R(H_2^+)}\right)^2 = 0.487 \times 74\,\text{K} = \boxed{36\,\text{K}}$