Physics 4E — Quiz 3 redo solutions — Spring 2015

Problem 1: $L = 12\,\mathring{A}$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L} x\right) \quad ; \text{ let } x_1 = 3\mathring{A},\ x_2 = 7\mathring{A}$$

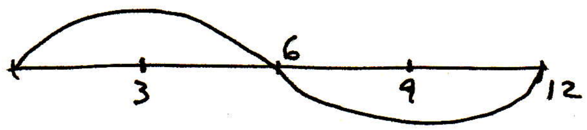
$$\frac{P(x_1)}{P(x_2)} = 4 = \frac{\sin^2\left(\frac{\pi n}{L} x_1\right)}{\sin^2\left(\frac{\pi n}{L} x_2\right)} = \frac{\sin^2 \frac{\pi}{4} n}{\sin^2 \frac{7\pi}{12} n}$$

$n=1$: $$\frac{\sin^2 \frac{\pi}{4}}{\sin^2 \frac{7\pi}{12}} = \frac{0.5}{0.93} \neq 2$$

$n=2$: $$\frac{\sin^2 \frac{\pi}{2}}{\sin^2 \frac{7\pi}{6}} = \frac{1}{0.25} = 4 \Rightarrow \boxed{n=2}$$

$$\Rightarrow \quad E_n = E_2 = \frac{\hbar^2 \pi^2}{2mL^2} \cdot 4 = \boxed{1.045\ \text{eV}}$$

(b)



probability to find electron in region $0 < x < 3$ is $\boxed{0.25}$

(c) $$E_1 = \frac{\hbar^2 \pi^2}{2mL^2} = 0.261\ \text{eV}$$

$$E_2 - E_1 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_2 - E_1} = \boxed{15{,}819\,\mathring{A}}$$

This transition is allowed by the electric dipole selection rule because $n=2$ is odd, $n=1$ is even.

Problem 1(d) (in class)

Electron in $n=2$ can make transitions absorbing photons to n's odd $\Rightarrow$

$n=3,\ n=5,\ n=7, \ldots$

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} = 0.261\,eV\,n^2$$

$$E_3 - E_2 = 0.261\,eV(9-4) = 1.31\,eV \ ; \ \lambda_{23} = \frac{hc}{E_3 - E_2} = \boxed{9493\,\mathring{A}}$$

$$E_5 - E_2 = 0.261\,eV(25-4) = 5.481\,eV \ ; \ \boxed{\lambda_{25} = 2262\,\mathring{A}}$$

$$E_7 - E_2 = 0.261\,eV(49-4) = 11.75\,eV \Rightarrow \lambda_{27} = 1056\,\mathring{A} < 2000\,\mathring{A}$$

So the wavelengths are $\boxed{9493\,\mathring{A},\ 2262\,\mathring{A}}$

Problem 2

(a) $\psi(x) = A\sin(kx)$ since $\psi(x=0)=0$

k is found from $E = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{4}{3.81}}\ \text{Å}^{-1} = \boxed{1.025\ \text{Å}^{-1}}$

(b) $\frac{P(x=1)}{P(x=0.5)} = \frac{\sin^2(k \cdot 1)}{\sin^2(k \cdot 0.5)} = \frac{\sin^2(1.025)}{\sin^2(0.5125)} = \boxed{3.04 \text{ more likely}}$

(c) $V_0 = 10^{10}\,\text{eV} \sim \infty$. For an infinite square well,

$$E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2 = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \frac{\pi}{L} n \Rightarrow \boxed{L = \frac{\pi n}{k}}$$

So smallest L is $n=1$, next smallest is $n=2$. $\boxed{L = 3.065\ \text{Å}}$ smallest

(d) $\boxed{L = 6.13\ \text{Å}}$ next smallest

(e) For $V_0 < \infty$, match boundary conditions. Same algebra as in quiz 3 $\Rightarrow \tan(kL) = -\frac{k}{\alpha}$

$$\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} = \sqrt{\frac{385-4}{3.81}}\ \text{Å}^{-2} = 10\ \text{Å}^{-1}$$

$\Rightarrow \tan(kL) = -\frac{k}{10\ \text{Å}^{-1}} = -0.1025 \Rightarrow kL = \tan^{-1}(-0.1025) = -0.102$

But we can't have $kL < 0$, need to add π or 2π (tan is periodic)

Smallest L: add π. $kL = \pi - 0.102 = 3.039 \Rightarrow \boxed{L = 2.97\ \text{Å}}$

Next smallest L: add 2π: $kL = 2\pi - 0.102 = 6.18 \Rightarrow \boxed{L = 6.03\ \text{Å}}$

Problem 2 (in class)

(g) Add 3π to $kL = -0.102 \to 9.32$

$$\Rightarrow \boxed{L = \frac{9.32}{1.025} \text{Å} = 9.095 \text{Å}}$$

(h) Smallest L corresponds to

$$kL = \frac{\pi}{2}, \quad E = V_0$$

For $E = \dfrac{\hbar^2 k^2}{2m} = V_0 \Rightarrow k = \sqrt{\dfrac{2m}{\hbar^2} V_0} = 10.05 \text{Å}^{-1}$

$$\Rightarrow \boxed{L = \frac{\pi}{2k} = 0.156 \text{Å}}$$

The energy of the electron would be 385 eV.

Problem 3

$$\Psi(x) = Cxe^{-\lambda x^2} \quad , \quad \lambda = 0.5\,\text{Å}^{-2}$$

(a) $1 = \int dx\, |\Psi(x)|^2 = C^2 \int dx\, x^2 e^{-2\lambda x^2} = \frac{1}{2}\pi^{1/2}(2\lambda)^{-3/2}C^2 \Rightarrow$

$$\Rightarrow C^2 = \frac{2\,(2\lambda)^{3/2}}{\pi^{1/2}} = 1.128 \Rightarrow \boxed{C = 1.062\,\text{Å}^{-3/2}}$$

(b) $\frac{d\Psi}{dx} = Ce^{-\lambda x^2} - 2C\lambda x^2 e^{-\lambda x^2}$

$$\boxed{\frac{d^2\Psi}{dx^2} = -6C\lambda x e^{-\lambda x^2} + 4C\lambda^2 x^3 e^{-\lambda x^2}}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi = E\Psi \Rightarrow$$

$$\frac{3\hbar^2\lambda x}{m} - \frac{2\hbar^2\lambda^2}{m}x^3 + V(x)\,x = Ex \quad \left(Ce^{-\lambda x^2} \text{ cancelled out}\right)$$

$$\Rightarrow \boxed{V(x) = E - \frac{3\hbar^2\lambda}{m} + \frac{2\hbar^2\lambda^2}{m}x^2}$$

(c) $V(x=1) - V(x=0) = \frac{2\hbar^2\lambda^2}{m} = 4 \times 3.81\,\text{eV}\,\text{Å}^2 \times (0.5\,\text{Å})^2 \Rightarrow$

$$\boxed{V(x=1) - V(x=0) = 3.81\,\text{eV}}$$

(d) If $V(x=0) = 0 \Rightarrow E = \frac{3\hbar^2\lambda}{m} = \frac{6\hbar^2}{2m}\lambda = 11.43\,\text{eV}$

$$\Rightarrow \boxed{E = 11.43\,\text{eV}}$$

(e) This electron is in the first excited state of a harmonic oscillator potential

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) \Rightarrow E = \tfrac{3}{2}\hbar\omega = 11.43\,eV \Rightarrow$$

$$\boxed{\hbar\omega = 7.62\,eV}$$

Check: wavefunction is $\psi(x) \propto x\, e^{-\frac{m\omega}{2\hbar}x^2} \Rightarrow$

$$\lambda = \frac{m\omega}{2\hbar} = \frac{m\hbar\omega}{2\hbar^2} = \frac{1}{4}\frac{2m}{\hbar^2}\hbar\omega = \frac{1}{4\times 3.81} \times 7.62\,Å^{-2} = 0.5\,Å^{-2}$$

Wavelength of photon: $\frac{hc}{\lambda} = \hbar\omega \Rightarrow \boxed{\lambda = 1627.3\,Å}$

(f) Use average potential energy = average kinetic energy

$$\left\langle \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \right\rangle = \frac{3}{2}\hbar\omega \Rightarrow$$

$$\frac{1}{2}m\omega^2\langle x^2\rangle = \frac{3}{4}\hbar\omega \Rightarrow \langle x^2\rangle = \frac{2}{m\omega^2}\cdot\frac{3}{4}\hbar\omega = \frac{3\hbar}{2m\omega} \Rightarrow$$

$$\Rightarrow \langle x^2\rangle = \frac{3\hbar^2}{2m\hbar\omega} = \frac{3\times 3.81}{7.62}\,Å^2 = \frac{3}{2}Å^2 \Rightarrow$$

$$\Rightarrow \boxed{\Delta x = \sqrt{\tfrac{3}{2}}\,Å = 1.22\,Å}$$

(g) $$\frac{\langle p^2\rangle}{2m} = \frac{3}{4}\hbar\omega \Rightarrow \langle p^2\rangle = \frac{3}{2}m\hbar\omega \Rightarrow$$

$$\Rightarrow \frac{\langle p^2\rangle}{\hbar^2} = \frac{3}{2}\frac{m}{\hbar^2}\hbar\omega = \frac{3}{4}\frac{2m}{\hbar^2}\hbar\omega = 1.5\,Å^{-2} = \frac{3}{2}Å^{-2} \Rightarrow$$

$$\Rightarrow \frac{\Delta p}{\hbar} = \sqrt{\tfrac{3}{2}}\,Å^{-1} = 1.22\,Å^{-1}$$

(h) $$\Delta p\,\Delta x = \sqrt{\tfrac{3}{2}}\cdot\sqrt{\tfrac{3}{2}}\,\hbar = \frac{3}{2}\hbar = 1.5\hbar > \frac{\hbar}{2}$$

Problem 3 (in class)

(i) $\psi(x) \propto x e^{-\lambda x^2}$

$$\psi' = e^{-\lambda x^2} - 2\lambda x^2 e^{-\lambda x^2} = 0 \Rightarrow$$

$$x^2 = \frac{1}{2\lambda} \Rightarrow x = \pm\sqrt{\frac{1}{2\lambda}} \quad ; \quad \lambda = 0.5\,\mathring{A}^{-2}$$

$$\Rightarrow \boxed{x = \pm 1\,\mathring{A}}$$

(j) Classical amplitude

$$\frac{1}{2} m\omega^2 A_c^2 = E \Rightarrow A_c^2 = \frac{2E}{m\omega^2} = \frac{2 \cdot \frac{3}{2}\hbar\omega}{m\omega^2} =$$

$$= \frac{3\hbar}{m\omega} = \frac{3\hbar^2}{m\hbar\omega} = \frac{6 \times 3.81\,\mathring{A}^2}{7.62} = 3\,\mathring{A}^2 \Rightarrow$$

$$\boxed{A_c = \sqrt{3}\,\mathring{A} = 1.73\,\mathring{A}}$$