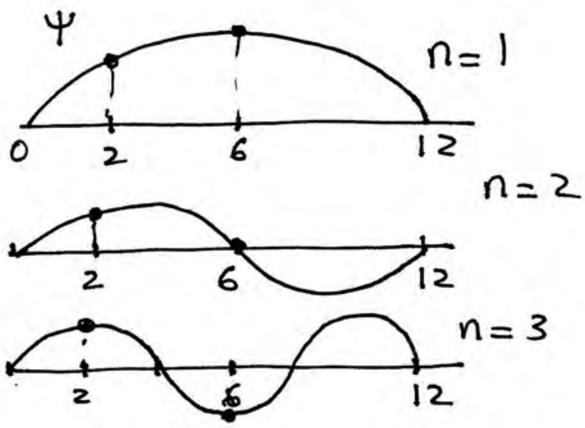
Physics 4E   Quiz 3 Solutions   Spring 2015

Problem 1



Consider lowest energy states

The probability is $\propto |\Psi|^2$

For $n=3$, $|\Psi(x=2)|^2 = |\Psi(x=6)|^2$

For $n=1$, $n=2$ they are not the same

$\Rightarrow$ electron is in state $n=3$.

$$E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2 = 3.81\,eV\mathring{A}^2 \frac{\pi^2}{(12\mathring{A})^2} n^2 = 0.261 n^2\,eV$$

$\Rightarrow \boxed{E_3 = 2.35\,eV}$ (a)   For (c) we will need $E_2 = 1.045\,eV$, $E_1 = 0.261\,eV$

(b) The wavefunction is $\Psi_3(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi}{L}x\right)$

Let $x_1 = 2 = L/6$; $x_2 = 1 = L/12$

$$\frac{\Psi(x_1)}{\Psi(x_2)} = \frac{\sin(\pi/2)}{\sin(\pi/4)} = \frac{1}{\sqrt{2}/2} = \sqrt{2} \Rightarrow \frac{|\Psi(x_1)|^2}{|\Psi(x_2)|^2} = 2$$

$\Rightarrow$ it is twice as likely to be at $x=2$ than at $x=1$

(c) If it emits a photon $\Rightarrow$ makes a transition to a lower energy state, either $n=2$ or $n=1$:

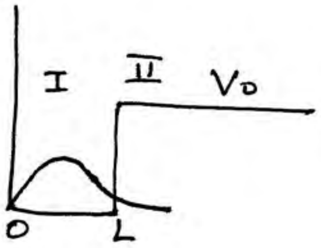
$$E_3 - E_2 = \frac{hc}{\lambda_{32}} \Rightarrow \lambda_{32} = \frac{hc}{E_3 - E_2} = \frac{12,400\,eV\mathring{A}}{2.35\,eV - 1.045\,eV} = \boxed{9502\,\mathring{A}}$$

$$E_3 - E_1 = \frac{hc}{\lambda_{31}} \Rightarrow \lambda_{31} = \frac{hc}{E_3 - E_1} = \frac{12,400}{2.35 - 0.261}\,\mathring{A} = \boxed{5936\,\mathring{A}}$$

Problem 2

$E = 5\,eV$

$\psi(x) = A\sin(kx)$ for $0<x<L$



(a) 10^{20} eV is much much larger than 5 eV, so it's practically infinity. For an infinite square well, in the lowest energy state, $k = \frac{\pi}{L}$. Since here $k = 0.95\frac{\pi}{2}$, we know V_0 is not infinite, so it must be much smaller than 10^{20} eV.

(b) The wavefunction in regions I $(0<x<L)$ and II $(x>L)$ is

$\psi_I(x) = A\sin(kx)$ $\qquad k = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow E = \hbar^2k^2/2m$

$\psi_{II}(x) = Be^{-\alpha x}$ $\qquad \alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

as follows from the Schrödinger eq. $-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$.

Continuity of ψ at $x=L$: $A\sin(kL) = Be^{-\alpha L}$ → dividing,

Continuity of ψ' at $x=L$: $kA\cos(kL) = -\alpha Be^{-\alpha L}$

$$\frac{\tan(kL)}{k} = -\frac{1}{\alpha} \Rightarrow \alpha = -\frac{k}{\tan(kL)} \Rightarrow \alpha^2 = \frac{k^2}{\tan^2(kL)} \Rightarrow$$

$$\Rightarrow \frac{2m}{\hbar^2}(V_0 - E) = \frac{k^2}{\tan^2(kL)} \Rightarrow V_0 = E + \underbrace{\frac{\hbar^2k^2}{2m}}_{E}\frac{1}{\tan^2(kL)} \Rightarrow$$

$$\Rightarrow V_0 = E\left(1 + \frac{1}{\tan^2(kL)}\right) = E\cdot\frac{1}{\sin^2(kL)} = E\cdot\frac{1}{\sin^2(0.95\pi)}$$

$$\boxed{V_0 = \frac{E}{\sin^2(0.95\pi)} = 40.86\,E = 204.3\,eV}$$

Problem 3

The energy of the harmonic oscillator is

$$E = \frac{1}{2} m v^2 + \frac{1}{2} m \omega^2 x^2 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

Taking averages:

$$E = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m \omega^2 \langle x^2 \rangle = E_{kinetic} + E_{potential}$$

In the lowest energy state, $E = E_0 = \frac{\hbar\omega}{2}$. Since the average potential and kinetic energies are equal,

$$\frac{1}{2} m \omega^2 \langle x^2 \rangle = \frac{\hbar\omega}{4} \Rightarrow \langle x^2 \rangle = \frac{\hbar}{2m\omega}$$

From $E_0 = \frac{\hbar\omega}{2} \Rightarrow \omega = \frac{2E_0}{\hbar} \Rightarrow \langle x^2 \rangle = \frac{\hbar}{2m \cdot \frac{2E_0}{\hbar}} = \frac{\hbar^2}{4mE_0} \Rightarrow$

$$\Rightarrow \langle x^2 \rangle = \frac{1}{2} \cdot \frac{3.81 \text{eV} \text{Å}^2}{3 \text{eV}} = 0.635 \text{Å}^2$$

Since $\langle x \rangle = 0 \Rightarrow \boxed{\Delta x = \sqrt{\langle x^2 \rangle} = 0.797 \text{Å}}$

(b) $\frac{\langle p^2 \rangle}{2m} = \frac{\hbar\omega}{4} \Rightarrow \langle p^2 \rangle = \frac{m\hbar\omega}{2} = mE_0$

Since $\langle p \rangle = 0$, $\Delta p = \sqrt{\langle p^2 \rangle} = \sqrt{mE_0} \Rightarrow$

$$\Rightarrow \frac{\Delta p}{\hbar} = \sqrt{\frac{mE_0}{\hbar^2}} = \sqrt{\frac{1}{2} \frac{2m}{\hbar^2} E_0} = \sqrt{\frac{1}{2} \cdot \frac{3 \text{eV}}{3.81 \text{eV} \text{Å}^2}} = 0.627 \text{Å}^{-1}$$

$$\Rightarrow \boxed{\frac{\Delta p}{\hbar} = 0.627 \text{Å}^{-1}}$$

(c) $\Delta x \Delta p = \Delta x \cdot \frac{\Delta p}{\hbar} \cdot \hbar = 0.797 \text{Å} \cdot 0.627 \text{Å}^{-1} \cdot \hbar = 0.5\hbar$

$\Rightarrow \Delta x \Delta p = \frac{1}{2}\hbar$, in agreement with the uncertainty principle.