

Uncertainty, Measurement, and Models

Overview Exp #1

Lecture # 2

Physics 2BL

Summer Session I 2015

Outline

- What uncertainty (error) analysis can for you
- Issues with measurement and observation
- What does a model do?
- General error propagation formula with example
- Overview of Experiment # 1
- Homework

What is uncertainty (error)?

- Uncertainty (or error) in a measurement is not the same as a mistake
- Uncertainty results from:
 - Limits of instruments
 - finite spacing of markings on ruler
 - Design of measurement
 - using stopwatch instead of photogate
 - Less-well defined quantities
 - composition of materials

Understanding uncertainty is important

- for comparing values
- for distinguishing between models
- for designing to specifications/planning

Measurements are less useful (often useless) without a statement of their uncertainty

An example

Batteries

rated for 1.5 V potential difference across
terminals

in reality...

Utility of uncertainty analysis

- Evaluating uncertainty in a measurement
- Propagating errors – ability to extend results through calculations or to other measurements
- Analyzing a distribution of values
- Quantifying relationships between measured values

Evaluating error in measurements

- To measure height of building, drop rock and measure time to fall: $d = \frac{1}{2}gt^2$
- Measure times
2.6s, 2.4s, 2.5s, 2.4s, 2.3s, 2.9s
- What is the “best” value
- How certain are we of it?

Uncertainty in time

- Measured values - (2.6s, 2.4s, 2.5s, 2.4s, 2.3s, 2.9s)
- By inspection can say uncertainty < 0.4 s
- Calculate **standard deviation**
$$\sigma = \sqrt{\sum (t_i - \bar{t})^2 / (n-1)}$$
$$\sigma = 0.2137288 \text{ s}$$
$$\sigma = 0.2 \text{ s} \quad (\text{But what does this mean???)}$$

Propagation of error

- Same experiment, continued...
- From best estimate of time, get best estimate of distance: 31 meters
- Know uncertainty in time, what about uncertainty in distance?
- From error analysis tells us how errors propagate through mathematical functions
(2 meters)

Expected uncertainty in a calculated sum $a = b + c$

– Each value has an uncertainty

- $b = \bar{b} \pm \delta b$

- $c = \bar{c} \pm \delta c$

– Uncertainty for a (δa) is **at most** the sum of the uncertainties

$$\delta a = \delta b + \delta c$$

– Better value for δa is

$$\delta a = \sqrt{(\delta b)^2 + (\delta c)^2}$$

– Best value is

- $a = \bar{a} \pm \delta a$

Expected uncertainty in a calculated product $a = b * c$

– Each value has an uncertainty

- $b = b \pm \delta b$

- $c = c \pm \delta c$

– Relative uncertainty for a (ϵ_a) is **at most** the sum of the RELATIVE uncertainties

$$\epsilon_a = \delta a/a = \epsilon_b + \epsilon_c$$

– Better value for δa is

$$\epsilon_a = \sqrt{(\epsilon_b^2 + \epsilon_c^2)}$$

– Best value is

- $a = a \pm \epsilon_a$ (fractional uncertainty)

What about powers in a product

$$a = b * c^2$$

- Each value has an uncertainty
 - $b = b \pm \delta b$
 - $c = c \pm \delta c$
 - $\epsilon a = \delta a/a$ (relative uncertainty)
- powers become a prefactor (weighting) in the error propagation
 - $\epsilon a^2 = (\epsilon b^2 + (2 * \epsilon c)^2)$

How does uncertainty in t effect the calculated parameter d ?

$$- d = \frac{1}{2} g t^2$$

$$\epsilon d = \sqrt{(2 * \epsilon t)^2} = 2 * \epsilon t$$

$$\epsilon d = 2 * (.09 / 2.52) = 0.071$$

$$\delta d = .071 * 31 \text{ m} = 2.2 \text{ m} = 2 \text{ m}$$

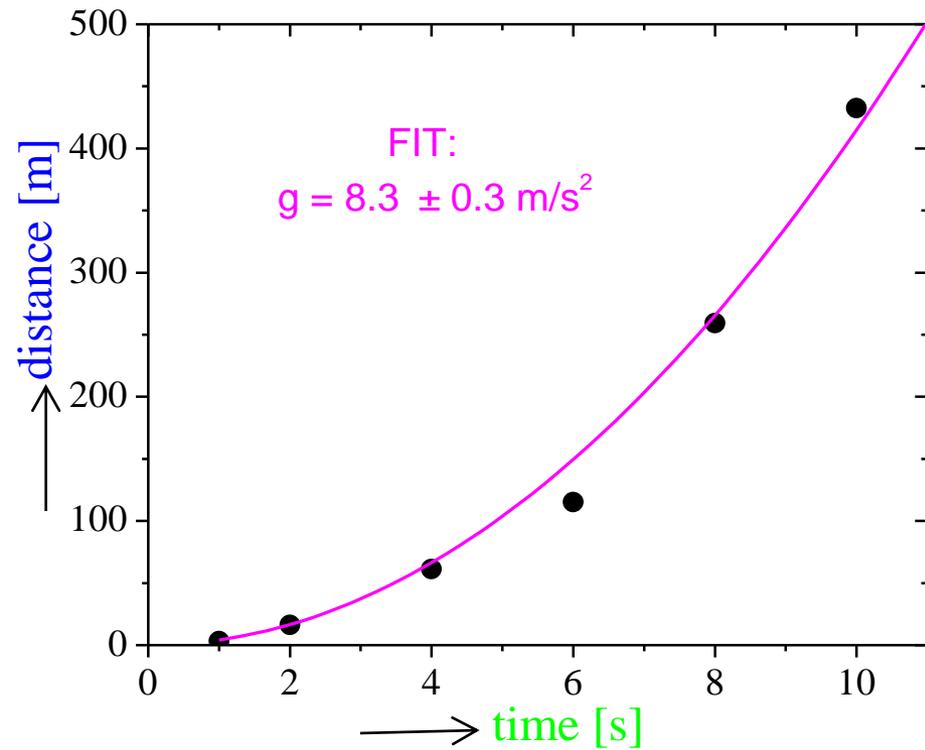
Statistical error

Relationships

- Know there is a functional relation between d and t $d = \frac{1}{2} g t^2$
- d is directly proportional to t^2
- Related through a constant $\frac{1}{2} g$
- Can measure time of drop (t) at different heights (d)
- plot d versus t to obtain constant

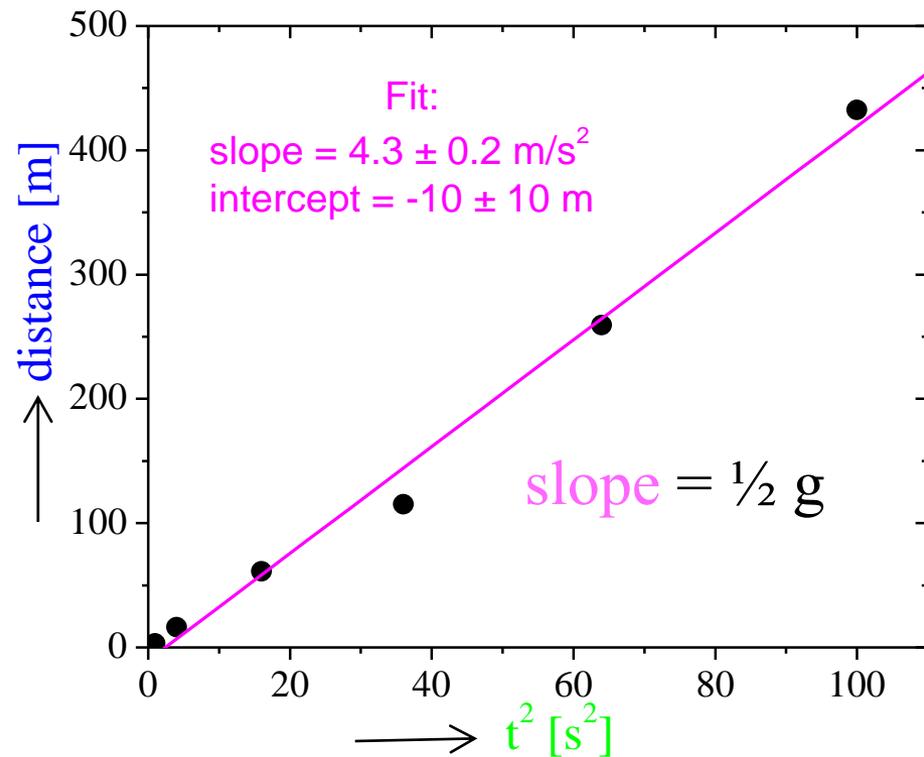
Quantifying relationships

$$d = \frac{1}{2} g t^2$$



$$g = 8.3 \pm 0.3 \text{ m/s}^2$$

$$d = \frac{1}{2} g (t^2)$$



$$g = 8.6 \pm 0.4 \text{ m/s}^2$$

General Formula for error propagation

For independent, random errors

$$\delta q = \left| \frac{dq}{dx} \right| \delta x$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x} \delta x \right)^2 + \left(\frac{\partial q}{\partial y} \delta y \right)^2}$$

Measurement and Observation

- Measurement: deciding the amount of a given property by observation
- Empirical
- Not logical deduction
- Not all measurements are created equal...

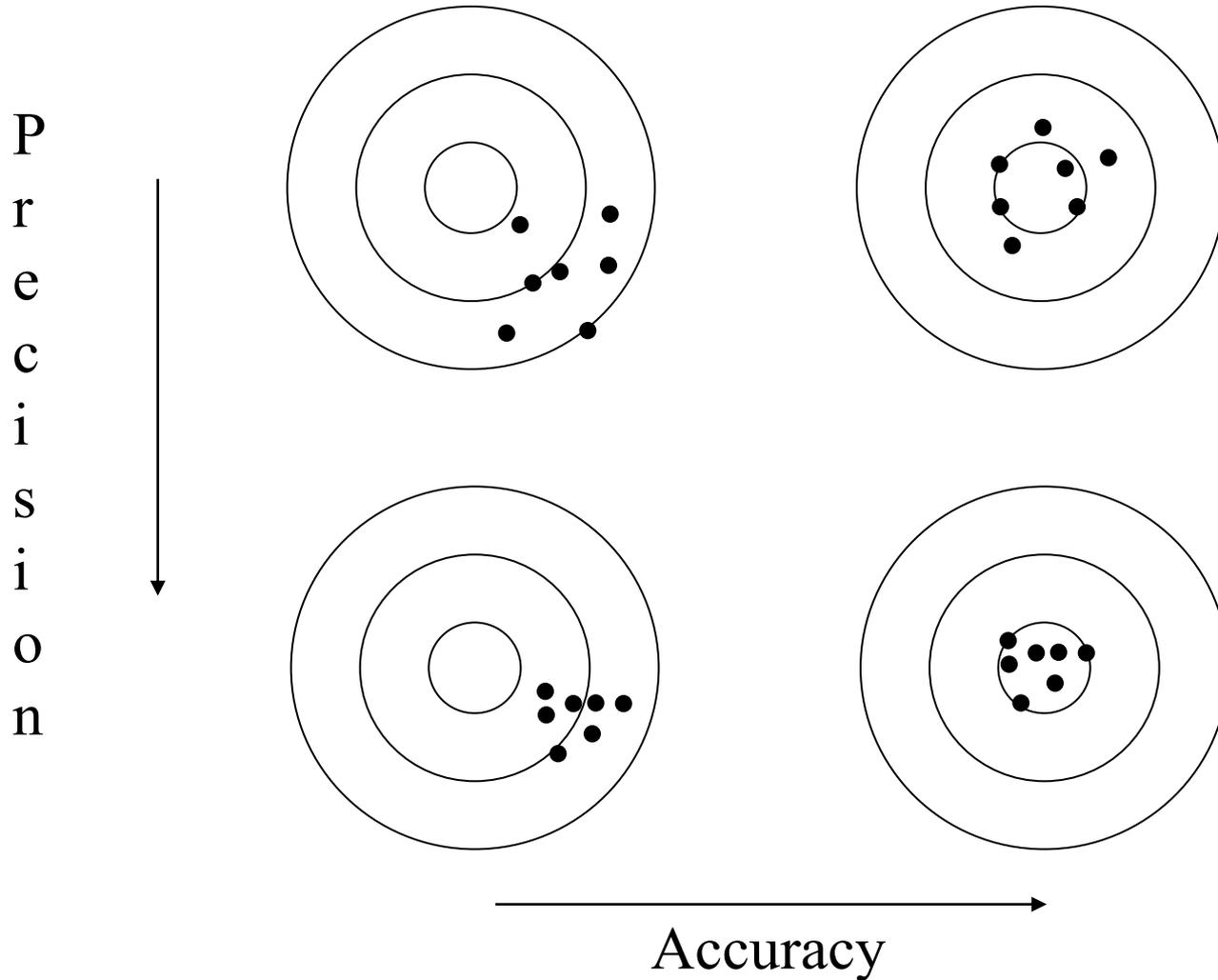
Reproducibility

- Same results under similar circumstances
 - Reliable/precise
- ‘Similar’ - a slippery thing
 - Measure resistance of metal
 - need same sample purity for repeatable measurement
 - need same people in room?
 - same potential difference?
 - Measure outcome of treatment on patients
 - Can’t repeat on same patient
 - Patients not the same

Precision and Accuracy

- Precise - reproducible
- Accurate - close to true value
- Example - temperature measurement
 - thermometer with
 - fine divisions
 - or with coarse divisions
 - and that reads
 - 0 C in ice water
 - or 5 C in ice water

Accuracy vs. Precision



Random and Systematic Errors

- Accuracy and precision are related to types of errors
 - random (thermometer with coarse scale)
 - can be reduced with repeated measurements, careful design
 - systematic (calibration error)
 - difficult to detect with error analysis
 - compare to independent measurement

Observations in Practice

- Does a measurement measure what you think it does? Validity
- Are scope of observations appropriate?
 - Incidental circumstances
 - Sample selection bias
- Depends on model

Models

- Model is a construction that represents a subject or imitates a system
- Used to predict other behaviors (extrapolation)
- Provides context for measurements and design of experiments
 - guide to features of significance during observation

Testing model

- Models must be consistent with data
- Decide between competing models
 - elaboration: extend model to region of disagreement
 - precision: prefer model that is more precise
 - simplicity: Ockham's razor

The Earth

Volume – radius

Mass

Density

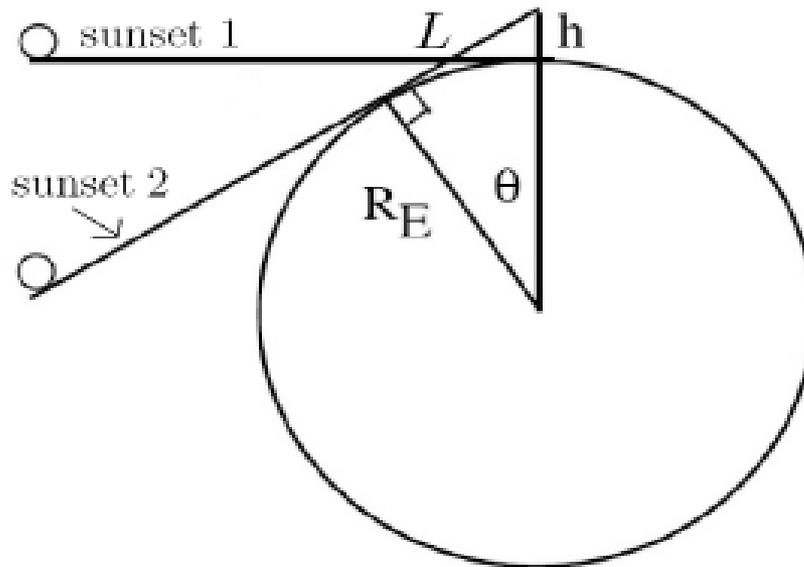


Experiment 1 Overview:

Density of Earth

density

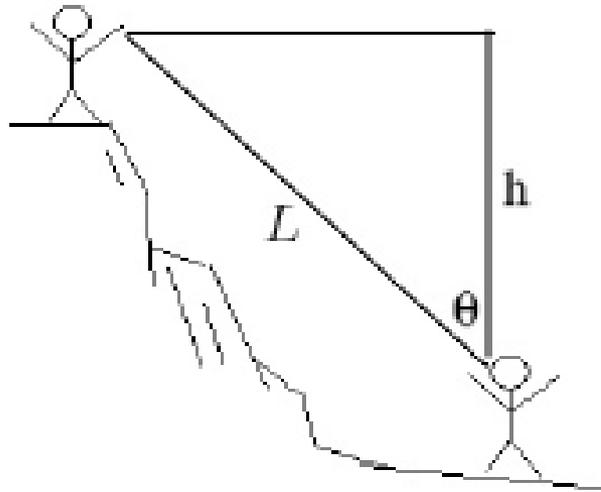
$$\rho = \frac{M_E}{\frac{4}{3}\pi R_E^3} = \frac{3g}{4\pi G R_E} = \frac{GM_E m}{R_E^2} = mg$$



$$R_E = \frac{2h}{\omega^2(\Delta t)^2}$$

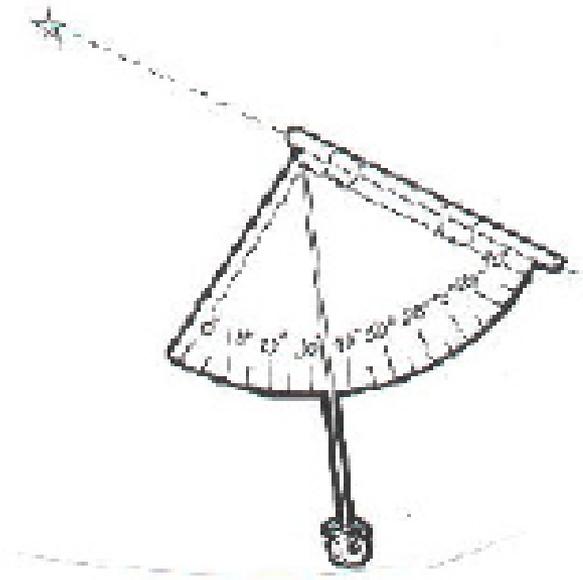
measure Δt between sunset on cliff and at sea level

Experiment 1: Height of Cliff



rangefinder to get L

Wear comfortable shoes



Sextant to get θ

Make sure you use
 θ and not $(90 - \theta)$

Measure Earth's Radius using Δt Sunset

Now, is this time delay measurable?

h - height above the sea level

L - distance to the horizon line

$$t = \frac{L}{2\pi R_e} T = \frac{T}{2\pi} \sqrt{\frac{2h}{R_e}}$$

$$T = 24 \text{ hr} = 24 \cdot 60 \cdot 60 \text{ s}$$

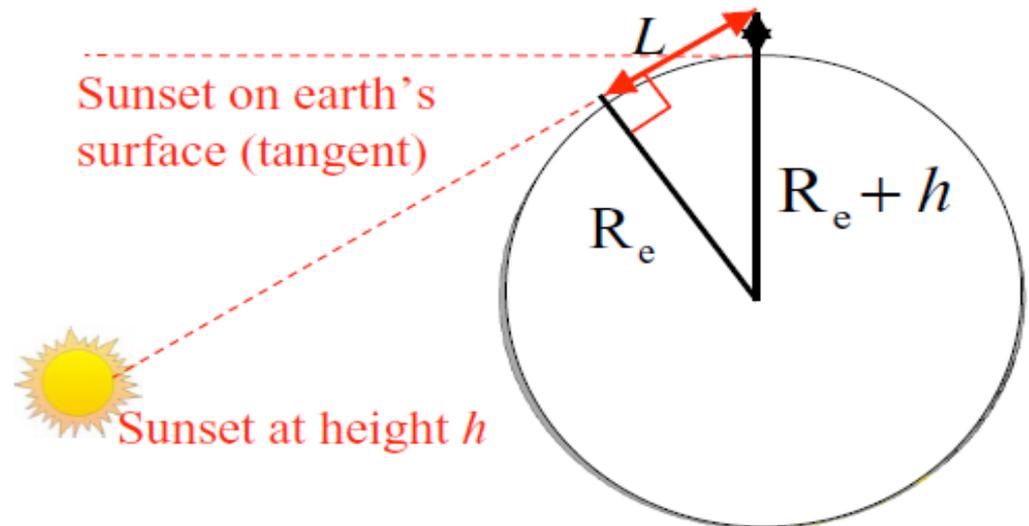
$$= 86400 \text{ s}$$

$$R_e = 6,000,000 \text{ m}$$

$$h \sim 100 \text{ m} \quad \text{- our cliff}$$

$$t = \frac{86400 \text{ s}}{2\pi} \sqrt{\frac{200}{6 \times 10^6}} \approx 80 \text{ s}$$

Looks doable!



Have we forgotten something?

"The Equation" for Experiment 1a

$$t = \frac{T}{2\pi} \sqrt{\frac{2Ch}{R_e}} = \frac{1}{\omega} \sqrt{\frac{2Ch}{R_e}}$$

from previous page.

$$\omega = \frac{2\pi}{24 \text{ hr}}$$

Which are the variables that contribute to the error significantly?

$$\Delta t = t_1 - t_2 = \frac{1}{\omega} \sqrt{\frac{2C}{R_e}} (\sqrt{h_1} - \sqrt{h_2})$$

Time difference between the two sunset observers.

$$C \equiv \frac{1}{\cos^2(\lambda)\cos^2(\lambda_s) - \sin^2(\lambda)\sin^2(\lambda_s)}$$

Season dependant factor slightly greater than 1.

What other methods could we use to measure the radius of the earth?

The formula for your error analysis.

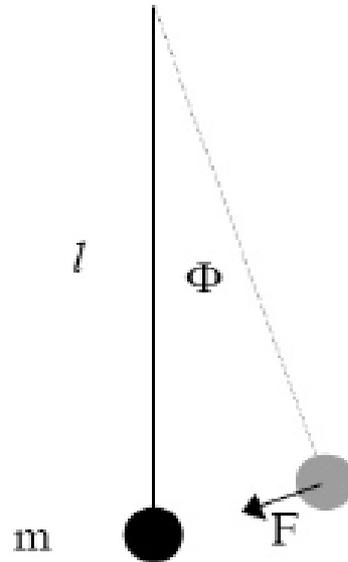
$$R_e = \frac{2C}{\omega^2} \left(\frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t} \right)^2$$

Eratosthenes

angular deviation = angle subtended

Experiment 1: Determine g

pendulum



$$F = -mg\sin(\phi) = -mg\phi$$

$$F = m\alpha = ml\ddot{\phi}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

period

Experiment 1: Pendulum

- For release angle θ_i , you should have a set of time data $(t_1^p, t_2^p, t_3^p, \dots, t_N^p)$.
- Calculate the average, \bar{t}^p , and the standard deviation, σ_{t^p} , of this data.
- Divide \bar{t}^p and σ_{t^p} by p to get average time of a *single* period, \bar{T} and standard deviation of a single period σ_T .
- Calculate SDOM, $\sigma_T = \frac{\sigma_{t^p}}{\sqrt{N}}$.
- Now you should have $T \pm \sigma_T$ for you data at θ_i .
- Repeat these calculations for data at each release angle.

Grading rubric uploaded on website

Error Propagation - example

We saw earlier how to determine the acceleration of gravity, g .

Using a simple pendulum, measuring its length and period:

-Length l : $l = l_{best} \pm \delta l$

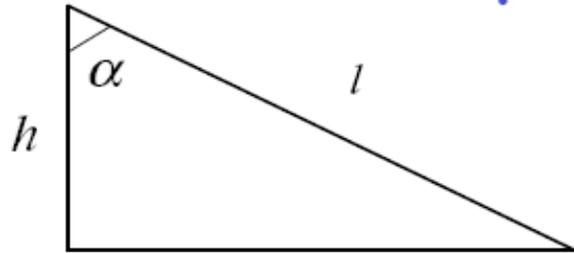
-Period T : $T = T_{best} \pm \delta T$

Determine g by solving:

$$g = l \cdot (2\pi / T)^2$$

The question is what is the resulting uncertainty on g , δg ??

Example



Given that: $l = 10 \pm 0.1 \text{ m}$
 $\alpha = 20 \pm 3^\circ$

=> Find h .

$$h = l \cdot \cos \alpha = 10 \cdot \cos 20^\circ = 10 \cdot 0.94 = 9.4 \text{ m}$$

$$\delta h = \sqrt{\left(\frac{\partial h}{\partial l} \delta l\right)^2 + \left(\frac{\partial h}{\partial \alpha} \delta \alpha\right)^2}$$

$$\frac{\partial h}{\partial l} = \cos \alpha$$

$$\frac{\partial h}{\partial \alpha} = l \cdot (-\sin \alpha)$$

$$\delta h = \sqrt{(\cos \alpha \cdot \delta l)^2 + (l \cdot (-\sin \alpha) \cdot \delta \alpha)^2} = \sqrt{(0.94 \cdot 0.1)^2 + (10 \cdot [-0.34] \cdot 0.05)^2} = 0.2 \text{ m}$$

$$h = 9.4 \pm 0.2 \text{ m}$$

always use radians when calculating the errors on trig functions

$$\delta \alpha = 3^\circ = \frac{2\pi \text{ rad}}{360^\circ} \cdot 3^\circ = 0.05 \text{ rad}$$



Propagating Errors for Experiment 1

$$\rho = \frac{3}{4\pi} \frac{g}{GR_e} \quad \text{Formula for density.}$$

$$\sigma_\rho = \frac{3}{4\pi} \frac{1}{GR_e} \sigma_g \oplus \frac{-3}{4\pi} \frac{g}{GR_e^2} \sigma_{R_e} \quad \text{Take partial derivatives and add errors in quadrature}$$

Or, in terms of relative uncertainties: $\frac{\sigma_\rho}{\rho} = \frac{\sigma_g}{g} \oplus \frac{\sigma_{R_e}}{R_e}$

shorthand notation for quadratic sum: $\sqrt{a^2 + b^2} = a \oplus b$

Propagating Errors for R_e

$$R_e = \frac{2C}{\omega^2} \left(\frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t} \right)^2$$

basic formula

$$\sigma_{R_e} = \frac{\partial R_e}{\partial \Delta t} \sigma_{\Delta t} \oplus \frac{\partial R_e}{\partial h_1} \sigma_{h_1} \oplus \frac{\partial R_e}{\partial h_2} \sigma_{h_2}$$

Propagate errors (use shorthand for addition in quadrature)

$$\sigma_{R_e} = \frac{2R_e}{\Delta t} \sigma_{\Delta t} \oplus \frac{R_e}{\sqrt{h_1} (\sqrt{h_1} - \sqrt{h_2})} \sigma_{h_1} \oplus \frac{R_e}{\sqrt{h_2} (\sqrt{h_1} - \sqrt{h_2})} \sigma_{h_2}$$

Note that the error blows up at $h_1=h_2$ and at $h_2=0$.

Reminder

- Prepare for lab
- Read Taylor chapter 4
- Homework due next meeting - Taylor 4.6, 4.14, 4.18, 4.26