225B - Winter 2014

REVIEW

Maps of Manifolds

$\phi : \mathcal{M} \to \mathcal{M}'$ is a map between manifolds

(is a C^r map if the corresponding map of coordinates is C^r

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\phi"] \arrow[d, "x"'] & \mathcal{M}' \arrow[d, "y"'] \\ \mathbb{R}^n & R^m \end{tikzcd}$$

so $y \circ \phi \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is C^r)

Notes:

- In gnl not one-to-one
- Even if one-to-one may not have an inverse

So it goes one way.

let $f : \mathcal{M}' \to \mathbb{R}$ a fction on $\mathcal{M}'$

(a "scalar field")

then ϕ defines a fction on $\mathcal{M}$, $\phi^* f$

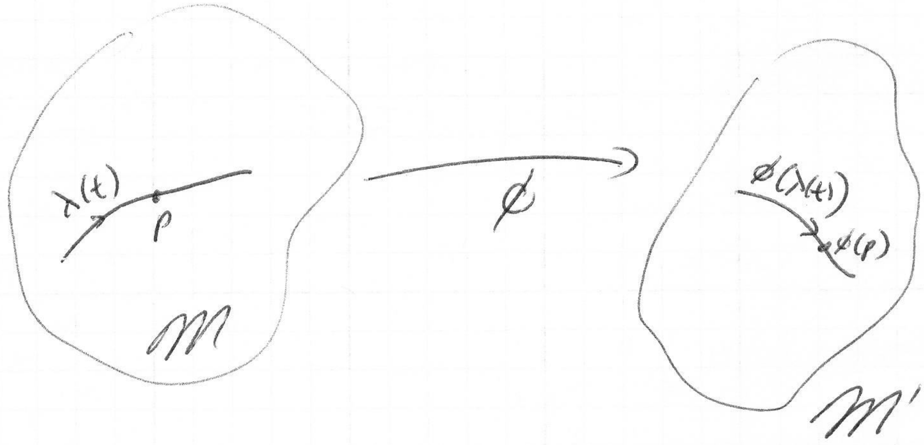
$$\phi^* f : \mathcal{M} \to R$$

defined by $\mathcal{M} \xrightarrow{\phi} \mathcal{M}' \to \mathbb{R}$

ie if $p \in \mathcal{M}$ then $\phi(p) \in \mathcal{M}'$ and $f(\phi(p))$ is defined.

(This is a "pull-back" of a zero form)

Go the other direction



By mapping curves $\lambda(t)$ in $\mathcal{M}$ into $\mathcal{M}'$ we can get maps of tangent vectors.

If $T_p(\mathcal{M})$ is the tangent space to $\mathcal{M}$ at p then

$$\phi_* : T_p(\mathcal{M}) \to T_{\phi(p)}(\mathcal{M}') \qquad \text{push-forward}$$

defined by mapping $\left(\frac{\partial}{\partial t}\right)_\lambda \longrightarrow \left(\frac{\partial}{\partial t}\right)_{\phi(\lambda)}$ $\longleftarrow$ denote this by $\phi_*\left(\frac{\partial}{\partial t}\right)_\lambda\Big|_{\phi(p}$

This is a linear transformation between the vector spaces: if x^μ and y^a are local coordinates on patches of $\mathcal{M}$ & $\mathcal{M}'$, then the curve is $x^\mu(t)$, mapped into $y^a(x^\mu(t))$ and

$$\frac{dy^a}{dt}\Big|_0 = \frac{\partial y^a}{\partial x^\mu}\Big|_p \frac{dx^\mu}{dt}\Big|_0$$

or $$N^a = \frac{\partial y^a}{\partial x^\mu}\Big|_p M^\mu \qquad \text{where } \vec{N} \in T_{\phi(p)}(\mathcal{M}'), \quad \vec{M} \in T_p(\mathcal{M})$$

so ϕ_* is just the matrix $\frac{\partial y^a}{\partial x^\mu}\Big|_p$. and we write $\vec{N} = \phi_* \vec{M}$

Since a vector $\vec{M}$ is a directional derivative, we have $\vec{M}(f)$ defined.

(A vector gives a map of any function f at p into a number.
If $\vec{M} = \frac{\partial}{\partial t}$ then $\vec{M}(f) = \frac{\partial f}{\partial t}\Big|_{p=\lambda(t_0)}$ ie the derivative along $\lambda(t)$

Explicitly $\frac{df(x^\gamma(t))}{dt} = \frac{\partial f}{\partial x^\mu}\dot{x}^\mu$ so the action of the vector $\vec{A}$ with coordinates a^μ on f is $\vec{A}(f) = a^\mu \partial_\mu f|_p$).

$$\begin{tikzcd}
T_p(M) \arrow[r, "\phi_*"] & T_{\phi(p)}(M') & \\
M \arrow[r, "\phi"] \arrow[rr, bend right, "\phi^* f"'] & M' \arrow[r, "f"] & R
\end{tikzcd}$$

defines $\vec{N}(f) = \phi_* \vec{M}(f)$

defines $\vec{M}(\phi^* f)$

$$\vec{M}(\phi^* f)\Big|_p = \phi_* \vec{M}(f)\Big|_{\phi(p)}$$

check:

$$m^\mu \partial_\mu (f(\phi(x)))\Big|_p = m^\mu \frac{\partial f}{\partial y^a}\Big|_{\phi(p)} \frac{\partial y^a}{\partial x^\mu}\Big|_p = \left(m^\mu \frac{\partial y^a}{\partial x^\mu}\right)\frac{\partial f}{\partial y^a}\Big|_{\phi(p)}$$

n^a components of $\phi_* \vec{M}$.

Students: should always flesh out relations in terms of coordinate patches, to make sure they understand.

Go on to 1-forms: define pull-back

$$\phi^*: \; T^*_{\phi(p)}(M') \to T^*_p(M)$$

by requiring the contraction is mapped properly: $\phi^*: \tilde{\omega} \to \phi^*\tilde{\omega}$

$$\left(\phi^*: \quad \tilde{\omega} \in T^*_{\phi(p)}(M') \longrightarrow \phi^*\tilde{\omega} \in T^*_p(M) \right)$$

with $\boxed{\tilde{\omega}(\phi_*\vec{M}) = \phi^*\tilde{\omega}(\vec{M})}$ ~~defines $\phi^*\tilde{\omega}$~~

~~Recall~~

$$\begin{tikzcd} T_p \arrow[r, "\phi_*"] & T_{\phi(p)} \\ M \arrow[r, "\phi"] & M' \\ T^*_p & T^*_{\phi(p)} \arrow[l, "\phi^*"'] \end{tikzcd}$$

Recall $\tilde{\omega}(\vec{N})$ is a number, ie $\tilde{\omega}$ is a map from $T_p \to \mathbb{R}$.
(In components $\tilde{\omega}(\vec{N})|_p = \omega_\mu|_p N^\mu|_p$, the index contraction.
Some texts write $\langle \tilde{\omega}, \vec{N} \rangle$).

So the ~~def~~ above gives the action of $\phi^*\tilde{\omega}$ on vectors $\vec{M} \in T(M)$ in terms of the action of $\tilde{\omega}$ on vectors $\vec{N} \in T(M')$. ~~which is~~

(In components $(\phi^*\tilde{\omega})_\mu M^\mu = \omega_a N^a = \omega_a \dfrac{\partial y^a}{\partial x^\mu} M^\mu$

that is $(\phi^*\tilde{\omega})_\mu = \omega_a \dfrac{\partial y^a}{\partial x^\mu}$).

In particular $\phi^*(df) = d(\phi^* f)$

(In components $df = f_{,a}\, dy^a$ $\quad \phi^*(df)$ ~~has~~ $f_{,a} \dfrac{\partial y^a}{\partial x^\mu} dx^\mu$

while $d(\phi^* f) = df(y(x)) = \left(\dfrac{\partial f}{\partial y^a} \dfrac{\partial y^a}{\partial x^\mu} \right) dx^\mu$ ✓✓).

Clearly this can be extended to tensors of type T^r_0 and T^0_r

$$\phi_* : T^r_0(p) \to T^r_0(\phi(p))$$

recall $T \in T^r_0(p)$ acts on r 1-forms $T(\tilde\omega^1, \dots, \tilde\omega^r) \in \mathbb{R}$

so $T \to \phi_* T$ by $T(\phi^*\tilde\omega^1, \dots, \phi^*\tilde\omega^r) = \phi_* T(\tilde\omega^1, \dots, \tilde\omega^r)$

And $T \in T^0_r$ acts on r vectors so $\phi^* : T^0_r(\phi(p)) \to T^0_r(p)$

$$\phi^* T(\vec{M}_1, \dots, \vec{M}_r) = T(\phi_*\vec{M}_1, \dots, \phi_*\vec{M}_r)$$

In components ϕ_*: $T^{a_1 \dots a_r} = \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} T^{\mu_1 \dots \mu_r}$

$$\phi^* : \quad T_{\mu_1 \dots \mu_r} = \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} T_{a_1 \dots a_r}$$

Defs

Rank: ϕ ~~is~~ $\mathcal{M} \to \mathcal{M}'$ is rank k (at p) if the dimension of the tangent space at $\phi(p)$ $(\phi_*(T_p(\mathcal{M}))$ is k.

Injective ϕ above is injective (at p) if rank = dimension of $\mathcal{M}$

$k = n$.

(In this case $n \leq n'$).

Exercise: If ϕ is injective then no non-zero vectors in $T_p(\mathcal{M})$ are mapped to zero by ϕ_*

__Surjective__: ϕ is surjective at p if rank of ϕ = dimension of M'

$k = n'$

(So that $n \geq n'$).

(~~__Immersion__: ϕ is an immersion if it has an inverse ϕ^{-1},~~
~~(with same differentiability as ϕ) such that~~
~~for each $p \in M$ there is $U \subset M$ with $p \in U$~~
~~$\ni$ $\phi^{-1} : \phi(U) \to U$~~
(Skip immersion: it ~~dep~~ is subtle only when C^r properties matter)

If ϕ is injective $\forall p \in M$ we say ϕ is an immersion (actually, def'n of immersion is ~~of~~ given in terms of existence of differentiable inverse of ϕ, and then equivalence of statements is proved) $\Longrightarrow$ $\phi_* : T_p \to \phi_*(T_p) \subset T_{\phi(p)}$ is an isomorphism.

Then $\phi(M) \subset M'$ is an n-dimensional __immersed__ submanifold in M'.

This is one-one locally, but may not be so globally.

An imbedding is, basically, an immersion that is one-one (actually a homeomorphism onto its image).

Diffeomorphism: one-to-one map $\phi : \mathcal{M} \to \mathcal{M}'$ with inverse $\phi^{-1} : \mathcal{M}' \to \mathcal{M}$.

Then $n = n' = k$, ϕ is injective and surjective.

Thm: If ϕ_* is injective and surjective at p then there is an open $U \in \mathcal{M}$, $p \in U$ + $\phi : U \to \phi(U)$ is a diffeomorphism.

That is if $\phi_* : T_p \to T_{\phi(p)}$ is an isomorphism

then ϕ is a local diffeomorphism.

With a diffeomorphism we can go with $\phi_* : T_p(\mathcal{M}) \to T_{\phi(p)}(\mathcal{M}')$

and with $(\phi^{-1})^* : T^*_p(\mathcal{M}) \to T^*_{\phi(p)}(\mathcal{M}')$

So for any tensor in T^r_s

$$T(\tilde{\omega}^1, \dots, \tilde{\omega}^s, \vec{M}_1, \dots, \vec{M}_r)\Big|_p = \phi_* T((\phi^{-1})^*\tilde{\omega}^1, \dots, (\phi^{-1})^*\tilde{\omega}^s, \phi_* M_1, \dots \phi_* M_r)\Big|_{\phi(p)}$$

Differentiation without a connection

Two types arise naturally:

- Exterior derivative
- Lie Derivative

Exterior derivative $d : \Omega_s \rightarrow \Omega_{s+1}$

Ω_s : linear space of s-forms $\tilde{a} = a_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots dx^{\mu_n}$

($\Omega_s \subset T^0_s$, is the totally antisymmetric T^0_s tensors).

Recall if $\tilde{a}$ & $\tilde{b}$ are p & q forms , $\tilde{a} \wedge \tilde{b} = (-1)^{pq} \tilde{b} \wedge \tilde{a}$.

d acts by

$$d\tilde{a} = da_{\mu_1 \cdots \mu_s} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_s}$$

$$= \frac{\partial a_{\mu_1 \cdots \mu_s}}{\partial x^{\sigma}} dx^{\sigma} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_s}$$

Exercise: show

- this is indeed a T^0_{s+1} (tensor) (obvious from first line).
- $d(a \wedge \tilde{b}) = d\hat{a} \wedge \tilde{b} + (-1)^s a \wedge db$ if a is an s-form
- $d(d\tilde{a}) = 0$
- $d(\phi^* \hat{a}) = \phi^* (d\hat{a})$

Useful integration results (reminder)

if ϕ is a diffeomorphism and $\hat{a}$ is an n-form ($n = \dim \mathcal{M}$)

$$\int_{\mathcal{M}} \hat{a} = \int_{\mathcal{M}' = \phi(\mathcal{M})} \phi_* \hat{a}$$

If $\hat{b}$ is an n-1 form

$$\int_{\partial \mathcal{M}} \tilde{b} = \int_{\mathcal{M}} d\tilde{b}$$

Stoke's theorem.

Lie derivative

~~Let~~ $\vec{M}$ vector field on $\mathcal{M}$

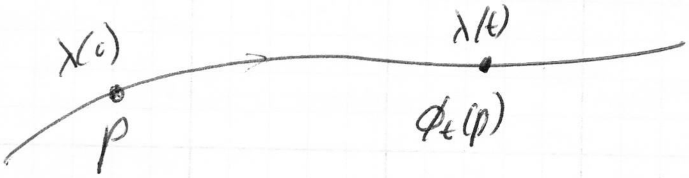
Thm. (Fund. thm of diff. eqs) $\Longrightarrow$ unique curve $\lambda(t)$ ~~point~~ through p with $\lambda(0) = p$ and $\vec{M} = \frac{\partial}{\partial t}$

~~With~~ Locally, with coordinates x^μ, $\lambda(t)$ is $x^\mu(t)$ ~~and~~ with tangent $\frac{dx^\mu}{dt}$; so the theorem above is the statement of uniqueness of solution of

$$\frac{dx^\mu(t)}{dt} = M^\mu(x(t))$$

$\lambda(t)$ is the "integral curve of $\vec{M}$"

Given $\vec{M}$ we can construct a diffeomorphism ϕ_t of $\mathcal{M}$ into itself (actually from small open neighborhoods $U \ni p$ into $\mathcal{M}$), that maps p into the point along the curve a ~~distance t~~ parameter t away

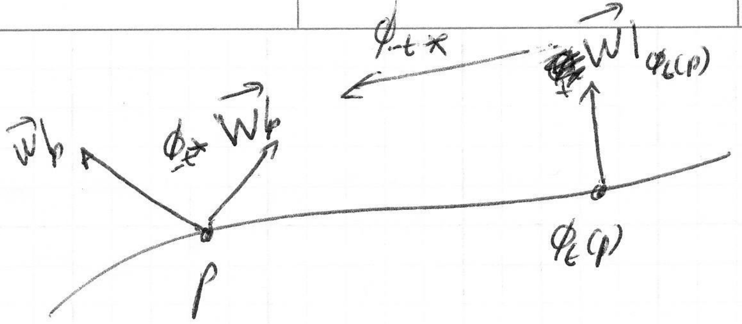


(Note ϕ_t forms a one parameter local group of diffeomorphisms

$$\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t \qquad \phi_{-t} = (\phi_t)^{-1} \qquad \phi_0 = e \text{ identity})$$

From ϕ_t construct $\phi_{t*} : T^r_s(p) \to T^r_s(\phi(p))$

$$T|_p \mapsto \phi_{t*} T|_{\phi(p)}$$



Since ϕ_{t*} is a diffeomorphism, ϕ_{t*} is an isomorphism, we can directly compare $\phi_{t*}\vec{W}$ with T. Define Lie derivative at p by

$$\mathcal{L}_{\vec{M}} T = \lim_{t\to 0} \frac{1}{t}\left[\phi_{-t*}T|_p - T|_p\right]$$

Note: both at p.

Properties:

(i) If $T \in T^r_s(p) \Rightarrow \mathcal{L}_M T \in T^r_s(p)$

(ii) $\mathcal{L}_{\vec{M}}$ is linear

(iii) $\mathcal{L}_{\vec{M}}$ preserves contractions

(iv) $\mathcal{L}_{\vec{M}}(T\otimes S) = \mathcal{L}_{\vec{M}} T \otimes S + T\otimes \mathcal{L}_{\vec{M}} S$

(v) $\mathcal{L}_{\vec{M}} f = \vec{M}(f)$ (f a fctn $f: M\to\mathbb{R}$).

$\mathcal{L}_V W$.

Start from
$$\begin{tikzcd}\mathcal{M} \arrow[r, "\phi_t"'] \arrow[d, "x"'] & \mathcal{M} \arrow[d, "y"'] \\ \mathbb{R}^n & R^n\end{tikzcd}$$

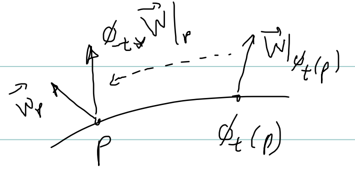
with ϕ_t the integral curve of $\vec{V}$, $\frac{dx^\mu}{dt} = V^\mu(x(t))$

For small t this is $x^\mu(t) = x^\mu(0) + tV^\mu(x^\nu(0))$.

Starting from x^μ, the coordinate for p, this is $x^\mu(t) = x^\mu + tV^\mu(x^\nu)$.

So $y^\mu = x^\mu + tV^\mu(x^\nu)$ (to order t).

Now for $\mathcal{L}_V W$ we need:



$$\begin{tikzcd}\mathcal{M} \arrow[r, "\phi_{-t}"] \arrow[d, "y"'] & \mathcal{M} \arrow[d, "x"'] \\ \mathbb{R}^n & R^n\end{tikzcd}$$

$x^\mu = y^\mu - tV^\mu(x)$
$\uparrow$
(or y, difference is higher order in t).

So we take the vector field $\vec{W}$ at $\phi_t(p)$, $W^\mu(x^\mu + tV^\mu)$ and push forward by ϕ_{-t}: $(\phi_{-t*}W)^a\Big|_p = \frac{\partial x^a}{\partial y^\mu}\Big|_{\phi_t(p)} W^\mu\Big|_{\phi_t(p)}$, where $x^a = y^a - tV^a(x)$

with y the coordinate of $\phi_t(p)$.

That is, $\frac{\partial x^a}{\partial y^\mu}\Big|_{\phi_t(p)} = \delta^a_\mu - tV^a{}_{,\mu}\Big|_{\phi_t(p)}$

So in terms of coordinates at p,

$$(\phi_{-t*}W)^a\Big|_p = \left(\delta^a_\mu - tV^a{}_{,\mu}(x+tV)\right)W^\mu(x+tV)$$

Now $\mathcal{L}_V W = \lim_{t\to 0}\frac{1}{t}\left[(\phi_{-t*}W^a)\Big|_p - W^a\Big|_p\right) = V^\mu \partial_\mu W^a - \partial_\mu V^a W^\mu$

Note, with $\vec{V} = V^\mu\partial_\mu$ and $\vec{W} = W^\mu\partial_\mu$ then

$[V^\lambda\partial_\lambda, W^\nu\partial_\nu] = V^\lambda\partial_\lambda W^\nu\partial_\nu - W^\lambda\partial_\lambda V^\nu\partial_\nu = (V^\lambda\partial_\lambda W^\nu - W^\lambda\partial_\lambda V^\nu)\partial_\nu$

$\Rightarrow \mathcal{L}_{\vec{V}}\vec{W} = [\vec{V}, \vec{W}]$

Look closely at $\mathcal{L}_V(f)$

Recall $\mathcal{M} \xrightarrow[\phi]{} \mathcal{M}' \xrightarrow[f]{} \mathbb{R} \Rightarrow \phi^* f : \mathcal{M} \to \mathbb{R}$ is $\phi^* f = f \circ \phi$

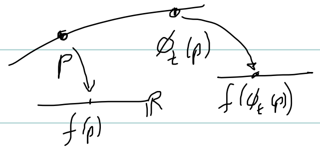
or $\phi^* f(p) = f(\phi(p))$.

Also if ϕ is a diffeomorphism then $\mathcal{M} \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} \mathcal{M}'$

The push forward of the inverse is just the pull-back:

$(\phi^{-1})_* f : \mathcal{M} \to \mathbb{R}$. is $\phi^* f : \mathcal{M} \to \mathbb{R}$

For $\mathcal{L}_V f$ we need $(\phi_{-t*} f)\big|_p$:



$(\phi_{-t*} f\big|_p$ just says its the function that maps p to the image under f of $\phi(p)$

or $f(\phi_t(p)) \Rightarrow \mathcal{L}_V(f) = \frac{1}{t}\Big(f(\phi_t(p)) - f(p)\Big) = \frac{1}{t}\Big(f(x+Vt) - f(x)\Big) = V^\mu \partial_\mu f$

Formally $$\begin{tikzcd} \mathcal{M} \arrow[r, "\phi_t"'] & \mathcal{M} \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$ defines $\phi_t^* f(p) = f(\phi_t(p))$

The def of $\mathcal{L}_V$: $\mathcal{L}_V T = \lim_{t \to 0} \frac{1}{t}\Big[\phi_{-t*} T\big|_p - T\big|_p\Big]$

Just recall that $(\phi_{-t})_*$ is a push-forward from the neighborhood of $\phi_t(p)$ to p.

Which is the same as a pull-back from p to $\phi(p)$. Which is the pic above: $\mathcal{L}_V(f) = V^\mu \partial_\mu f$

Finally $\mathcal{L}_V(\omega_\mu W^\mu) = V^\mu \partial_\mu(\omega_\nu W^\nu) = \mathcal{L}_V(\omega_\mu) W^\mu + \omega_\mu \mathcal{L}_V(W^\mu)$

$\mathcal{L}_V(\omega_\mu) W^\mu = V^\mu \partial_\mu(\omega_\nu W^\nu) - \omega_\mu\Big(V^\nu \partial_\nu W^\mu - \partial_\nu V^\mu W^\nu\Big) = \Big(V^\mu \partial_\mu \omega_\nu + \partial_\nu V^\mu \omega_\mu\Big) \; W^\nu$

Get $\mathcal{L}_{\vec{M}} \vec{W}$ explicitly:

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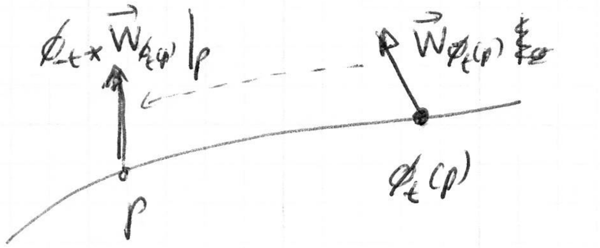
superseeded by previous two



Recall $$\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$$

then $$\vec{W}_p \to \phi_* \vec{W}|_{\phi(p)}$$

means $$W^\mu \to (\phi_* W)^a\Big|_{\phi(p)} = \frac{\partial y^a}{\partial x^\mu}\Big|_p W^\mu\Big|_p$$

Moreover, for our case ϕ_{-t*} at p is what? Take

$$\mathcal{M} \xrightarrow{\phi_{-t}} \mathcal{M}'$$

$$\vec{W}_{\phi_t(p)} \to \phi_{-t*}\vec{W}|_p$$

$$(\phi_{-t*}W)^a|_p = \frac{\partial y^a}{\partial x^\mu}\Big|_{\phi_t(p)} W^\mu\Big|_{\phi_t(p)}$$

But $y^a(x^\mu)$ is just the shift in coordinates along the curve: x^a are the coordinates of p, in terms of x^μ, the coordinates of $\phi_t(p)$.

If the curve is the integral of $\dfrac{dx^\mu}{dt} = M^\mu$ ($\vec{M}$ a vector field).

then, $x^\mu(t) = x^\mu(0) + tM^\mu$ to order t, and $y^a(x^\mu)$ is just

$$x^\mu(0) = x^\mu(t) - tM^\mu \quad \text{so} \quad \frac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi_t(p)} = \delta^\nu_\mu - M^\nu_{,\mu}\, t$$

$W^\mu|_{\phi_t(p)}$ is just $W^\mu(x^\alpha(t)) = W^\mu(x^\mu(0) + tM^\mu) = W^\mu|_p + t M^\nu W^\mu_{,\nu}|$

so $$\phi_{-t*}W|_p - W|_p = (\delta^\nu_\mu - M^\nu_{,\mu}\, t)(W^\mu + tM^\nu W^\mu_{,\nu}) - W^\nu$$

and $$(\mathcal{L}_{\vec{M}}\vec{W})^\mu = M^\nu W^\mu_{,\nu} - W^\nu M^\mu_{,\nu} = [M, W]^\mu$$

"Lie bracket"
"commutator"

In particular, this shows $\mathcal{L}_{\vec{M}} \vec{W} = -\mathcal{L}_{\vec{W}} \vec{M}$

From this, ~~two~~ one can obtain ~~the res~~ the action of $\mathcal{L}_{\vec{M}}$ on other tensors:

$$\mathcal{L}_{\vec{M}} (\tilde{\omega} \otimes \vec{W}) = \mathcal{L}_{\vec{M}} \tilde{\omega} \otimes \vec{W} + \tilde{\omega} \otimes \mathcal{L}_{\vec{M}} \vec{W}$$

now, contracting ⇒

The rest of this page has been done above, albeit a little differently... ignore

$$\mathcal{L}_{\vec{M}} (\tilde{\omega}(\vec{W})) = \mathcal{L}_{\vec{M}} \tilde{\omega}(\vec{W}) + \tilde{\omega}(\mathcal{L}_{\vec{M}}(\vec{W}))$$

Now if we use $\vec{W} = \vec{E}_\mu$, a basis vector we can get $\mathcal{L}_{\vec{M}} \tilde{\omega}$.

In particular, if $\vec{E}_\mu = \frac{\partial}{\partial x^\mu}$, the coordinate basis, then

$$\mathcal{L}_{\vec{M}}(\tilde{\omega})(\vec{E}_\mu) = (\mathcal{L}_{\vec{M}}(\tilde{\omega}))_\mu$$ the components we are looking for.

$$\mathcal{L}_{\vec{M}}(\tilde{\omega}(E_\mu)) = \mathcal{L}_{\vec{M}}(\omega_\mu) = \vec{M}(\omega_\mu)$$ (property (4))

$$= \frac{\partial \omega_\mu}{\partial X^\nu} M^\nu = \omega_{\mu,\nu} M^\nu$$

and $$\tilde{\omega}(\mathcal{L}_{\vec{M}}(E_\mu))^\nu = \frac{\partial (E_\mu)^\nu}{\partial X^\rho} M^\rho - \frac{\partial M^\nu}{\partial X^\rho}(E_\mu)^\rho = -\frac{\partial M^\nu}{\partial X^\mu}$$

so $$\tilde{\omega}(\mathcal{L}_{\vec{M}}(E_\mu)) = -\omega_\nu M^\nu{}_{,\mu}$$

$$\Rightarrow \quad (\mathcal{L}_{\vec{M}}(\tilde{\omega}))_\mu = \omega_{\mu,\nu} M^\nu + M^\nu{}_{,\mu} \omega_\nu$$

Exercise: Show

$$\mathcal{L}_{\vec{M}} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = M^\sigma \partial_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$$

$$- (\partial_\lambda M^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$$

$$- (\partial_\lambda M^{\mu_2}) T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$$

$$\vdots$$

$$+ (\partial_{\nu_1} M^\lambda) T^{\mu_1\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_\ell}$$

$$+ (\partial_{\nu_2} M^\lambda) T^{\mu_1\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_\ell}$$

In particular

$$\mathcal{L}_{\vec{M}} g_{\mu\nu} = M^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu M^\lambda g_{\lambda\nu} + \partial_\nu M^\lambda g_{\mu\lambda}$$

Since these are tensors equations, we can replace ∂ by ∇.

$$\Rightarrow \quad \mathcal{L}_M g_{\mu\nu} = \cancel{M} \; M^\lambda{}_{;\mu} g_{\lambda\nu} + M^\lambda{}_{;\nu} g_{\mu\lambda} = M_{\nu;\mu} + M_{\mu;\nu}$$

or

$$\boxed{\mathcal{L}_M g_{\mu\nu} = 2 M_{(\mu;\nu)}}$$

~~Easy~~ This is useful stuff. We will use it for symmetries later, but ~~the~~ here is a simple application. Assume the action for GR breaks down into

$$S = S_G(g_{\mu\nu}) + S_M(g_{\mu\nu}, \psi) \qquad (\star)$$

ψ = matter fields

S_G = "Hilbert" action (gives Einstein's eqs ... we'll see this later, it came).

~~Consider~~ This theory is "diffeomorphism invariant": given $\phi: M \to M$

$(M, g_{\mu\nu}, \psi)$ and $(M, \phi^* g_{\mu\nu}, \phi^*\psi)$

represent the same physics. The change in S_M under a diffeomorphism

$$\delta S_M = \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int d^4x \, \frac{\delta S_M}{\delta \psi} \delta \psi$$

Since we could have set $\psi = 0$, δS_G can be considered separately (it is invariant by itself; here is where the separation assumption in $(\star)$ comes in).

But $\frac{\delta S_M}{\delta \psi} = 0$ for <u>any</u> variation. So while here we look only at variations from diffeomorphisms, that term vanishes separately for any variation. Left with first term, we consider diffeomorphisms generated by a vector field U^μ:

$$\delta g_{\mu\nu} = \mathcal{L}_U g_{\mu\nu} = 2 U_{(\mu;\nu)}$$

$$\Rightarrow \quad \delta S_M = 0 = \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} 2 U_{(\mu;\nu)} = 4 \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} U_{\mu;\nu}$$

or

$$= 4 \int d^4x \left(\left[\frac{\delta S_M}{\delta g_{\mu\nu}} U_\mu \right]_{;\nu} - U_\mu \left(\frac{\delta S_M}{\delta g_{\mu\nu}} \right)_{;\nu} \right)$$

Dropping the surface term and multiplying by $\frac{\sqrt{-g}}{\sqrt{-g}}$ we have

$$\int dV \; U_\mu \nabla_\nu \left[\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right] = 0$$

~~If~~ Since this holds for arbitrary U_μ (diffeomorphisms generated by arbitrary vector fields) it must be that

$$\nabla_\nu \left(\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right) = 0$$

But $T^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}$ is the energy-momentum tensor.

Symmetries, Isometry, Killing Vectors

$\phi : \mathcal{M} \to \mathcal{M}$ a diffeomorphism, T a tensor.

ϕ is a symmetry of T if

$$\boxed{\phi^* T = T}$$

T symmetric

Some symmetries are discrete. But for continuous symmetries there is a one parameter set of diffeomorphism ϕ_t, and then T is symmetric iff

$$\boxed{\mathcal{L}_U T = 0}$$

T symmetric, continuous symmetry.

(Clearly U generates the curve, $U = \frac{\partial}{\partial t}$).

Note that one can choose coordinates locally so that t itself is one of the coordinates. In such coordinates

$$\mathcal{L}_U T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \partial_t T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$$

so $\mathcal{L}_U T = 0 \Rightarrow$ all components of T are independent of t.

(Converse is obviously true!)

→ (a diffeomorphism ϕ...)

An isometry is a symmetry of the metric tensor,

$$\phi^* g_{\mu\nu} = g_{\mu\nu}$$

~~If~~ A vector field $\vec{K}$ that generates an isommetry is called a Killing vector field:

continuous isommetry, Killing vector

$$\boxed{\mathcal{L}_K g_{\mu\nu} = 0}$$

then $\vec{K}$ satisfies

$$K_{(\mu;\nu)} = 0$$

One can show the ~~op~~ reverse: if $K_{(\mu;\nu)} = 0$ then $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$ where ϕ_t is generated by ~~ma~~ $K = \frac{\partial}{\partial t}$. This is done by integration (see Hawking & Ellis).

Again, one can choose local coordinates that include t, and then $g_{\mu\nu}$ is independent of t.

Now, in first quarter (Schutz, 7.4) we saw that the geodesic equation can be written in terms of $\vec{p} = m\vec{U}$ as

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$$

so if $g_{\nu\alpha}$ is independent of one coordinate (say "t"), then the corresponding p_β is conserved, $\frac{dp_t}{d\tau} = 0$

This can be done in a covariant language as follows: assume p_μ satisfies geodesic equation:

$$p^\mu p^\nu{}_{;\mu} = 0 \qquad (\nabla_p p = 0)$$

Then

$$p^\mu \nabla_\mu (p^\nu K_\nu) = p^\mu p^\nu \nabla_\mu K_\nu + K_\nu p^\mu \nabla_\mu p^\nu = 0$$

But LHS is just $\frac{d}{d\tau}(p^\nu K_\nu)$ so $\boxed{p^\nu K_\nu}$ is constant along particle path → a conserved quantity, as before.

Exercise: If $K_{\mu_1 \cdots \mu_r}$ is a Killing tensor, i.e., it satisfies

$$\nabla_{(\mu} K_{\mu_1 \cdots \mu_r)} = 0,$$

show that $K_{\mu_1 \cdots \mu_r} p^{\mu_1} \cdots p^{\mu_r}$ is conserved.

We can see this more generally with our Killing field technology:

let
$$P^\mu = T^{\mu\nu} K_\nu$$

then
$$P^\mu{}_{;\mu} = T^{\mu\nu}{}_{;\mu} K_\nu + T^{\mu\nu} K_{\nu;\mu}$$

$$= T^{\nu\mu}{}_{;\mu} K_\nu + \tfrac{1}{2} T^{\mu\nu} K_{(\nu;\mu)} = 0$$

So the vector P^μ is a "conserved current". ~~By Gauss's the~~

<u>Example</u>: In flat space (which is highly symmetric):

Killing vectors
$$\vec{p}_{(\alpha)} = \frac{\partial}{\partial x^\alpha}$$ (a vector for each $\alpha = 0, 1, 2, 3$)

and

$$\vec{m}_{(i)} = x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}$$ $i = 1, 2, 3$

$$\vec{l}_{(ij)} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$$ $i, j = 1, 2, 3$

10 - isommetries generate 10 parameter Lie-Group of isommetries of flat space time, the "inhomageneous Lorentz group"

To see how this works ~~choose obvious coordinates~~ write components out: $\vec{p}_{(0)}{}^\mu = (1, 0, 0, 0)$ $p^\mu_{(1)} = (0, 1, 0, 0)$ -- $(p^\mu_{(\alpha)} = \delta^\mu_\alpha)$

So $p^{(\alpha)}_{\mu,\nu} = 0$ trivially for all (α).

Less trivial: $m^\mu_{(1)} = (x^1, x^0, 0, 0) \rightarrow m_{(1)\mu} = (-x^1, x^0, 0, 0)$

$$m_{(1)\mu,\nu} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ so $m_{(1)(\mu,\nu)} = 0$

etc.

Then $P^\mu = T^{\mu\nu} K_\nu$ gives conservation of $E, \vec{P}$, $\vec{J}$ and $\underline{\underline{?}}$ (like $tP + xE$?).
DON'T KNOW

It is clear from the example that ~~manifolds~~ space-times may admit ~~more~~ several (or none) killing vectors.

Since ~~trans for~~ symmetry transformations generally form groups (group multiplication = composition of transformations, ie $\phi_t \circ \phi_u$). and these are continuous transformations generated by $\vec{K}$'s, we expect there to ~~be~~ ~~a~~ Lie groups & the $\vec{K}$'s to form Lie algebras. This is indeed the case, with the Lie bracket being just the commutator, ie.

$$[K_1, K_2] = \mathcal{L}_{K_1} K_2$$

Maximally Symmetric Spaces

Spaces with high degree of symmetry are easier to analyze.

What is the highest degree of symmetry?

Consider $\mathbb{R}^n$ - Euclidean space. Then we had

n translations

$\frac{1}{2}n(n-1)$ rotations

$= \frac{1}{2}n(n+1)$ symmetries in total

~~Rota~~ Symmetry under rotations at a point p is called "isotropy" at p

Symmetry under translation is called "homogeneity" of the spacetime.

This is as much as we can have, and we define a

"maximally symmetric space" = one with $\frac{1}{2}n(n+1)$ killing vector fields

Let's find them.

At $p \in M$ choose locally inertial coordinates, so that $g_{\mu\nu}$ is given by $\eta_{\mu\nu}$. Obviously (by construction) this is invariant under local Lorentz transformations. But isotropy means, in this coordinates, at this point p, $R_{\mu\nu\rho\sigma}$ should also be invariant,

$$R_{\mu\nu\rho\sigma} \propto \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$$

only tensor with proper symmetries and invariant.

NOTE: A local Lorentz transformation acts only on $T_p(M)$, ie, it is a change of basis vectors $\vec{E}^{(a)}$. It is these vectors that are used to define the components of $\mathbf{R}$.

If we write this as

~~$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \{ (g_{\rho\mu} g_{\sigma\nu}$$~~

$$R_{\mu\nu\rho\sigma} = \kappa \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

since this is a tensorial relation it holds at P in any coordinate system. But then use homogeneity ⇒ it holds everywhere on M with same constant κ.

Contracting indices

$$\boxed{R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)}$$

So, in particular, the Ricci scalar is a constant (should be obvious by homogeneity: same R everywhere).

A maximally symmetric space is determined by

- dimension
- signature
- R
- additional topological considerations (global issues).

Warm-up:

$n = 2$, $\eta = (++)$ (n=2 almost trivial, since only one component of $R^{\mu}{}_{\nu\rho\sigma}$).

$R > 0$ the sphere S^2 $ds^2 = a^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$ $R = \frac{2}{a^2}$

$R = 0$ is $\mathbb{R}^2$ $ds^2 = dx^2 + dy^2$

$R < 0$ less familiar, the hyperboloid H^2 $ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2)$ $y > 0$

Exercise: For H^2 show

(i) $R = -\frac{2}{a^2}$ (ii) The distance between y_1 & y_2 along $x =$ constant is $a \ln \frac{y_2}{y_1}$

(iii) Geodesics satisfy $(x - x_0)^2 + y^2 = b^2$ for x_0, b constants.

Now do $n=4$ with $(-+++)$

$R>0$ de Sitter space

$R=0$ (flat) Minkowski space

$R<0$ anti-de Sitter space

Study these. Study casual structure too.

Minkowski Space-time: (trivial, but will help us understand key concepts for other spacetimes).

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

$$= -(dt)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$(r \geq 0)$
$(0 \leq \theta \leq \pi)$
$(0 \leq \phi < 2\pi)$

Null coordinates:

$$v = t + r$$
$$w = t - r$$

$\infty > v \geq w > \infty$

$$ds^2 = -dv\,dw + \tfrac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$v = $ const and $w = $ const are null hypersurfaces.

Can we change coordinates to have only finite ranges? Let

$$W = \arctan w$$
$$V = \arctan v$$

$W < V$ and both in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Now

$$ds^2 = \frac{1}{\omega^2}\left[-4\,dV\,dW + \sin^2(V-W)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

where $\omega \equiv 2\cos W \cos V$

Finally write $T = V + W \quad R = V - W$

$0 \leq R < \pi$
$|T| + R < \pi$

so

$$ds^2 = \frac{1}{\omega^2}(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2)$$

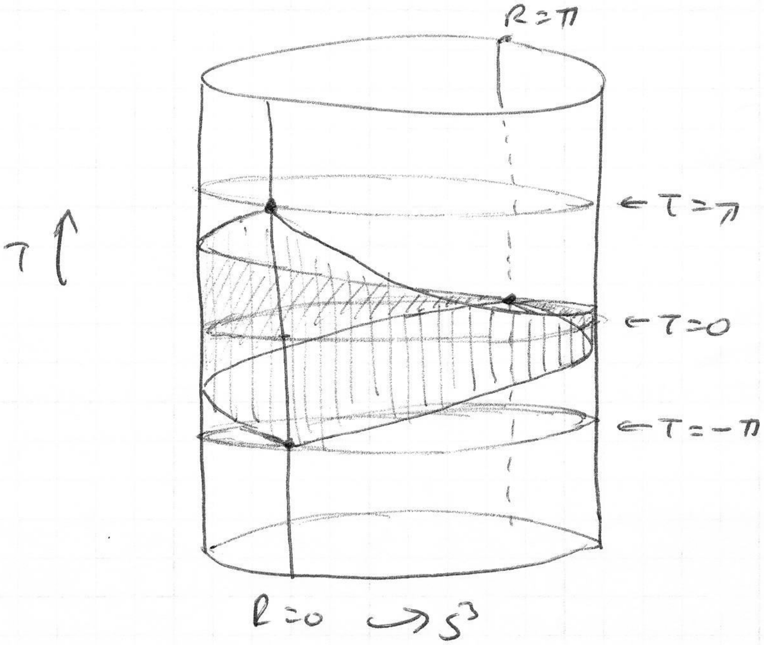
with $\omega = \cos T + \cos R$ (kind of irrelevant for us).

$$ds^2 = \frac{1}{\omega^2}\, ds_E^2$$

where $ds_E^2 = -dT^2 + dR^2 + \sin^2 R\, d\Omega^2$ "the metric for Einstein's static universe"

So Minkowski space is conformal to (a part of) the Einstein static univers

(A conformal transformation is a local change of scale; $\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$)



Conformal Diagrams (or Penrose Diagrams)

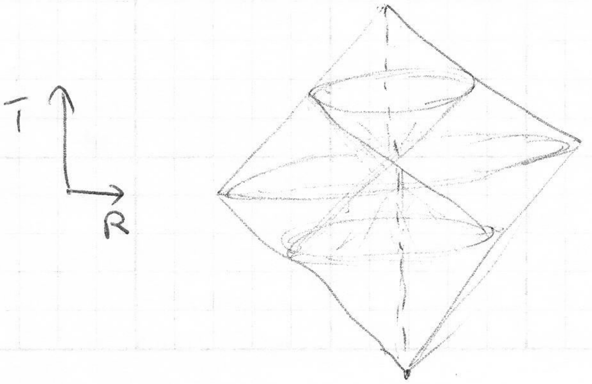
Space-time diagram for space-time (M, g). It has a "time" coordinate and a "radial" coordinate, with light-cones always at 45°. Also infinity is at finite coordinate distance (so we can fit it on a page).

Conformal transformation leave light-cones invariant

$$\left(\text{if } d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = 0 \quad \text{then} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0\right)$$

They are useful in displaying the causal structure of spacetime.

Drawing a circle for each sphere in θ, φ



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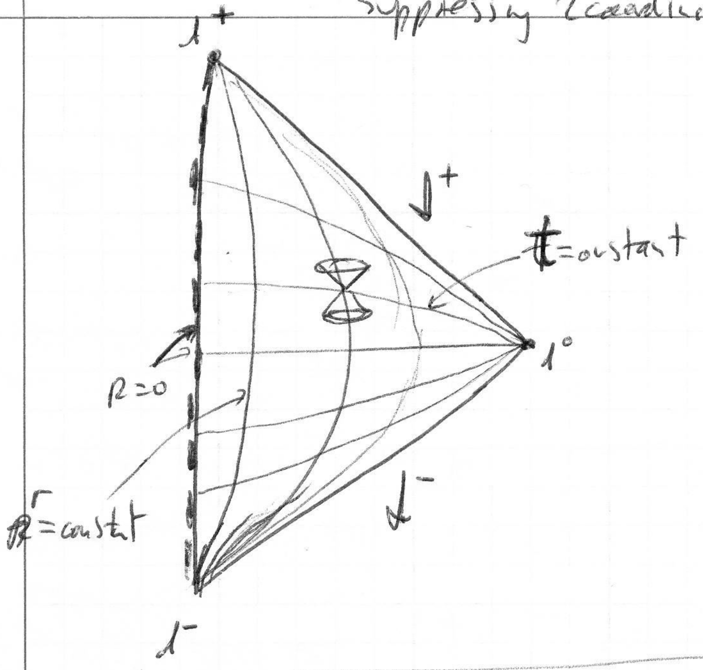
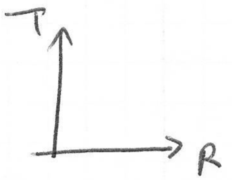
suppressing 2 coordinates (always done when with spherical symmetry):





Note: light cones always  everywhere
(i.e. 45°)

SKIP IN CLASS

The constant t, r surfaces are obtained from

$$T = V + W = tg^{-1}v + tg^{-1}w = tg^{-1}(t+r) + tg^{-1}(t-r)$$

$$R = V - W = tg^{-1}(t+r) - tg^{-1}(t-r)$$

More easily: for t=constant, eliminate r $\Rightarrow$ $w + v = 2t$ fixed

$\Rightarrow$ $\Rightarrow$ $\tan V + \tan W = 2t$

$\Rightarrow$ $\tan(\frac{1}{2}(T+R)) + \tan(\frac{1}{2}(T-R)) = 2t$

etc.

---

i^+ = future timelike infinity

i^- = past 〃 〃

i^0 = spatial infinity

$\mathscr{I}^+$ = ("scri-plus") future null infinity

$\mathscr{I}^-$ = past 〃 〃

Features:

(i) light cones at 45°

ii) i's are points, $\mathscr{I}$'s are surfaces (null) with topology $R \times S^2$

iii) timelike geodesics: from i^- to i^+; spacelike from i^0 to i^0

null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$

Some obvious things are more obvious in the Penrose diagram.

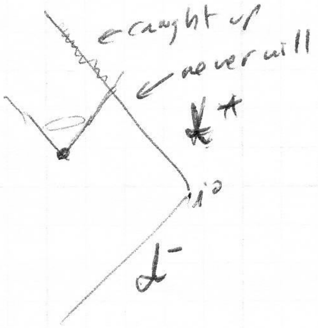
- i^+ is in the future lightcone of any event
- i^- is in the past lightcone of any event

i.e. you can reach any point, no matter how far, with a signal if you are willing to wait enough

- i^0 is neither in the future nor past light cone of an event

ie, you can not reach space-like infinity with a signal in finite time.

If you are willing to wait an infinite time, then a signal can reach spatial infinity, but will not catch up with other signals emitted by you earlier:

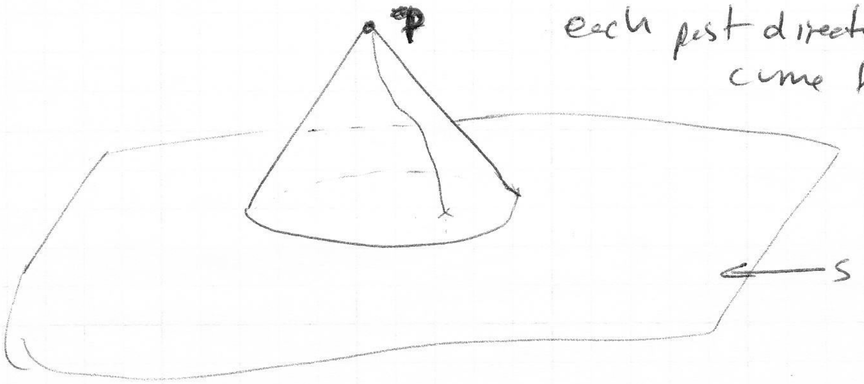


Cauchy Surfaces

$D^+(S)$: "future Cauchy development" of S :

If S is a space-like 3-surface then

$D^+(S) = \{ p \in \mathcal{M} \mid$ each past directed inextendible non-spacelike curve through p intersects $S \}$

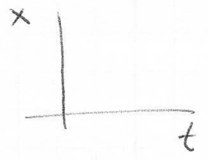


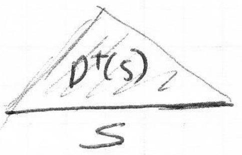
each past directed ~~timelike~~ non-spacelike curve through p intersects S.

$\Rightarrow p$ is in $D^+(S)$

---

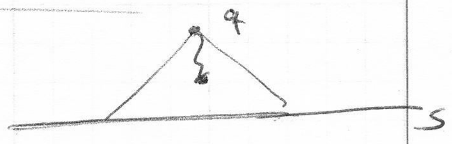
so in Minkowski 1+1 space





Notes:

- inextendible so that we avoid:



- non-spacelike, we want the part of space that can be causaly affected by S.

~~Similarly define $D^-(S)$, "future directed curves" giving~~

Since signals can ~~not~~ only travel on non-spacelike curves, if $p \in D^+(S)$ then knowing data (value of fields and first derivatives, or particle velocities, etc) on S is enough to predict them at p.

Similarly, if we want to evolve back into the past, but have information only on S, we can only infer the state in $D^-(S)$ (defined by "future" → "past" in def above).

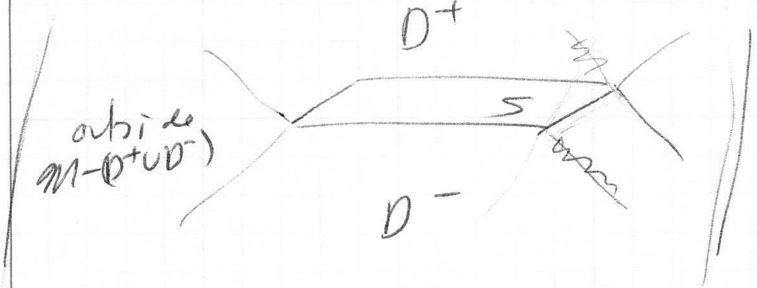
If $D^+(S) \cup D^-(S) = \mathcal{M}$

S is called a Cauchy surface

→ In words, ~~if~~ every inextendible non-spacelike curve in $\mathcal{M}$ intersects S. ~~S is Cauchy~~

In Minkowski space $t=0$ is a Cauchy surface. In fact $t = c$ = a constant is a collection of Cauchy surfaces that cover the whole of $\mathcal{M}$.

Not every spacelike surface in ~~$\mathcal{M}$~~ (Minkowski space) is Cauchy. Clearly extendible surfaces are not



More interestingly, some inextendible surfaces are not Cauchy. For example the surfaces

~~$-(x^0)^2 + (x^1)^2 +$~~

$$-t^2 + x^2 + y^2 + z^2 = \sigma \in \mathbb{R}$$

are spacelike if $\sigma < 0$. Let S_σ be the surface with $t<0$

These S_σ are not Cauchy, ~~and their~~ but are inextendible spacelike. The collections fills the past lightcone of the origin.

de Sitter space-time

This is $R>0$. Note that this means (R=const)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{4}Rg_{\mu\nu} \qquad R=\text{const}>0$$

Comparing with Einstein's equation, (8.8) in Schutz

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = kT^{\mu\nu}$$

we see that either $kT^{\mu\nu} = -\frac{R}{4}g^{\mu\nu}$ (a very odd fluid??)

or $\Lambda = \frac{1}{4}R > 0$. That is, de Sitter spacetime is a solution to Einstein's equations with a ~~cosmological~~ positive cosmological constant, but no matter.

Since we have observed $\Lambda \neq 0$, with $\Lambda \sim \rho_{\text{dark matter}}$, and since Λ is const while ρ is decreasing with the (true) expansion of the universe, soon ρ will be negligible and the future of the universe will be described by ~~apattoten~~ (approximately) de Sitter spacetime.

It is defined by embedding the hyperboloid ~~(5 dimensions)~~

$$-U^2 + X^2 + Y^2 + Z^2 + W^2 = \alpha^2$$

defined in 5-dim Minkowski space, $ds_5^2 = -dU^2 + dX^2 + dY^2 + dZ^2 + dW^2$

~~in 4 dimensions~~

Let

$$U = \alpha \sinh(t/\alpha)$$

$$W = \alpha \cosh(t/\alpha) \cos\chi$$

$$X = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$

$$Y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

$$Z = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$

Then

$$ds^2 = -dt^2 + \alpha^2\cosh^2\left(\tfrac{t}{\alpha}\right)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

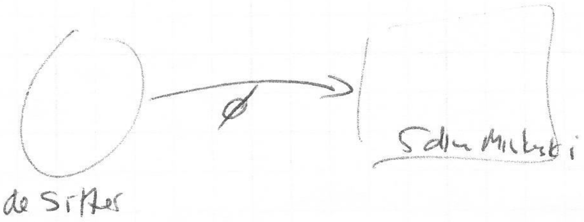
(Exercise: but not for class time!

(1) $-U^2 + X^2 + Y^2 + Z^2 + W^2 = -\alpha^2\sinh^2\left(\tfrac{t}{\alpha}\right) + \alpha^2\cosh^2\left(\tfrac{t}{\alpha}\right)\left[\cos^2\chi + \sin^2\chi(\cos^2\theta + \sin^2\theta(\cdots)\right]$

$$= \alpha^2 \quad ✓✓$$

(2)

$$g_{\mu\nu} = \frac{\partial y^a \partial y^b}{\partial x^\mu \partial x^\nu} g_{ab} \quad \text{is a pull back} \quad \phi^*: g \to \phi^* g$$



ϕ given by the above eqs, ie, $U = \alpha\sinh(t/\alpha)$ etc.

So $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} g_{ab}\, dx^\mu dx^\nu$

$$= -\frac{\partial U}{\partial x^\mu}\frac{\partial U}{\partial x^\nu}dx^\mu dx^\nu + \frac{\partial W}{\partial x^\mu}\frac{\partial W}{\partial x^\nu}dx^\mu dx^\nu + \cdots + \frac{\partial Z}{\partial x^\mu}\frac{\partial Z}{\partial x^\nu}dx^\mu dx^\nu$$

$$= -\cosh^2\frac{t}{\alpha}dt^2 + \sinh^2\frac{t}{\alpha}(\cos^2\chi + \sin^2\chi(\cos^2\theta + \cdots))dt^2$$

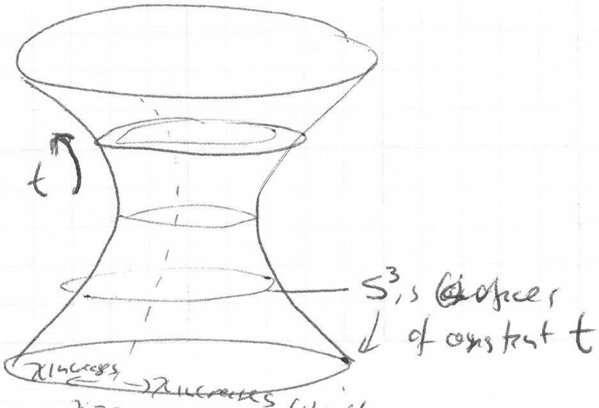
$$+ \alpha^2\cosh^2\frac{t}{\alpha}\left[\sin^2\chi + \cos^2\chi\left(\cos^2\theta + \sin^2\theta(\cos^2\phi + \sin^2\phi)\right)\right]d\chi^2$$

$$+ \alpha^2\cosh^2\frac{t}{\alpha}\sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

)

Note $\quad d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \quad$ metric on a 2sphere

Similarly $\quad d\Omega_3^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2 \to$ metric on a 3sphere



de Sitter space: spatial 3-sphere that shrinks to a minimum radius α, then re-expands.

topology: $\mathbb{R}^1 \times S^3$

Another coordinate system that is common is

$$\hat{t} = \alpha \log\left(\frac{w+u}{\alpha}\right) \qquad \hat{x} = \frac{\alpha x}{w+u} \qquad \hat{y} = \frac{\alpha y}{w+u} \qquad \hat{z} = \frac{\alpha z}{w+u}$$

restricted to the hyperboloid (you can simply write a ~~ed~~ $\hat{t}, \dots, \hat{z}$ as a function of t, χ, θ, ϕ). In terms of these

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}\left(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\right)$$

NOT IN LECT [To see this, write

$$w+u = \alpha e^{\hat{t}/\alpha}$$

$$x = \hat{x} e^{\hat{t}/\alpha}$$

$$y = \hat{y} e^{\hat{t}/\alpha}$$

$$z = \hat{z} e^{\hat{t}/\alpha}$$

and insist on the hyperboloid, $(w-u)(w+u) + r^2 = \alpha^2 \Rightarrow w-u = \frac{\alpha^2 - \hat{r}^2 e^{2\hat{t}/\alpha}}{\alpha e^{\hat{t}/\alpha}}$ or

$$w-u = \alpha e^{-\hat{t}/\alpha} - \frac{1}{\alpha}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) e^{\hat{t}/\alpha}$$

Now proceed with the pull-back of η^{ab}:

$$ds^2 = d(w+u)d(w-u) + dx^2 + dy^2 + dz^2 = (e^{\hat{t}/\alpha} d\hat{t})\Big[\Big(-e^{-\hat{t}/\alpha} - \frac{\hat{r}^2}{\alpha^2} e^{\hat{t}/\alpha}\Big) d\hat{t} - \frac{\hat{\vec{r}} \cdot d\hat{\vec{r}}}{\alpha} e^{\hat{t}/\alpha}$$

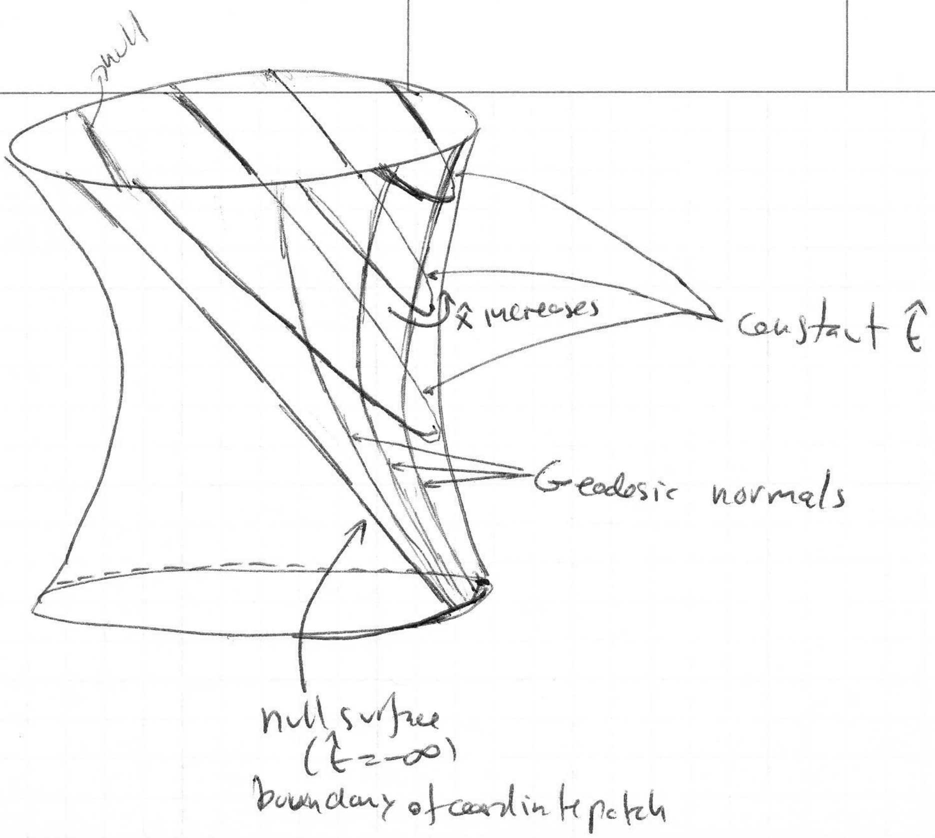
$$+ \left(d\hat{\vec{r}} - \frac{\hat{\vec{r}}}{\alpha} d\hat{t}\right)^2 e^{2\hat{t}/\alpha}$$

cross terms cancel!]

This coordinates cover only the region

$$w+u \geq 0$$

of the hyperboloid



[Check $w+v=0 \iff \alpha\sinh(\frac{t}{\alpha}) + \alpha\cosh(\frac{t}{\alpha})\cos\chi = 0$

$$\Rightarrow \quad \cos\chi = -\tanh(\tfrac{t}{\alpha})$$

As $t \to \mp\infty$, $\tanh(\frac{t}{\alpha}) \to \mp 1$ so $\cos\chi \to \pm 1$ or $\chi \to 0$ or π]

Penrose diagram for de Sitter:

~~Let~~ ~~t'=~~ Change coordi from t to t' by

$$\tan\left(\tfrac{1}{2}t' + \tfrac{\pi}{4}\right) = e^{t/\alpha}$$

with $t' \in (-\pi/2, \pi/2)$

Not needed [then

$$dt^2 \frac{e^{2t/\alpha}}{\alpha^2} = \left(\frac{1/2}{\cos^2(\frac{1}{2}t' + \frac{\pi}{4})}\right)^2 dt'^2$$

or

$$dt^2 = \frac{\alpha^2}{4} \frac{1}{\cos^4(\frac{t'}{2}+\frac{\pi}{4})} \frac{\cos^2}{\sin^2} (dt')^2 = \frac{\alpha^2}{4\cos^2\sin^2} dt'^2$$

$$= \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})} dt'^2$$

and

$$\cosh\frac{t}{\alpha} = \frac{1}{2}\left(\tan + \frac{1}{\tan}\right) = \frac{1}{2} \frac{\sin^2 + \cos^2}{\sin\cos} = \frac{1}{\sin(t'+\frac{\pi}{2})}$$

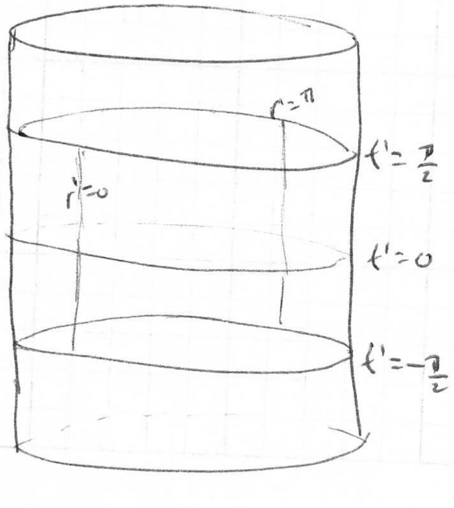
so]

$$ds^2 = \frac{\alpha^2}{\sin^2(t'+\frac{\pi}{2})} d\bar{s}^2$$

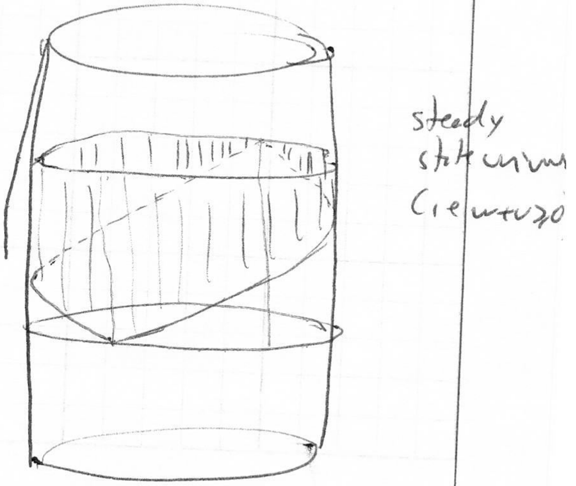
($d\bar{s}^2 = ds_E^2$ in previous notation)

where $d\bar{s}^2 = -dt'^2 + d\chi^2 + d\Omega_2^2 = -dt'^2 + d\Omega_3^2$

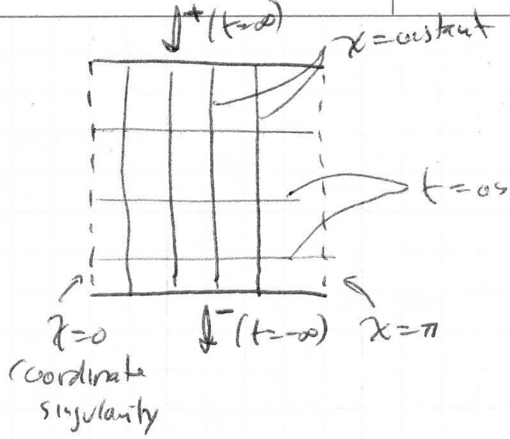
So de-Sitter is conformal to the metric $d\bar{s}^2$ = Einstein Static familiar from Minkowski. Now



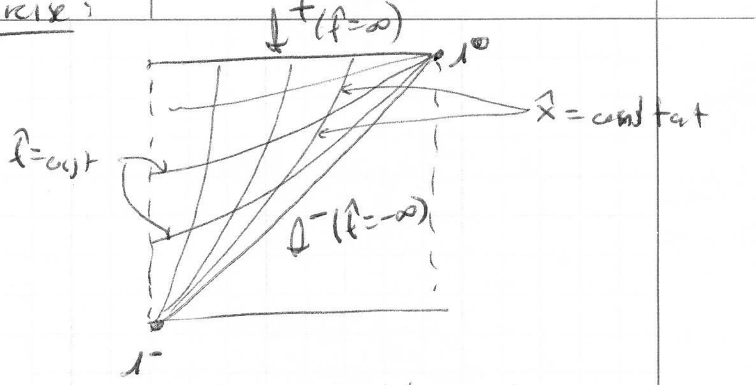
and



Exercise:



de-Sitter



Steady-state universe
of Bondi & Gold, and Hoyle (circa 1948)

$\mathcal{I}^\pm$ spacelike future/past infinity.

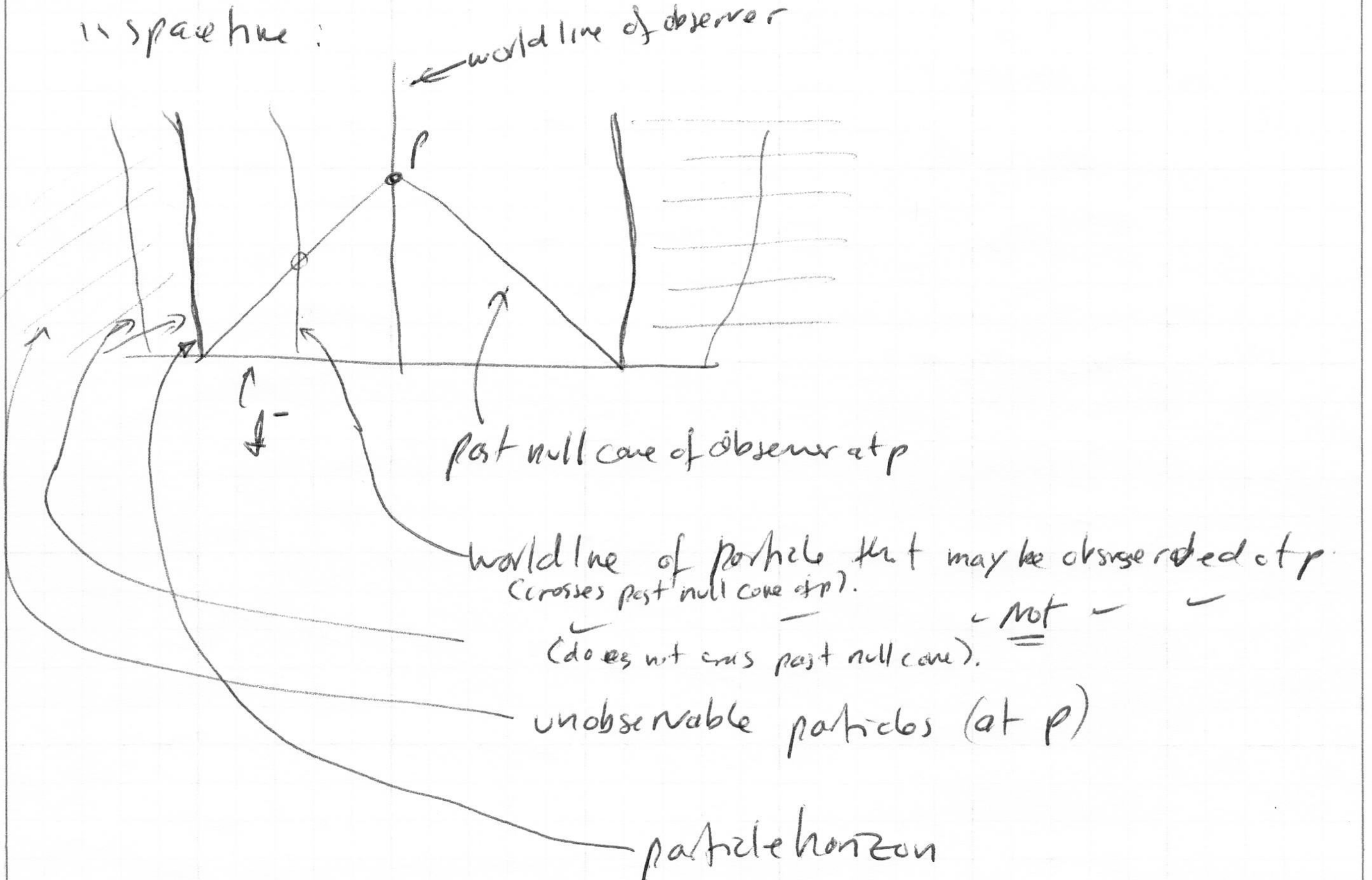
Horizons: (NEXT PAGE)

Horizons

[illegible] deSitter future & past infinities are spacelike (Contrast with Minkowski's timelike).

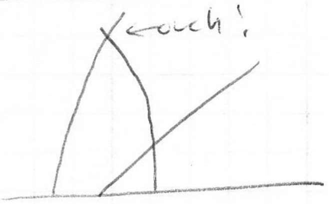
This gives rise to both particle & event horizons

Particle Horizon: defined for an observer at some event p in spacetime:



~~So~~ The particle horizon separates the region of spacetime occupied by particles that may have been seen at p from those that can not be seen at p.

Particle Horizons are defined with respect to a *congruence of world-lines*. Problem is ↙

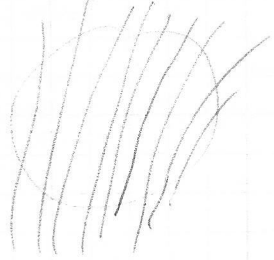


→ so we wouldn't be able to separate space into two pieces → no "horizon".

~~So we~~

Congruence is a set of ~~lines~~ curves such that each point p (in some ~~open~~ open set $U \subset \mathcal{M}$) is in exactly one ~~the~~ curve.

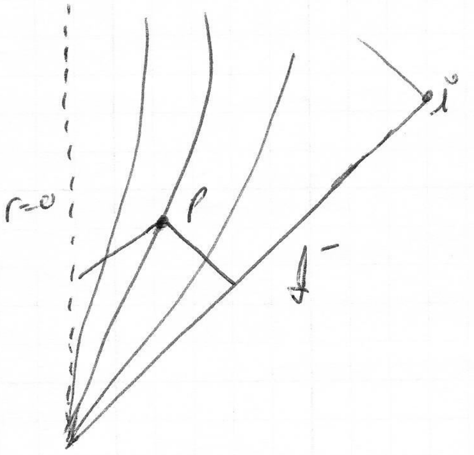
~~image:~~



By definition, curves in a congruence do not cross.

Examples:

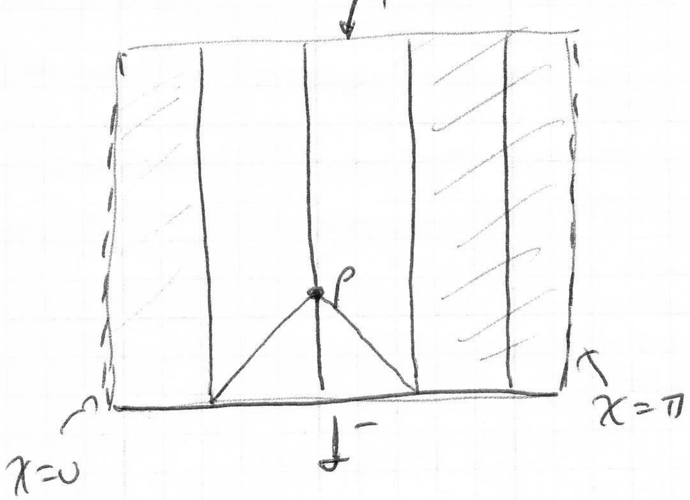
(i) There are no particle horizons in Minkowski space



every timelike geodesic crosses the past light cone of p.

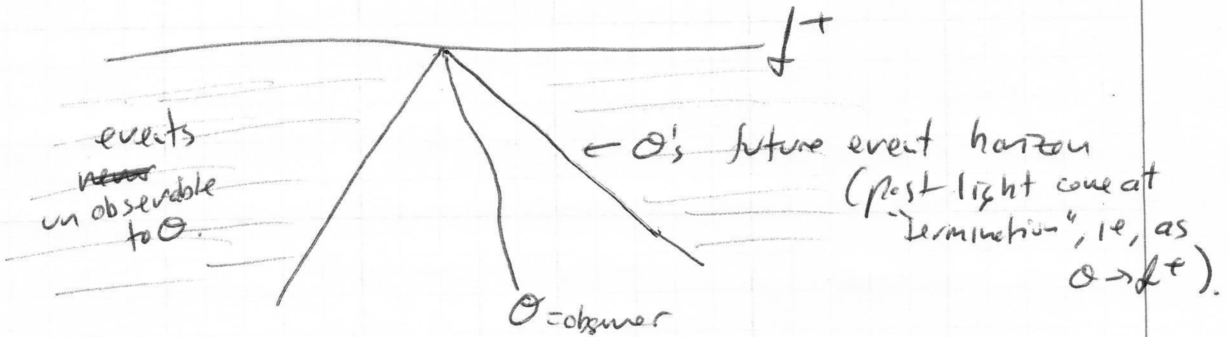
More generally, this is true if $\mathcal{I}^-$ is null.

(ii) de-Sitter does have particle horizons. Consider the congruence of $\chi=$ constant in the Penrose diagram

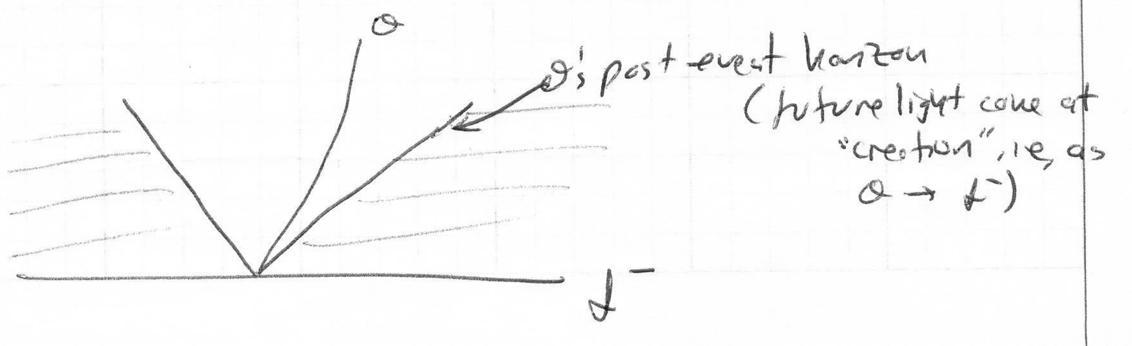


Event Horizons: ~~while~~ Particle horizon tells us which ~~'observers~~ 'particles' may have been seen at p_0. We may ask instead which particles may influence p at all throughout its whole history. That is, if the spacetime is expanding faster than the speed of light then if some observers far away from us, light sent to us will never reach us. We want to characterize this situation with an "event horizon" separating those events that can never influence us from those that can. ~~Clearl~~

Clearly, at any event p_* the events inside its past light cone are observable, while those outside are not. The ~~future~~ future event horizon is the limiting light cone of an observer as it goes into future infinity, $\mathcal{I}^+$.

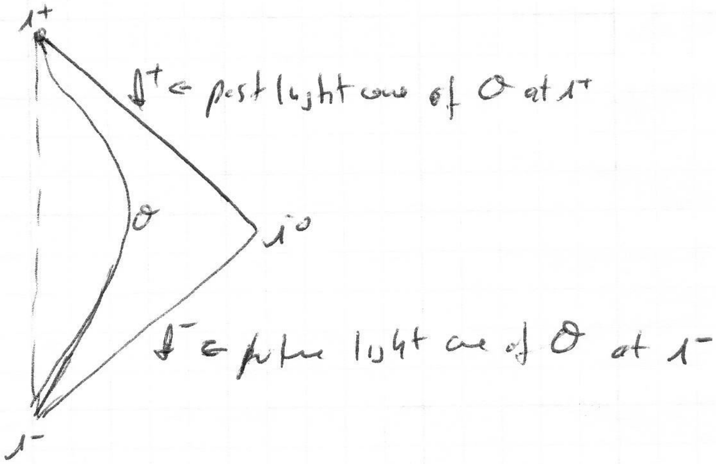


Similarly, past event horizon is defined to separate events that O will be able to influence in its history from those it won't:

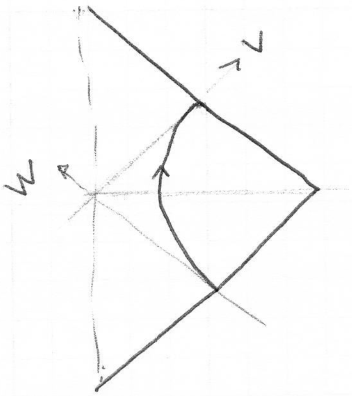


Examples: (i) Minkowski space time.

If $\mathcal{O}$ is a geodesic (free falling) observer ⇒ no event horizons



(ii) Uniformly accelerated 'observer' in Minkowski space-time



picture is $r^2 - t^2 = a^2$

has both future and past event horizons.

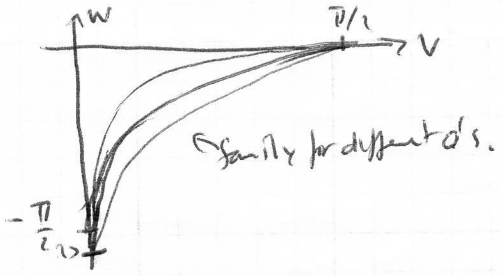
[Work it out: Recall $ds^2 = \frac{1}{\omega^2} ds_E^2$, see above,

and uniformly accelerated → $r^2 - t^2 = a^2$ or $vw = -a^2$

⇒ $\mathrm{tg} W \, \mathrm{tg} V = a^2$ ⇒ $\mathrm{tg}(\frac{1}{2}(T+R)) \, \mathrm{tg}(\frac{1}{2}(T-R)) = -a^2$

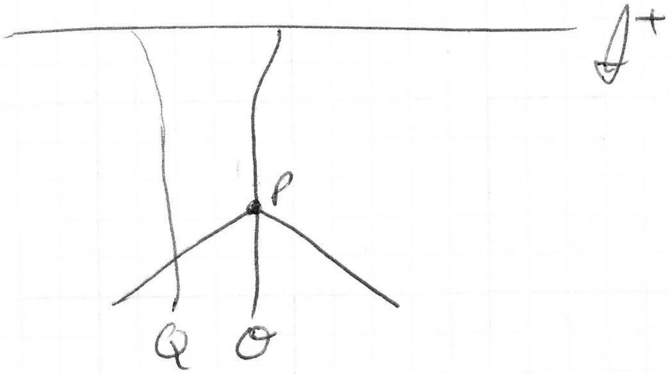
Here $ds_E^2 = -dT^2 + dR^2 + \sin^2 R \, d\Omega^2$ $\quad 0 \le R \le \pi \quad |T| + R < \pi$

Now $\mathrm{tg} W \, \mathrm{tg} V = -a^2$ is easy to draw

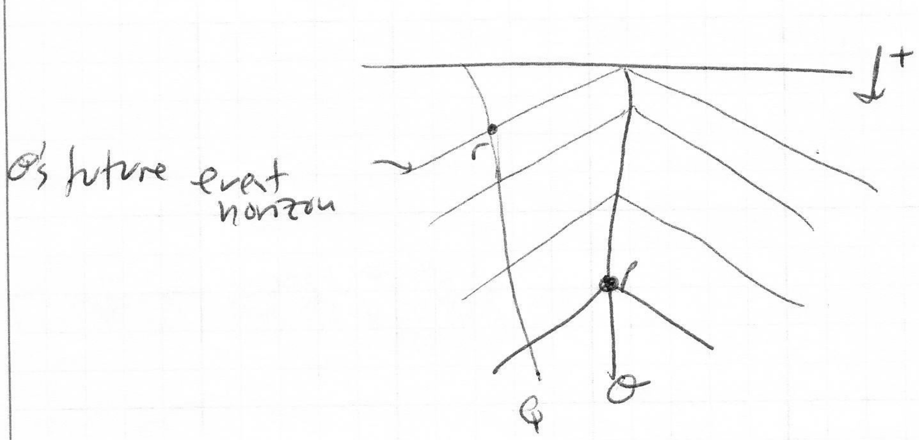


]

Consider (in de-Sitter space, or any space with $\mathscr{I}^+$ spacelike) an observer $\mathcal{O}$ and a particle worldline Q. Suppose Q intersects the past light cone of event p on $\mathcal{O}$:



⇒ Q is observable to $\mathcal{O}$ at any time after p:



But note, there is a point r on Q that lies on $\mathcal{O}$'s future event horizon ⇒ Events on Q after r are NOT observable to $\mathcal{O}$.

Since r is seen at $\mathscr{I}^+$, it takes ∞ proper time from any event on $\mathcal{O}$ until observation of r on $\mathcal{O}$.

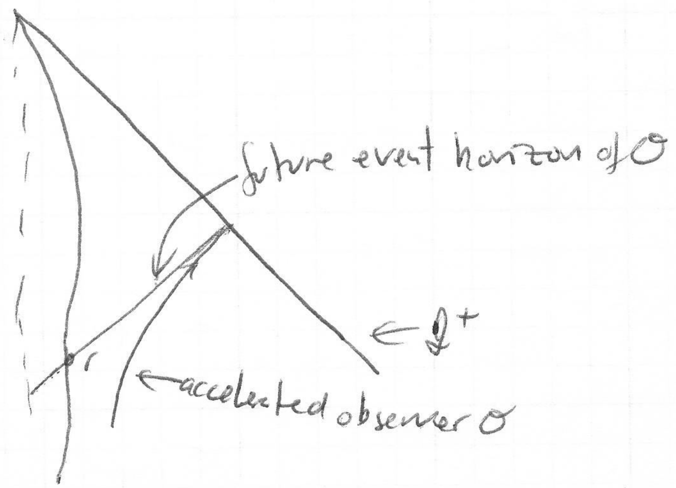
On Q, of course, it takes finite proper time from any past event to r.

It takes an infinite time in $\mathcal{O}$ to see a finite part of Q's history

(eg, $\mathcal{O}$ observes infinite redshift of light from Q as it approaches r).

Likewise, Q will see finite history of $\mathcal{O}$ in infinite time.

Even in Minkowski space if we have non-geodesic observers:



which seems perfectly logical (∞ redshifted light from ~~the~~ accelerated light source?; light from r appears ∞ redshifted as $O \to \mathscr{I}^+$).

anti-de Sitter space

(R < 0 case) we now will have $\Lambda = \frac{1}{4}R < 0$.

Cosider hyperboloid

$$-U^2 - W^2 + x^2 + y^2 + z^2 = -\alpha^2$$

in flat R^5 with $--+++$ signature

$$ds^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$$

(compare signs with de-Sitter! both w^2 & a^2 (& dw^2) flipped).

Let

$$U = \alpha \sin t' \cosh\rho$$

$$W = \alpha \cos t' \cosh\rho$$

$$x = \alpha \sinh\rho \sin\theta \cos\phi$$

$$y = \alpha \sinh\rho \sin\theta \sin\phi$$

$$z = \alpha \sinh\rho \cos\theta$$

} spherical coordinates in R^3 with radius $\alpha \sinh\rho$

This defines a map from the Hyperboloid H^4 to R^5

$$\varphi : H^4 \to R^5$$

with induced metric $\varphi^* g$ (pull back of g).

Then $$ds^2 = \alpha^2 \left[-\cosh^2\rho \, dt'^2 + d\rho^2 + \sinh^2\rho (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Exercise: Check this

$$\left[\frac{1}{\alpha^2} ds^2 = -dt'^2 \{ \cosh^2\rho (\cos^2 t' + \sin^2 t') + d\rho^2 [-\sinh^2\rho (\sin^2 t' + \cos^2 t') + \cosh^2\rho (\cos^2\theta + \sin^2\theta (s_\varphi^2 + c_\varphi^2) \right.$$

$$\left. + \sinh^2\rho \, d\theta^2 [\sin^2\theta + \cos^2\theta (s_\varphi^2 + c_\varphi^2))] + \sin^2\theta d\phi^2 \right] ✓$$

Note that with $\rho \geq 0$ a radius-like coordinate, the ~~spa~~ $t' =$ constant sections are R^3 (topologically).

But for ρ, θ, ϕ fixed, t' lines are periodic $t' \to t' + 2\pi$

→ Space has closed timelike curves (a no-no). (maybe... see later, causality).

Another coordinate system:

$$U = \alpha \sin t$$

$$V = \alpha \cos t \cosh r$$

$$x = \alpha \cos t \ \sinh r \sin\theta \cos\varphi$$
$$y = \alpha \cos t \ \sinh r \sin\theta \sin\varphi$$
$$z = \alpha \cos t \sinh r \cos\theta$$

Now $\varphi^* g$ is

$$\Big[\ \frac{1}{\alpha^2} ds^2 = \left(-\cos^2 t - \sin^2 t(\cosh^2 r - \sinh^2 r(\cos^2\theta + \sin^2\theta))\right) dt^2$$
$$+ \cancel{\$} + \cos^2 t(-\sinh^2 r + \cosh^2 r(-)) \ dr^2 \ + \cos^2 t \sinh^2 r \ (\sin^2\theta + \cos^2) d\theta^2 -$$

$$\frac{1}{\alpha^2} ds^2 = -dt^2 + \cos^2 t \ [dr^2 + \sinh^2 r \, d\Omega_2^2]$$

As we'll see this system has simple geodesics: $(r, \theta, \varphi) =$ constant. So these lines are orthogonal to $t =$ constant surface.

But note that at $t = \pm\frac{1}{2}\pi$ there are singularities. Clearly these are only coordinate singularities, but this frame can only be used for one piece of the space.

So the space described so far is one with topology $S^1 \times R^3$.

We take de-Sitter space to be the universal covering space of this, meaning, take $t' \in [-\infty, \infty]$ and keep the metric as above (the embedding no longer makes sense).

Structure at infinity and Penrose diagram: Let's define (similar to the de-Sitter case)

$$\cos\chi = \frac{1}{\cosh\rho}$$

[so ~~$d\rho^2 =$~~ $\sinh\rho \, d\rho = \frac{\sin\chi}{\cos^2\chi} d\chi$

$$\Rightarrow \quad (-1+\cosh^2\rho) d\rho^2 = \frac{\sin^2\chi}{\cos^4\chi} d\chi^2 \qquad -1 + \frac{1}{\cos^2\chi} = \frac{1-\cos^2\chi}{\cos^2\chi} = \tan^2\chi$$

$$\Rightarrow \quad d\rho^2 = \frac{\cos^2\chi}{\sin^2\chi} \frac{\sin^2\chi}{\cos^4\chi} d\chi^2 = \frac{1}{\cos^2\chi} d\chi^2$$

and $ds^2 = \alpha^2 \left[-\frac{1}{\cos^2\chi} dt'^2 + \frac{1}{\cos^2\chi} d\chi^2 + \tan^2\chi \, d\Omega_2^2 \right]$]

which has $\chi \in [0, \frac{\pi}{2})$ and

$$ds^2 = \frac{\alpha^2}{\cos^2\chi} \left[-dt'^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2 \right] = \frac{\alpha^2}{\cos^2\chi} d\bar{s}^2$$

recognizing again the metric of Einstein-static universe.

Note that with $t' \in (-\infty, \infty)$ but $\chi \in [0, \frac{\pi}{2}]$ anti-de Sitter is conformally related to half of the Einstein-static universe (the $\chi \in [\frac{\pi}{2}, \pi]$ is missing).

Geodesics in anti de Sitter (not for class)

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Find geodesics? Start $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu})$

$\Gamma_{\rho\rho\rho} = 0$

$\Gamma_{tt\rho} = \Gamma_{t\rho t} = -\frac{1}{2}(\cosh^2\rho)_{,\rho} = -\cosh\rho\sinh\rho \quad \Rightarrow \Gamma^t{}_{t\rho} = \Gamma^t{}_{\rho t} = \frac{\sinh\rho}{\cosh\rho}$

$\Gamma_{\rho t t} = \cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\rho{}_{tt} = \cosh\rho\sinh\rho$

$\Gamma_{\theta\theta\rho} = \Gamma_{\theta\rho\theta} = \frac{1}{2}(\sinh^2\rho)_{,\rho} = \cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\theta{}_{\theta\rho} = \Gamma^\theta{}_{\rho\theta} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\theta\theta} = -\cosh\rho\sinh\rho \quad \Rightarrow \Gamma^\rho{}_{\theta\theta} = -\cosh\rho\sinh\rho$

Ignore ϕ: always look at $\phi = \text{const}$ plane (could have done that w/ θ?)
Then

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

To be sure, let's keep ϕ:

$\Gamma_{\phi\phi\rho} = \Gamma_{\phi\rho\phi} = \frac{1}{2}\sin^2\theta\; 2\sinh\rho\cosh\rho = \sin^2\theta\sinh\rho\cosh\rho \quad \Gamma^\phi{}_{\rho\phi} = \Gamma^\phi{}_{\phi\rho} = \frac{\cosh\rho}{\sinh\rho}$

$\Gamma_{\rho\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho \quad \Gamma^\rho{}_{\phi\phi} = -\sin^2\theta\sinh\rho\cosh\rho$

$\Gamma_{\phi\phi\theta} = \sin\theta\cos\theta\sinh^2\rho \quad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

$\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta\sinh^2\rho \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$

Conserved quantities $\quad g_{t\mu}\frac{dx^\mu}{d\tau} = -\cosh^2\rho\frac{dt}{d\tau} = T$

$\frac{d^2t}{d\tau^2} +$

but it works if $\Phi = 0$ see below $\leftarrow$ not this, since g depends on θ $\rightarrow$ $\boxed{g_{\theta\mu}\frac{dx^\mu}{d\tau} = \sinh^2\rho\frac{d\theta}{d\tau} = \Theta}$?? $\quad g_{\phi\mu}\frac{dx^\mu}{d\tau} = \sin^2\theta\sinh^2\rho\frac{d\phi}{d\tau} = \Phi$

So

$$\frac{d^2\rho}{d\tau^2} + \cosh\rho\sinh\rho\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\theta}{d\tau}\right)^2 - \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] = 0$$

$$\frac{d^2\rho}{d\tau^2} + \frac{\sinh\rho}{\cosh^3\rho}T^2 - \frac{\cosh\rho}{\sinh^3\rho}\Theta^2 - \frac{\cosh\rho}{\sin^2\theta\sinh^3\rho}\Phi^2 = 0$$

This equation has a 1st integral that is easy to find. But, even easier, use τ = proper time so

$$g_{\mu\nu} U^\mu U^\nu = -1$$

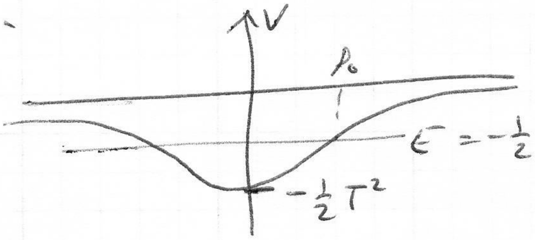
or

$$\left(\frac{d\rho}{d\tau}\right)^2 - \frac{T^2}{\cosh^2\rho} + \frac{\Theta^2}{\sinh^2\rho} + \frac{\Phi^2}{\sinh^2\rho \sin^2\theta} = -1$$

Look for solutions with $\Theta = \Phi = 0$. Then

$$\frac{d\rho}{d\tau} = \sqrt{\frac{T^2}{\cosh^2\rho} - 1} \qquad (\star)$$

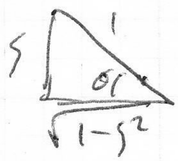
This is like motion in a potential $-\frac{T^2}{2\cosh^2\rho}$ with total energy $-\frac{1}{2}$.



And clearly there are "bound state" solutions, with turning points at $\cosh^2\rho_0 = T^2$ or $\rho_0 = \text{arccosh}\, T$. Now, it is easy to integrate $(\star)$

$$\int \frac{d\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}} = \int \frac{\cosh\rho \, d\rho}{\sqrt{T^2 - \cosh^2\rho}} = \int \frac{d \sinh\rho}{\sqrt{T^2 - 1 - \sinh^2\rho}}$$

Let $\sinh\rho = \sqrt{T^2-1}\,\xi \Rightarrow \quad = \int \frac{d\xi}{\sqrt{1-\xi^2}}$



$\xi = \sin\theta$

$$\Rightarrow \int \frac{\cos\theta \, d\theta}{\cos\theta} = \theta = \arcsin\xi = \text{arctg}\frac{\xi}{\sqrt{1-\xi^2}}$$

$$= \arcsin\left(\frac{\sinh\rho}{\sqrt{T^2-1}}\right)$$

or $\text{arctg}\left(\frac{\sinh\rho}{\sqrt{T^2-1-\sinh^2\rho}}\right) = \text{arctg}\left(\frac{\tanh\rho}{\sqrt{\frac{T^2}{\cosh^2\rho} - 1}}\right)$

Then $t(\tau)$ is obtained from

$$\frac{dt}{d\tau} = -\frac{T}{\cosh^2\rho}$$

For this we need

$$\boxed{\sin\tau = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

or $(T^2-1)\sin^2\tau = \sinh^2\rho = \cosh^2\rho - 1$

so

$$\frac{dt}{d\tau} = -\frac{T}{1+(T^2-1)\sin^2\tau}$$

We need

$$\int \frac{d\tau}{1+k^2\sin^2\tau} = \frac{tg^{-1}\left[\sqrt{1+k^2}\, tg\,\tau\right]}{\sqrt{1+k^2}}$$ (mathematica)

so

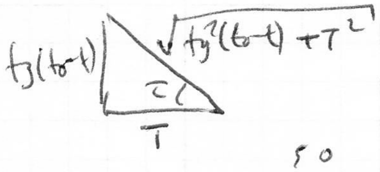
$$-\frac{(t-t_0)}{T} = \frac{1}{\sqrt{1+(T^2-1)}} \text{arctg}\left[T\, tg\,\tau\right]$$

or

$$\boxed{-tg(t-t_0) = T\, tg\,\tau}$$

(The sign is because ~~T(0) is proper distance, not proper time~~).

We can also obtain the trajectory. Since $tg\,\tau = \frac{1}{T} tg(t_0-t)$



$$\Rightarrow \sin\tau = \frac{tg(t_0-t)}{\sqrt{tg^2(t_0-t)+T^2}} = \frac{1}{\sqrt{1+T^2 ctg^2(t_0-t)}}$$

so

$$\boxed{\frac{1}{\sqrt{1+T^2 ctg^2(t_0-t)}} = \frac{\sinh\rho}{\sqrt{T^2-1}}}$$

In all these it's worth remembering $T = -\cosh\rho_0$

Check the θ piece (recall $g = g(\theta, \rho)$ so we were unjustified in using $g_{\theta\mu}\frac{dx^\mu}{d\tau}$ = constant)

Now

$$\frac{d^2\theta}{d\tau^2} + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

But if ϕ = constant ($\Phi = 0$) we have

$$\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) + 2\frac{\cosh\rho}{\sinh\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} = 0$$

Now, check: $\frac{d\theta}{d\tau} = \frac{\textcircled{L}}{\sinh^2\rho}$ gives $\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = -2\frac{\cosh\rho}{\sinh^3\rho}\textcircled{L}\frac{d\rho}{d\tau}$

while the 2nd term is $2\frac{\cosh}{\sinh\rho}\frac{d\rho}{d\tau}\frac{\textcircled{L}}{\sinh^2\rho}$

so they cancel ✓

Connecting both coordinate systems:

in (t, r, θ, ϕ) system geodesics are $r, \theta, \phi = \text{constant}$ with $r = \rho_0$

Comparing both systems:

u: $\sin t' \cosh\rho = \sin t$

v: $\cos t' \cosh\rho = \cos t \cosh r$

$$\boxed{\begin{aligned} z:&\quad \sinh\rho = \cos t \sinh r \\ u/v:&\quad tg\, t' = \frac{1}{\cosh r}\, tg\, t \end{aligned}}$$

(θ, φ remain the same)

Geodesics
$$\begin{cases} \sinh\rho = \sinh\rho_0 \sin\tau \\ tg(t' - t_0) = \cosh\rho_0\, tg\,\tau \end{cases}$$

Going to other system:

$$\sinh\rho_0 \sin\tau = \cos t \sinh r$$

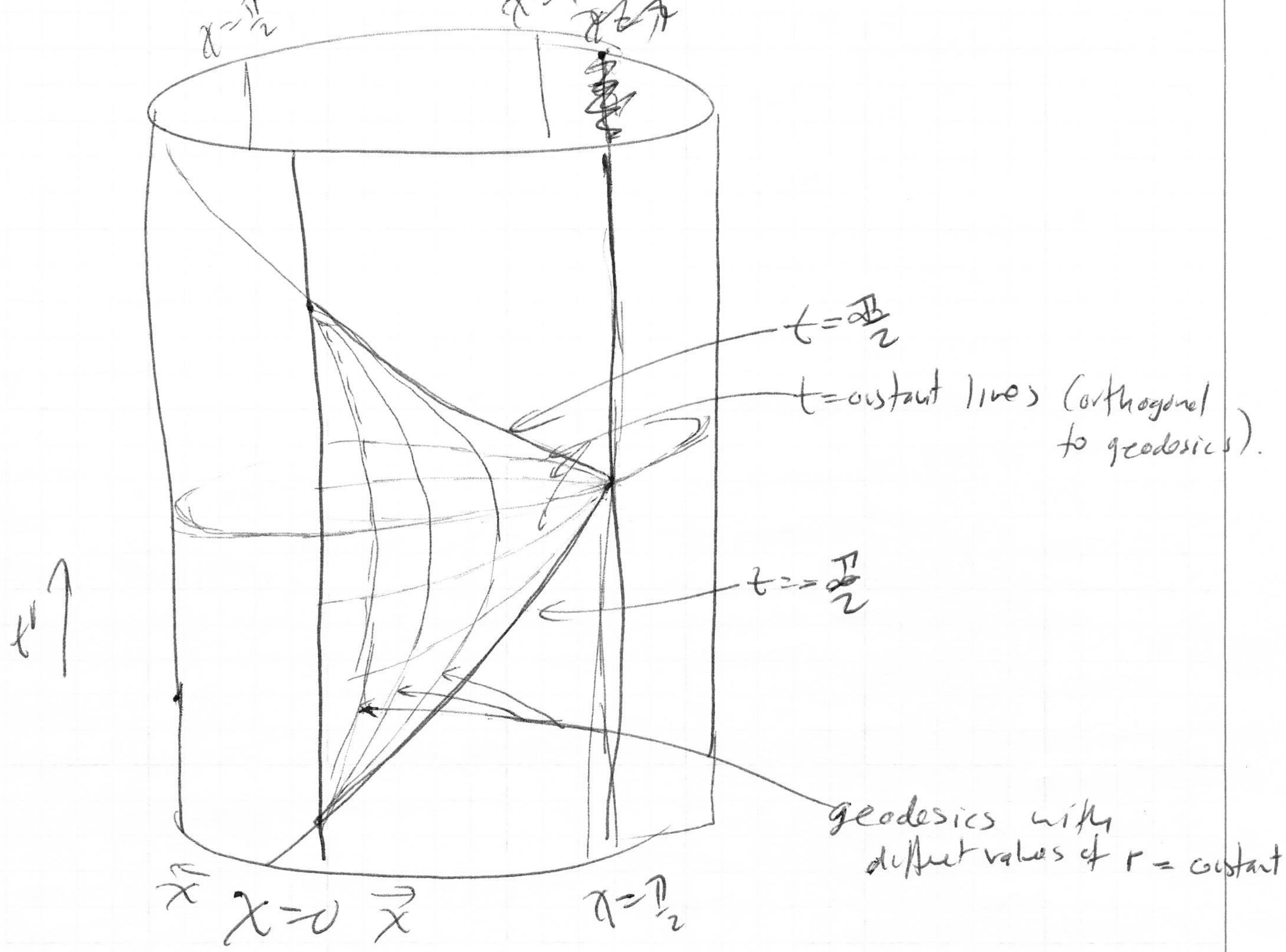
$\tau \to \tau + \frac{\pi}{2}$ ~~st~~ $\boxed{\rho_0 = r}$

and then

$$tg(t' - t_0) = \cosh\rho_0\, tg\left(\tau + \frac{\pi}{2}\right) = \cosh\rho_0 \frac{\cos\tau}{-\sin\tau}$$

$$ctg(t' - t_0) = -\frac{1}{\cosh\rho_0}\, tg\,\tau$$

$\Rightarrow t_0 = \frac{\pi}{2}$ $\Rightarrow$ it works



(The lines $t = \pm\frac{\pi}{2}$ are easy to understand. Since

$$\sin t = \sin t' \cosh \rho$$

we have $\pm 1 = \sin t' \cosh \rho = \sin t' \frac{1}{\cos\chi}$ where we introduced the variable χ for the conformal mapping

$\Rightarrow$

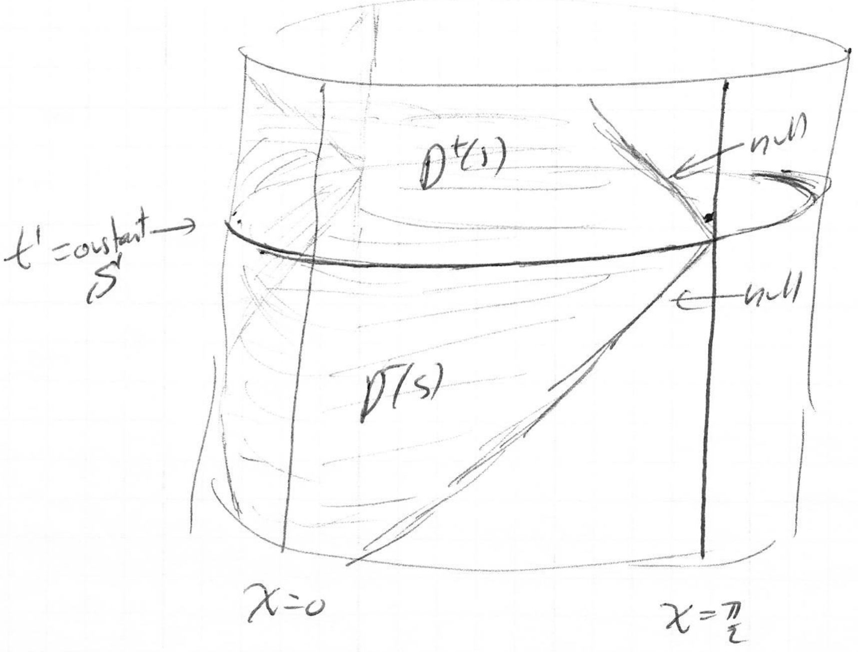
$$\cos\chi = \pm \sin t'$$

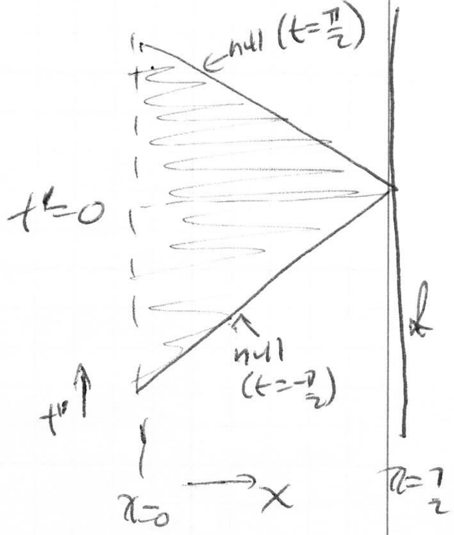
or $\chi = \frac{\pi}{2} \pm t'$).

Note that the apparent singularity in t, r, θ, φ coord's is related to convergence of geodesics.

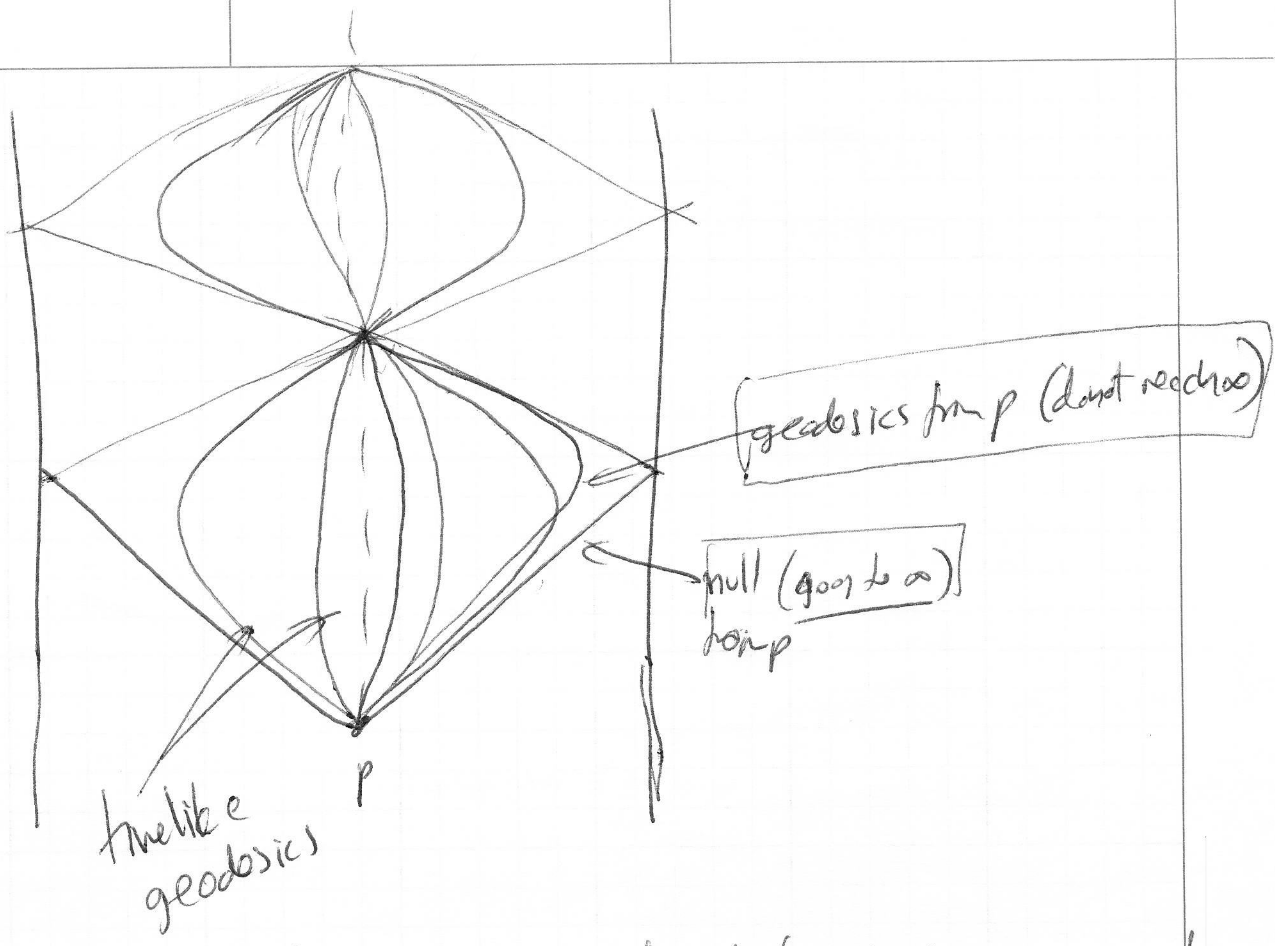
Causal structure of anti-de Sitter space:

$\boxed{\text{NO CAUCHY SURFACE}}$!



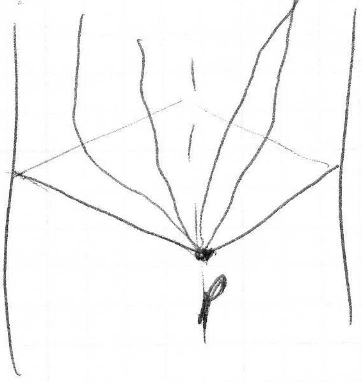


Evident ⇒ information flows in/out from boundary at ∞.

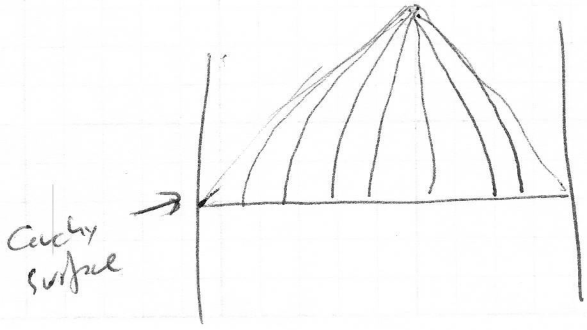


timelike geodesics from p are confined to infinite sequence of diamonds

But there are timelike curves (non-geodesic) from p that can reach any point future of the null-cone from p.



Also



every point in $D^+(S)$ can be reached by a unique geodesic from S, normal to S.

Maximally symmetric hypersurfaces: isotropic cosmology.

Since there is a separate course in cosmology, we won't study cosmology here. But we do set up the stage by analyzing the spacetime one obtains from the requirements of homogeneity and isotropy.

Why? In cosmology (the study of the history, dynamics, evolution of the Universe on a large scale) our observations are limited, because:

(i) can only be done from one specific point, Earth, at one ~~a~~ specific time, now (cosmologically the fact that we have been doing astronomy for ~ 1000 years is still basically an instantaneous ~~observation~~ event).

(ii) can only see part of the Universe; observations are limited by

- dust and other intervening stuff
- physical, ~~ai~~ eg the universe is opaque before recombination
- luminosity
- only see few bandwidths of light.

Hence, to make progress it is always the case that assumptions are made ~~to say~~ in building mathematical models ~~describing~~ of the ~~ent~~ spacetime that describes the universe.

Generally/commonly two assumptions are made: approximate:

(i) Homogeneity: that there is no preferred point in ~~the~~ space

ii) Isotropy – – – direction – –

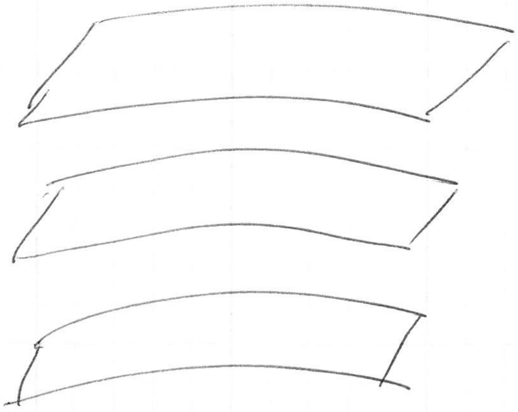
These are often ~~called~~ referred to as the Copernican Principle (since we would not occupy a preferred place in the Universe, just like Copernicus moved the center from Earth to Sun, now we are disposing with a center anywhere).

~~Proper~~ Notice that we are talking about approximate ~~sym~~ conditions. There is matter in the universe and it is not uniformly spread (there are galaxies, planets, bacteria...). But, on average, ~~using~~ smoothing over distance scales of several intergalactic ~~universes~~ lengths, the universe appears fairly smooth. So this is a starting ~~assum~~ approximation that has to be improved to account for the very interesting irregularities → whole course on cosmology.

We will restrict our attention to ~~determining~~ the spacetimes that are homogeneous & isotropic, and discuss briefly what Einstein's equations imply for them.

Technical def*: of

Homogeneity & Isotropy. For spacetime to be homogeneous:



Need
← foliation of spacetime by ~~space~~ 1-parameter family of spacelike surfaces Σ_t, and

for any Σ_t (any "time"), for any two points p, q in Σ_t there is an isometry taking $p \to q$.

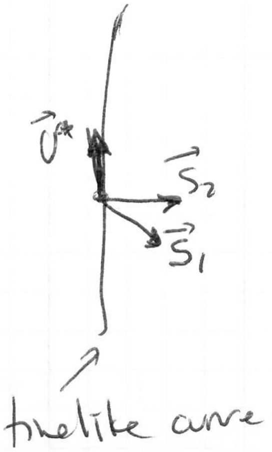
(Recall an isometry ϕ is $\phi : \mathcal{M} \to \mathcal{M}$ + $\phi^* g = g$).

In other words, there is some definition of time for which at each $t =$ constant hypersurface, the metric is the same at all points.

Isotropy: First define isotropy for an observer. We want to say that an observer sees same stuff in any direction.

So

~~For A spacetime is to be spatially isotropic abu~~



$\vec U$ = (timelike) tangent to worldline at p.

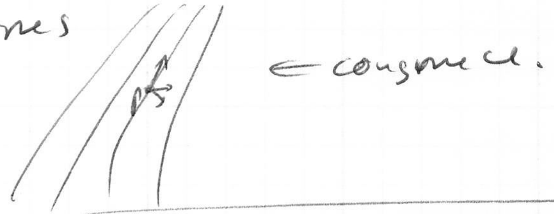
$\vec S_i$ = spacelike ~~tangent~~ vectors at p, unit magnitude, ~~[illegible]~~

(ie, $\vec S_i$ are $\perp$ $\vec U$)

Isotropy at p: an isometry leaving p and $\vec U$ fixed but taking (rotating) $\vec S_1 \to \vec S_2$ for any pair ~~#~~ $\vec S_1, \vec S_2$.

(So there is no preferred direction, ie, no preferred spatial vector $\perp$ to $\vec U$).

Isotropic space: if there is a congruence of timelike curves ~~such that every point p~~ and the spacetime has isotropy at every point on these curves



~~(So the~~

Def "isotropic observers" (Wald) those in this congruence

Notes:

- ~~For~~ ~~is~~ The 2 definitions (hom'y & iso'y) required a preferred collection of subspaces.
- If spacetime is homogeneous AND isotropic, then Σ_t are $\perp$ to $\vec U$. For if $\vec U$ had a component along Σ_t, say $\vec S$, then this would be a spatial vector in a preferred direction (we can project out the part that is not orthogonal to $\vec U$, to construct $\vec S$, a preferred vector with $\vec S \perp \vec U$).

(Actually, ~~one~~ must be careful; the previous note is true when the Σ_t and the isotropic observers ~~have~~ are unique. If not unique, one can choose $\Sigma_t \perp$ to $\vec{U}$. Example is flat Minkowski)

Now, use $\Sigma_t \to \mathcal{M}$ (embedding) to define $\boxed{h(t) \; \phi^* g}$. $h_{ij}(t)$ is Riemannian (sign$(+++)$). (This is the same as $h = g$ restricted to act on vectors tangent to Σ_t). ~~Space~~

So consider the space Σ_t with metric h_{ij}, inverse h^{ij}. We expect isotropy + homogeneity $\Rightarrow$ Σ_t is a 3-dim maximally symmetric space.

In fact isotropy is enough to show this (and so isotropy $\Rightarrow$ homogeneity). Consider the 3-d curvature tensor (field)

$$\bar{R}^{ij}{}_{kl} \qquad \text{(the bar for 3-D)}$$

With indexes raised as shown, this is a linear map ~~on~~ L on 2-forms, $L : \Omega^{(2)} \to \Omega^{(2)}$

$$~~\omega_{ij}~~ \quad \tilde{\omega} = \omega_{ij} \, d\tilde{x}^i \wedge d\tilde{x}^j$$

$$\tilde{\omega} \to L(\tilde{\omega}) = \omega_{ij} \bar{R}^{ij}{}_{kl} \, d\tilde{x}^k \wedge d\tilde{x}^l$$

Now, defining the positive, symmetric inner product on $\Omega^{(2)}$ by

$$(\tilde{\sigma}, \tilde{\omega}) = (\tilde{\omega}, \tilde{\sigma}) = \omega_{ij} \sigma_{lk} h^{il} h^{jk}$$

then L is self-adjoint; $(\tilde{\sigma}, L\tilde{\omega}) = (L\tilde{\sigma}, \omega)$ (which follows from $\bar{R}_{ijkl} = \bar{R}_{klij}$). $\to$ there is a basis of orthonormal eigenvectors of L. By isotropy the eigenvalues must be all the same (else, special direction), so ~~$L = K I$~~

so $$L \propto I$$ ~~Ka~~

or

$$\bar R^{ij}{}_{kl} = \kappa\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right)$$

$$\Rightarrow \qquad \bar R_{ijkl} = \kappa\left(h_{ik} h_{jl} - h_{il} h_{jk}\right)$$

→ ~~h_{ij} is maximally symmetric~~.

Now, this is the statement that h_{ij} is maximally symmetric if κ is constant (same everywhere on Σ_t). This follows from homogeneity. But it also follows from the Bianchy identity

$$\bar R_{ijkl;m} + \bar R_{ijmk;l} + \bar R_{ijlm;k} = 0$$

$$\Rightarrow \qquad K_{,m}(h_{ik}h_{jl} - h_{il}h_{jk}) + K_{,l}(h_{im}h_{jk} - \text{nept}) + K_{,k}(\,\cdots) = 0$$

Now contract with $h^{ik}h^{jl}$:

$$K_{,m}(3^2-3) + K_{,m}(1-3) + (1-3)K_{,m} = 0$$

$$\rightarrow K_{,m} = 0$$

So, isotropy ⇒ homogeneity <u>AND</u> h_{ij} is maximally symmetric.

Also $\kappa = \dfrac{\bar R}{6}$ a constant for each Σ_t

~~Finally we have we~~

~~$g_{ab} = -u_a u_b + h$~~

So we have a space which admits a congruence of isotropic observers (with tangent $\vec u$) with a corresponding foliation by spacelike surfaces Σ_t orthogonal to $\vec u$, ~~a which~~ which are Riemannian 3-dim maximally symmetric spaces with metric h. The full space time has metric g, ~~If~~ and if $\vec\xi', \vec\xi$ are on Σ_t then

$$g(\vec\xi, \vec\xi') = h(\vec\xi, \vec\xi').$$

Let $\tilde{v}(\cdot) = g(\vec{v}, \cdot)$. Then clearly, if we define $h(\vec{v}, \vec{x}) = 0$

$$g = h + \lambda \tilde{v} \otimes \tilde{v}$$

however, since $g(\vec{v}, \vec{v}) = -1$, $\alpha(\vec{v}) = -1$ and

$$-1 = 0 + \lambda \quad \Rightarrow \lambda = -1$$

$$\boxed{g = -\tilde{v} \otimes \tilde{v} + h}$$

In components $g_{\mu\nu} = -v_\mu v_\nu + h_{\mu\nu}$

~~Ex~~ Useful coordinates:

(i) Obvious choice on each Σ_t, ie, spherical coordinates if $\bar{R} > 0$ ..

(ii) Assign a fixed spatial coordinate label to each isotropic observer ("comoving coordinates")

(iii) Homogeneity $\Rightarrow$ all isotropic observers agree on proper time of Σ_t, so label Σ_t by proper time τ of isotropic observer.

$$\boxed{ds^2 = -dt^2 + a^2(t) \begin{cases} d\chi^2 + \sin^2\chi \, d\Omega_2^2 & \bar{R} > 0 \\ d\chi^2 + \chi^2 d\Omega_2^2 & \bar{R} = 0 \\ d\chi^2 + \sinh^2\chi \, d\Omega_2^2 & \bar{R} < 0 \end{cases}}$$

Robertson-Walker metric.

Note, there $\underline{\text{is}}$ a preferred set of observers: the isotropic observers. In comoving coordinates the distance between fixed points p_1 & p_2 on the hypersurface Σ_t evolves with t as a(t).

Einstein's Equations

~~We have $\bar{R}_{ij} = 2k h_{ij}$~~

Need to compute Einstein's tensor. It is fairly standard to introduce radial coordinate r by

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

with $k = +1, 0, -1$ for $\bar{R} > 0, = 0, < 0$. Then

$r = \sin\chi, \chi, \sinh\chi$ in each case. Then

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right]$$

As usual $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu})$ and $\Gamma^{\mu}_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda}$

so

$$\Gamma_{011} = -\frac{1}{2}\frac{2a\dot{a}}{1-kr^2} = -\Gamma_{110} = -\Gamma_{101}$$

$$\Gamma_{0\theta\theta} = -\frac{1}{2}2a\dot{a}r^2 = -\Gamma_{\theta\theta 0} = -\Gamma_{\theta 0\theta}$$

$$\Gamma_{0\phi\phi} = -a\dot{a}r^2\sin^2\theta = -\Gamma_{\phi\phi 0} = -\Gamma_{\phi 0\phi}$$

$$\Gamma_{111} = \frac{1}{2}a^2\frac{2kr}{(1-kr^2)^2}$$

$$\Gamma_{1\theta\theta} = -\Gamma_{\theta 1\theta} = -\Gamma_{\theta\theta 1} = -a^2 r$$

$$\Gamma_{1\phi\phi} = -\Gamma_{\phi 1\phi} = -\Gamma_{\phi\phi 1} = -a^2 r\sin^2\theta$$

$$\Gamma_{\theta\phi\phi} = -\Gamma_{\phi\theta\phi} = -\Gamma_{\phi\phi\theta} = -a^2r^2\sin\theta\cos\theta$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^1_{10} = \Gamma^1_{01} = \frac{\dot{a}}{a}$$

$$\Gamma^0_{\theta\theta} = a\dot{a}r^2 \qquad \Gamma^\theta_{\theta 0} = \frac{\dot{a}}{a} = \Gamma^\phi_{\phi 0}$$

$$\Gamma^0_{\phi\phi} = a\dot{a}r^2\sin^2\theta$$

$$\Gamma^1_{11} = \frac{kr}{1-kr^2}$$

$$\Gamma^1_{\theta\theta} = -r(1-kr^2) \qquad \Gamma^\theta_{1\theta} = \frac{1}{r}$$

$$\Gamma^1_{\phi\phi} = -r(1-kr^2)\sin^2\theta \qquad \Gamma^\phi_{1\phi} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \mathrm{ctg}\,\theta$$

and

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

so we have

$$R_{00} = -3\partial_t\left(\frac{\dot a}{a}\right) + 3\left(\frac{\dot a}{a}\right)^2 = \boxed{-3\frac{\ddot a}{a}}$$

$$-\partial_r\left(\frac{2}{r}\right)$$

$$R_{11} = \partial_t\left(\frac{a\dot a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) + \Gamma^\rho_{\rho 0}\Gamma^0_{11} + \Gamma^\rho_{\rho 1}\Gamma^1_{11}$$

$$-\Gamma^\rho_{1\lambda}\Gamma^\lambda_{\rho 1}$$

$$= \frac{\ddot a a + \dot a^2}{1-kr^2} + 3\left(\frac{\dot a}{a}\right)\frac{a\dot a}{1-kr^2} \overset{-\frac{2}{r^2}}{\wedge} + \frac{kr}{1-kr^2}\left(\frac{2}{r} + \frac{kr}{1-kr^2}\right) - \left[2\left(\frac{\dot a}{a}\right)\frac{a\dot a}{1-kr^2}\right.$$

$$\left. + \left(\frac{kr}{1-kr^2}\right)^2 + \frac{2}{r^2}\right] \cancel{(1-kr^2) - 2(1-kr^2)\sin^2\theta\}}$$

$$= \frac{1}{1-kr^2}\left[\ddot a a + \dot a^2 + 3\dot a^2 + 2k - 2\dot a^2\right] \cancel{-\frac{2}{r^2}(1-kr^2))}\Big]$$

$$= \boxed{\frac{\ddot a a + 2\dot a^2 + 2k}{1-kr^2}}$$

← ~~[illegible] like text~~

$$R_{\theta\theta} = \partial_t(a\dot a r^2) + \partial_r(-r(1-kr^2)) - \partial_\theta(\mathrm{ctg}\,\theta)$$

$$+ a\dot a r^2\left(3\frac{\dot a}{a}\right) + (-r(1-kr^2))\left[\frac{kr}{1-kr^2} + \frac{2}{r}\right]$$

$$- \left[2(a\dot a r^2)\left(\frac{\dot a}{a}\right) + 2(-r(1-kr^2))\frac{1}{r} + \mathrm{ctg}^2\theta\right]$$

$$= (a\ddot a + \dot a^2)r^2 - (1-kr^2) + 2kr^2 + \frac{1}{\sin^2\theta} + 3\dot a^2 r^2 - kr^2 - 2(1-kr^2)$$

$$-2\dot a^2 r^2 + 2(1-kr^2) - \frac{\cos^2\theta}{\sin^2\theta}$$

$$= (a\ddot a + 2\dot a^2)r^2 - 1 + 2kr^2 + \frac{1-\cos^2\theta}{\sin^2\theta} = \boxed{(a\ddot a + 2\dot a^2 + 2k)r^2}$$

$$R_{\phi\phi} = \partial_t(a\dot a r^2\sin^2\theta) + \partial_r(-r(1-kr^2)\sin^2\theta) + \partial_\theta(-\sin\theta\cos\theta)$$

$$+ (a\dot a r^2\sin^2\theta)\left(3\frac{\dot a}{a}\right) + (-r(1-kr^2)\sin^2\theta)\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) + (-\sin\theta\cos\theta)\,\mathrm{ctg}\,\theta$$

$$-\left[2\left(\frac{\dot a}{a}\right)(a\dot a r^2\sin^2\theta) + 2(-r(1-kr^2)\sin^2\theta)\left(\frac{1}{r}\right) + 2(-\sin\theta\cos\theta)\,\mathrm{ctg}\,\theta\right]$$

$$= (\ddot a a + \dot a^2)r^2\sin^2\theta - (1-3kr^2)\sin^2\theta - \cos^2\theta + \sin^2\theta + 3\dot a^2 r^2\sin^2\theta$$

$$- kr^2\sin^2\theta + \cos^2\theta = \boxed{(\ddot a a + 2\dot a^2 + 2k)r^2\sin^2\theta}$$

And
$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + \frac{1}{a^2}\left[(\ddot{a}a + 2\dot{a}^2 + 2k)\cdot 3\right]$$
$$= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

Now Einstein's equations are $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

or, since $-R = 8\pi G T$; $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$

Model energy & matter in the universe by a perfect fluid. For consistency with the isotropy and homogeneity of the metric we must choose the fluid to be homogeneous and isotropic, that is the fluid is at rest in comoving coordinates:

$$U^\mu = (1,0,0,0)$$

$$T_{\mu\nu} = (\rho+p)U_\mu U_\nu + p g_{\mu\nu} \qquad T^\mu{}_\nu = (\rho+p)U^\mu U_\nu + p\delta^\mu_\nu$$

Note that $T^{\mu}{}_{\nu;\mu} = 0$ and for $\nu = 0$

$\Rightarrow$ $\cancel{\frac{\partial}{\partial t}(\rho+p)}$ recall $T^\mu{}_{\nu;\lambda} = T^\mu{}_{\nu,\lambda} + \Gamma^\mu_{\rho\lambda}T^\rho_\nu - \Gamma^\rho_{\nu\lambda}T^\mu_\rho$

so $T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu_{\mu\rho}T^\rho_\nu - \Gamma^\rho_{\nu\mu}T^\mu_\rho$

and so
$$T^\mu{}_{0;\mu} = -(\rho+p)_{,0} + p_{,0} + (-\rho)\left(3\frac{\dot{a}}{a}\right) - \left(3\frac{\dot{a}}{a}\right)(p)$$

so
$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho+p) = 0}$$

This equation could be obtained from Einstein's, but this is simpler.

Now $T = g^{\mu\nu} T_{\mu\nu} = -(\rho+p) + 4p = 3p-\rho$

$$R_{00} = 8\pi G\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$$

$$-3\frac{\ddot a}{a} = 8\pi G\left(\rho + \tfrac{1}{2}(3p-\rho)\right) = 8\pi G\left(\tfrac{1}{2}\rho + \tfrac{3}{2}p\right)$$

or

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho+3p)}$$

and $$R_{11} = 8\pi G\left(T_{11} - \tfrac{1}{2} g_{11} T\right)$$

$$\frac{\ddot a a + 2\dot a^2 + 2k}{1-kr^2} = 8\pi G\left[\frac{a^2}{1-kr^2}p - \frac{1}{2}\frac{a^2}{1-kr^2}(3p-\rho)\right]$$

$$= \frac{1}{1-kr^2}\, 8\pi G a^2\left(p - \tfrac{1}{2}(3p-\rho)\right)$$

$\Rightarrow$

$$\frac{\ddot a}{a} + 2\left(\frac{\dot a}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

Eliminate $\frac{\ddot a}{a}$ using above, $\cancel{\frac{\ddot a}{a}}$ and $\frac{4\pi G}{3}\left(\rho + 3p + 3\rho - 3p\right) = \frac{4^2\pi G\rho}{3}$

$\Rightarrow$

$$\boxed{\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho}$$

These are Friedmann equations. Metrics that obey these are called FRW metrics (Friedmann-Robertson-Walker).

We have two equations for three unknowns, $a(t)$, $\rho(t)$ and $p(t)$. But ρ and p are not independent if we know what constitutes the matter/energy in the universe. For example, a collisionless fluid (dust) has $p=0$, while radiation has $p = \frac{1}{3}\rho$. So we write an "equation of state" $p = w\rho$.

We will take w to be a fixed number, and are particularly interested in the cases

$$w = \begin{cases} 0 & \text{dust (or "matter")} \\ \frac{1}{3} & \text{radiation} \\ -1 & \text{cosmological constant.} \end{cases}$$

The last one is just the statement that if we ~~add~~ modify Einstein's equations by adding Einstein's cosmological constant Λ:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

then we can rewrite

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}\right)$$

and think of $-\frac{\Lambda}{8\pi G} g_{\mu\nu}$ as a contribution to $T^{(\text{total})}_{\mu\nu} = T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}$.

Then, ~~the~~ $T^{\Lambda}_{\mu\nu}$ is of the form of ~~a~~ fluid with $\rho = +\frac{\Lambda}{8\pi G}$ and $p = -\rho$ (so $w = -1$).

Then $T^{\mu}{}_{0;\mu} = 0 \Rightarrow \dot\rho + 3\frac{\dot a}{a}(\rho + w\rho) = 0$ or

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a} \Rightarrow \frac{d}{dt}\ln\rho = -3(1+w)\frac{d}{dt}\ln a$$

$$\boxed{\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}}$$

Note, $w = 0 \Rightarrow \rho \sim \frac{1}{a^3}$ makes sense, $\rho \sim \frac{1}{\text{volume}}$

$w = \frac{1}{3} \Rightarrow \rho \sim \frac{1}{a^4}$ ✓ $\rho \sim \frac{1}{\text{volume}} \times$ redshift
↳ which we have not explaine

$w = -1 \Rightarrow \rho =$ constant, in fact $\rho = \frac{\Lambda}{8\pi G}$.

One can then solve Einstein's equations

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

or

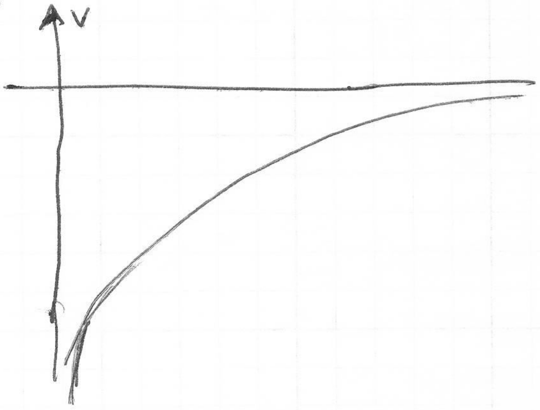
$$\cancel{\dot a^2 + \frac{k}{a^2} - \frac{8\pi G\rho_0 a_0}{3}}$$

$$\boxed{\dot a^2 - \frac{8\pi G}{3}\rho_0 a_0^{3(1+w)}\frac{1}{a^{1+3w}} = -k}$$

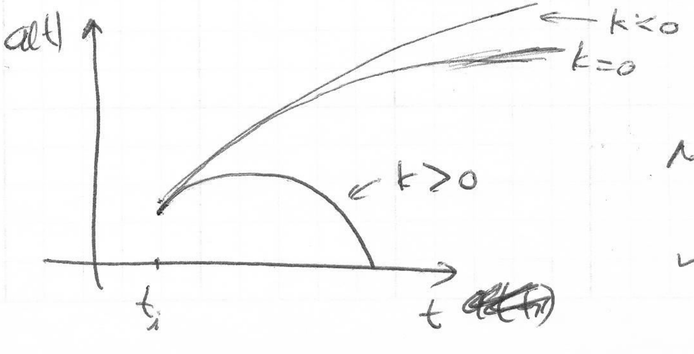
This is like the equation

$$E = \frac{1}{2}m\dot r^2 + V(r)$$

multiplied by $\frac{2}{m}$, so ~~it is a particle~~ the solution has same time dependence as a particle in a potential $V \sim \frac{(-1)}{r^{1+3w}}$ with engy $E \sim -k$. For $1+3w > 0$ $V \to -\infty$ as $r \to 0$ and $V \to 0$ as $r \to \infty$, so



Now if $E<0$ the motion has a ~~maximum~~ turning point at some maximum r and then eventually $r \to 0$. ~~and~~ For $E \geq 0$ the motion is unbounded provided $\dot r > 0$ initially. So for $w > \frac{1}{3}$ we have



Note that for $k=0$ $\dot a \to 0$ as $t \to \infty$ while for $k<0$ $\dot a > 0$ for $t \to \infty$.

Clearly it is of great (political) interest to know if the universe will expand for ever (and if so whether it will do so by slowing down to $\dot{a}=0$ asymptotically) or if it will collapse into a "big ~~g~~ crunch". $\boxed{\text{Need to know } k \gtreqless 0.}$

Note, however, that if we start with $\dot{a}>0$ at some point, running the clock back in any case gives $a\to 0$, so it looks like the universe grew out of a singular ($a=0$) condition, or better, started small at some t_0 and quickly grew. This is called "the big bang". However, it is *not* an explosion. Recall, comoving observers are separated by fixed comoving separation. It is just that the distance (space between any two of them) is $\to 0$ as $t\to t_{\text{big bang}}$.

To figure out whether $k>0$, $k=0$ or $k<0$ in our present universe we can measure each term in the left of

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^{3(1+w)} = -\frac{k}{a^2}$$

First, if this is evaluated today, then $a=a_0$ and ~~defining~~ we ha

$$\left(\frac{\dot{a}}{a}\right)^2\Big|_0 - \frac{8\pi G\rho_0}{3} = -\frac{k}{a_0^2}$$

Need $H_0 = \frac{\dot{a}}{a}\Big|_0$ the Hubble ~~parameter~~ constant (should be called Hubble parameter since $H=\frac{\dot{a}}{a}\neq$ constant)

and ρ_0 = energy density.

H_0 can be measured from redshift vs luminosity of standard candles (see below) while ρ_0 can be "counted"

Actually, we should be more carefull to include all types of matter (different equations of state) ~~and~~ possible in the analysis --- we have assumed one dominant type.

Write

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} - \frac{k}{a^2}$$

You will often see this written as

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

where one term, $i=k=$curvature has $\rho_k = \frac{3}{8\pi G}\left(-\frac{k}{a^2}\right)$ not a real energy density! (set aside "ρ_c" ~~as~~ for critical density).

Also dividing this by H^2,

$$1 = \sum_i \Omega_i$$

where $\Omega_i = \frac{8\pi G}{3H^2}\rho_i = \frac{\rho_i}{\rho_c}$

where $\rho_c = \frac{3H^2}{8\pi G}$ is a quantity depending only on the geometry, through H, which gives the critical value for which k changes sign: if we define $\Omega = \sum_{i\neq k}\Omega_i$ then we have

$$\Omega_k = 1-\Omega$$

and $\Omega_k > 0, =0, <0$ $(k<0, =0, >0)$ iff $\Omega<1, =1, >1$.

So we need to measure all components ρ_i of ρ and compare them with ρ_c, obtained from measuring H.

Note that the different components scale differently:

$$\Omega_\Lambda \sim a^0 \qquad \Omega_k \sim \frac{1}{a^2} \qquad \Omega_{matt} \sim \frac{1}{a^3} \qquad \Omega_{rad} \sim \frac{1}{a^4}$$

If they were all similar today, then in the past, as $a \to 0$, Ω_{rad} ~~would~~ would be dominant.

In fact, today it's found $\Omega_M \sim \frac{1}{2}\Omega_\Lambda$ ~~and~~ $\gg \Omega_k, \Omega_{rad}$ with $\Omega \simeq 1$.

Moreover, the evolution of a(t) is still given, as before, by

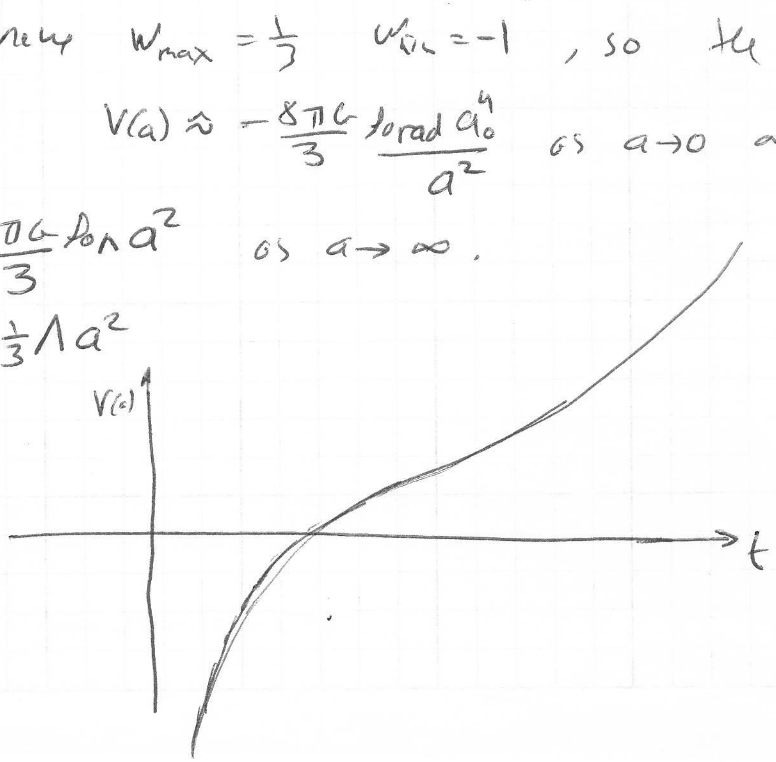
$$\ddot{a}^2 - \frac{8\pi G}{3}\sum_i \rho_{0i}\frac{a_0^{3(1+w_i)}}{a^{1+3w_i}} = -k$$

At small a, the largest w_i dominates; at large a the smallest w_i dominates. With matter, radiation and cosmological constant, we have $w_{max} = \frac{1}{3}$ $w_{min} = -1$, so the "potential" V(a) has $V(a) \approx -\frac{8\pi G}{3}\frac{\rho_{0rad}\, a_0^4}{a^2}$ as $a \to 0$ and

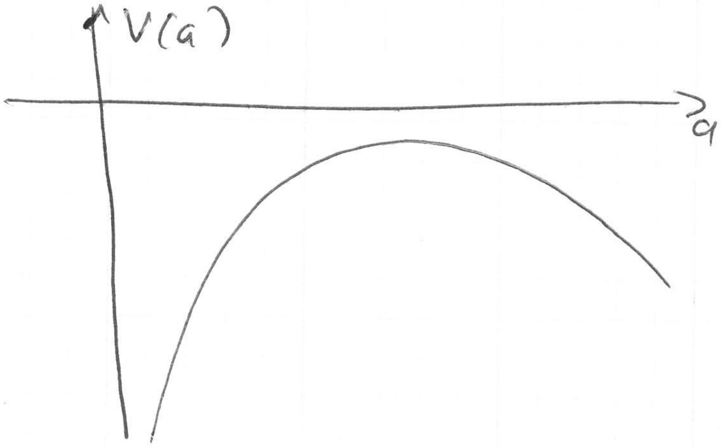
$$V(a) \approx -\frac{8\pi G}{3}\rho_{0\Lambda} a^2 \quad \text{as } a \to \infty.$$

$$= -\frac{1}{3}\Lambda a^2$$

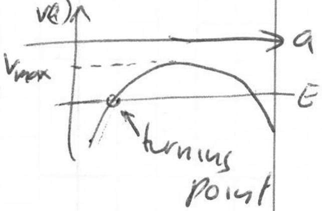
So, if $\Lambda < 0$



while, if $\Lambda > 0$



Let's look at this in more detail. For $k<0$ ("E">0) or $k=0$ ("E=0"), the "particle" motion is unbounded, describing an ever expanding universe. But for $k>0$ ("E"<0) there is a critical value of parameters beyond which the universe recollapses. This occurs if the maximum of the potential $V(a)$ is above the energy E.



<u>Recollapse condition</u>

$$\max_a \left[-\frac{8\pi G}{3} \sum_i \rho_{0i} \frac{a_0^{3(1+w_i)}}{a^{1+3w_i}} \right] > -k$$

or multiplying by a − sign and using $\Omega_{0i} = \frac{8\pi G}{3H_0^2}\rho_{0i}$

$$\min_a \left[H_0^2 \sum_i \frac{\Omega_{0i}\, a_0^{3(1+w_i)}}{a^{1+3w_i}} \right] < k = -H_0^2 a_0^2 \Omega_{0K}$$

or simply

$$\min_a \left[\frac{\Omega_{0Rad}\, a_0^4}{a^2} + \frac{\Omega_{0M}\, a_0^3}{a} + \Omega_{0\Lambda} a^2 \right] < -a_0^2 \Omega_{0K}$$

To simplify matter, let's ignore $\Omega_{radiation}$, since it is already negligible today. Then, taking one derivative:

$$\frac{d}{da}\left[\frac{\Omega_{0m}a_0^3}{a}+\Omega_{0\Lambda}a^2\right)=0$$

$$\Rightarrow \quad -\frac{\Omega_{0m}a_0^3}{a^2}+2\Omega_{0\Lambda}a=0$$

$$\Rightarrow \quad a=\left(\frac{\Omega_{0m}a_0^3}{2\Omega_{0\Lambda}}\right)^{1/3}=a_0\left(\frac{\Omega_{0M}}{2\Omega_{0\Lambda}}\right)^{1/3}$$

Now plug back into "potential" to find minimum

$$\text{minimum}=\frac{\Omega_{0m}a_0^3}{a_0(\Omega_{0m}/2\Omega_{0\Lambda})^{1/3}}+\Omega_{0\Lambda}a_0^2\left(\frac{\Omega_{0M}}{2\Omega_{0\Lambda}}\right)^{2/3}$$

$$=a_0^2\,\Omega_{0M}^{2/3}(2\Omega_{0\Lambda})^{1/3}\left(1+\tfrac{1}{2}\right)$$

and the condition for recollapse is

$$\boxed{\frac{3}{2}\Omega_{0M}^{2/3}(2\Omega_{0\Lambda})^{1/3}<-\Omega_{0K}}$$

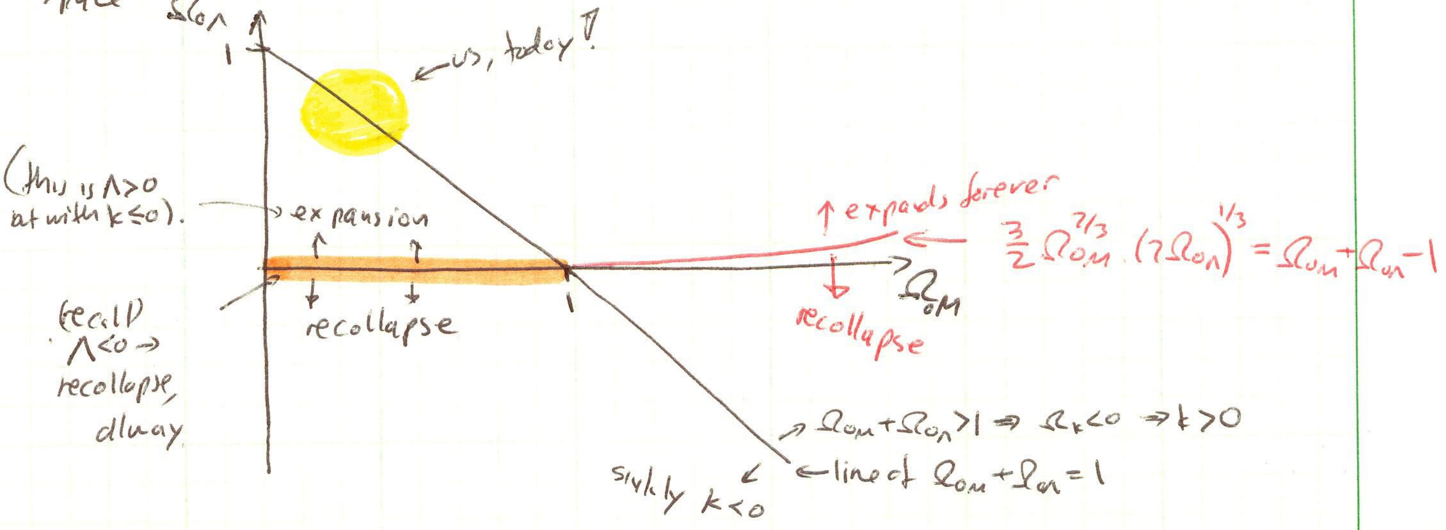
Moreover, recall that $\Omega_{0K}=1-\Omega_{0m}-\Omega_{0\Lambda}$, so the condition is on $\Omega_{0\Lambda}$ vs Ω_{0m}:

$$\boxed{\frac{3}{2}\Omega_{0M}^{2/3}(2\Omega_{0\Lambda})^{1/3}<\Omega_{0m}+\Omega_{0\Lambda}-1}$$

And keep in mind that we are doing the $k>0$ case, so $\Omega_{0k}<0$ (although, the treatment has been general).

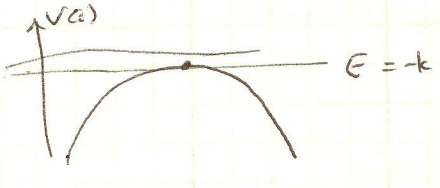
CAUTION: The solution to the inequality must be dealt with great care because of the cube root. There are two large (one positive and one negative) roots of the cubic (set the "<" to "="), and a small, positive root. Only the last is physical.

Let's put together our results in one graph: the Ω_{0M}, $\Omega_{0\Lambda}$ parameter space



— o — o —

Note that there is an unstable solution to $\dot{a}^2+V(a)=-k$ with $\dot{a}=0$ and $V(a)=-k$ at the top of the hill"



That is Einstein's static universe.

Sci Am March 2005 p36 has "misconceptions about cosmology"

Redshift and Distances (a la Carroll).

FRW has no timelike killing vector (the metric depends explicitly on t). But there is a Killing tensor. Let $U^\mu = (1, \vec{0})$, that is, $\vec{U}$ is the 4-vector tangent to isotropic observers in comoving coordinates (ie, their 4-velocity). Then let

$$K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu)$$

where $g_{\mu\nu}$ is the FRW metric with scale factor a.

Then $\nabla_{(\sigma} K_{\mu\nu)} = 0$ (see next page for check of this).

Now, take V^μ to be a tangent to a particle trajectory $V^\mu = \frac{dx^\mu}{d\lambda}$. This is the 4-velocity for a massive particle, or the wave 4-vector for a massless particle.

Along the geodesic

$$K^2 \equiv K_{\mu\nu} V^\mu V^\nu$$

is constant. Then, for a massive particle $V_\mu V^\mu = -1$

$$\frac{K^2}{a^2} = V_\mu V^\mu + (U_\mu V^\mu)^2$$

$$= -1 + (V^0)^2$$

But $V_\mu V^\mu = -1 \Rightarrow (V^0)^2 - g_{ij} V^i V^j = 1$ so

$$|\vec{V}|^2 \equiv g_{ij} V^i V^j = \frac{K^2}{a^2}$$

For massless particles $V_\mu V^\mu = 0$ and $U_\mu V^\mu = -\omega$

so $\frac{K^2}{a^2} = \omega^2$ or $\omega = \frac{K}{a}$

Check the $K_{(\mu\nu;\sigma)} = 0$

$$K_{\mu\nu;\sigma} = K_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma} K_{\lambda\nu} - \Gamma^{\lambda}_{\nu\sigma} K_{\mu\lambda}$$

Check

$$K_{00;0} = \cancel{K_{00,0}} - 2\Gamma^{\lambda}_{00} K_{\lambda 0} = 0$$

$$K_{00;i} = \cancel{K_{00,i}} - 2\Gamma^{\lambda}_{0i} K_{\lambda 0} = 0 \qquad (K_{i0} = 0 = K_{00})$$

$$K_{i0;0} = \cancel{K_{i0,0}} - \Gamma^{\lambda}_{i0} \overset{0}{\cancel{K_{\lambda 0}}} - \cancel{\Gamma^{\lambda}_{00}} K_{i\lambda}$$

$$K_{ij;0} = K_{ij,0} - \Gamma^{\lambda}_{i0} K_{\lambda j} - \Gamma^{\lambda}_{j0} K_{\lambda i}$$

Here $K_{ij} = a^2 g_{ij} = a^4 h_{ij}$

where h_{ij} is the metric on the hypersurface of constant k.

so $K_{ij,0} = 4\left(\frac{\dot a}{a}\right) K_{ij}$

Also $\Gamma^{\lambda}_{i0} K_{\lambda j} = \Gamma^{l}_{i0} K_{lj} = \frac{\dot a}{a} K_{ij}$

so $K_{ij;0} = 2\left(\frac{\dot a}{a}\right) K_{ij}$

$$K_{i0;j} = \overset{0}{\cancel{K_{i0,j}}} - \Gamma^{\lambda}_{ij} \overset{0}{\cancel{K_{\lambda 0}}} - \Gamma^{\lambda}_{0j} K_{\lambda i} = -\left(\frac{\dot a}{a}\right) K_{ij}$$

So $K_{(ij;0)} = (2-1-1)\left(\frac{\dot a}{a}\right) K_{ij} = 0$

Finally

$$K_{ij;l} = K_{ij,l} - \Gamma^{\lambda}_{il} K_{\lambda j} - \Gamma^{\lambda}_{jl} K_{\lambda i}$$

$$K_{ij,l} = a^4 h_{ij,l}$$

Recall $\Gamma^{m}_{il} = \frac{1}{2} g^{mp} (g_{ip,l} + g_{lp,i} - g_{il,p})$

$$= \frac{1}{2} h^{mn} (h_{in,l} + h_{ln,i} - g_{il,n})$$

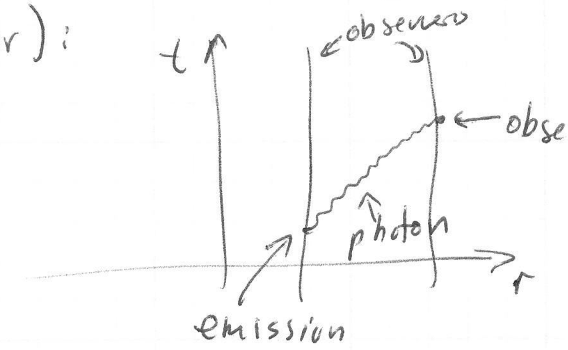
so $\Gamma^{m}_{il} K_{mj} = \frac{1}{2} a^4 h_{mj} \Gamma^{m}_{il} = \frac{1}{2} a^4 (h_{ij,l} + h_{lj,i} - h_{il,j})$

so

$$K_{ij;l} = a^4 \left[h_{ij,l} - \tfrac{1}{2}(h_{ij,l} + h_{lj,i} - h_{il,j}) - \tfrac{1}{2}(h_{ij,l} + h_{li,j} - h_{jl,i}) \right]$$

$= 0$ even before symmetrizing.

Consider two comoving observers (which have $\vec{U}$ as tangent vector):



then, since $k = \text{constant}$

$$\omega_{em} a_{em} = \omega_{obs} a_{obs}$$

or, since $\omega_{em} = \frac{1}{\lambda_{em}}$

$$\boxed{\frac{\lambda_{em}}{a_{em}} = \frac{\lambda_{obs}}{a_{obs}}}$$

That is $\lambda_{obs} = \frac{a_{obs}}{a_{em}} \lambda_{em}$

and since a is increasing $\lambda_{obs} > \lambda_{em} \Rightarrow$ redshift.

Define the redshift as

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{a_{obs}}{a_{em}} - 1$$

or

$$\boxed{\frac{a_{em}}{a_{obs}} = \frac{1}{1+z}}$$

$\Rightarrow$ Measuring z gives the factor by which the universe has grown since emission as $1+z$.

The instantaneous physical distance $d_p(t)$ between isotropic observers is the distance between them on a common t=constant surface. Recall

$$ds^2 = -dt^2 + a^2(t)\,[d\chi_2^2 + S_k^2(\chi)\,d\Omega_2^2]$$

where $S_{+1} = \sin\chi$ $S_0 = \chi$ $S_{-1} = \sinh\chi$. Then the distance between an isotropic observer at $\chi = 0$ and one at χ is

$$d_p(t) = a(t)\chi$$

Taking $\frac{\partial}{\partial t}$, we have $\dot{d}_p = \dot{a}\chi = \ddot{a}\left(\frac{d_p}{a}\right) = \left(\frac{\dot{a}}{a}\right) d_p$

So, interpreting $\dot{d}_p = V_p$ as the ~~ob~~"velocity of separation" of the isotropic observers, (really, the rate at which space is growing between them) we have

$$V_p = H d_p$$

which is Hubble's law. (if we evaluate that today we have

$$V_{p0} = H_0 d_{p0}).$$

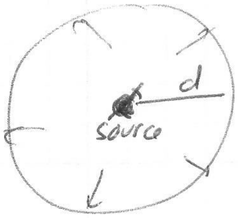
The problem at hand, though, is that H_0, which is of cosmological interest, cannot be directly determined from the above because we have no way of measuring d_{p0} or V_{p0} directly. The problem (beyond other accidental issues, like the fact that Galaxies are not necessarily isotropic observers) is that

(i) we have no ruler to measure d_{p0}, we have to infer it from other observations, like luminosity (see below)

(ii) we cannot observe V_{p0}, the velocity today of an observer far away, because light was emitted in the past. This is a small effect if the time T of light travel is much smaller than H_0^{-1}.

In flat space, the luminosity L (defined as energy/time emitted) of a source, and the flux F (defined as energy/time×area } received) are related by

$$L = 4\pi d^2 F$$



So we define a luminosity distance, d_L by

$$d_L^2 = \frac{L}{4\pi F}$$

This is useful if we can identify objects in the sky as "standard candles", ie, objects that have the same intrinsic luminosity. Then measuring the flux at Earth we can directly infer the relative distance, d_L, to Earth.

In a FRW background, ~~ph~~ photons from a source (at $\chi=0$) get redshifted by $(1+z)$. Moreover, since we are looking at energy/time ~~rece~~ emitted vs received, ~~if the~~ energy emitted over an ~~ph~~ time interval δt ~~a~~ is received over a time interval $(1+z)\delta t$. So $\frac{F}{L} = \frac{1}{(1+z)^2 A}$

where A is the area of a sphere centered at $\chi=0$ with comoving radius χ. Now, from ds^2 we have

$$A = 4\pi a_0^2 S_k^2(\chi)$$

So

$$d_L = \sqrt{\frac{L}{4\pi F}} = (1+z)\, a_0\, S_k(\chi)$$

(Note: check on δt argument. Emit two photons at $t=0$ and $t=\delta t$. They follow null ~~traject~~ geodesics from $\chi=0$ to χ:

$$ds^2=0=-dt^2+a^2d\chi^2$$

Or

$$\frac{d\chi}{dt}=a^{-1}$$

$$\Rightarrow \chi=\int_0^t a^{-1}(t')dt' = \int_{\delta t}^{t+\delta T} a^{-1}(t')dt'$$

and we want δT. But then, from the equality

$$\int_0^{\delta t} a(t')^{-1}dt' = \int_t^{t+\delta T} a(t')^{-1}dt'$$

and if δt is infinitesimal

$$a(0)^{-1}\delta t = a(t)^{-1}\delta T \quad \text{or} \quad \delta T = \left(\frac{a(0)}{a(t)}\right)^{-1}\delta t = \left(\frac{a_{em}}{a_{obs}}\right)^{-1}\delta t\Big).$$

Now, the expression for d_L is not very useful since it depends on χ explicitly, not an observable. However, as in the note above,

$$\chi=\int_0^t a^{-1}(t')dt' = \int \frac{dt'}{da}\frac{da}{a} = \int_{a_{em}}^{a_{obs}}\frac{da}{a\dot a}$$

Now using $\dfrac{a_{em}}{a_{obs}}=\dfrac{a}{a_0}=\dfrac{1}{1+z}$, where we have $a_{obs}=a_0$ (today) and $a_{em}=a$, the scale factor at emission corresponding to redshift z, we can change variables from a to z. Using $\dot a = Ha$, we have

$$\chi = \int_0^z \left[\frac{a_0}{(1+z)^2}dz'\right]\left[\frac{1}{a^2H}\right] = \frac{1}{a_0}\int_0^z \frac{dz'}{H(z')}$$

Note added: At this point a solution of Friedmann equations gives $a(t)$, the integral can be done if we invert $t=t(a)$, and then express the result in terms of the redshift. We instead write the integral as an integral over z:

To perform the integral we need a solution to Friedmann equations, which give $H(z)$. Of course,

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

and we know $\rho_i = \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} = \rho_{0i}(1+z)^{3(1+w_i)}$

Moreover, recall that evaluating this today and dividing by H_0^2 we get

$$1 = \sum_i \Omega_{0i}$$

So

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\sum_i \rho_{0i}(1+z)^{3(1+w_i)} = \sum_i \Omega_{0i}(1+z)^{3(1+w_i)}$$

Let $E(z) = H(z)/H_0$. Then

$$\chi = \frac{1}{a_0 H_0}\int_0^z \frac{dz'}{E(z')} \quad \text{with} \quad E(z) = \sqrt{\sum_i \Omega_{0i}(1+z)^{3(1+w_i)}}$$

and this can be plugged into $d_L = (1+z)a_0 S_k(\chi)$ to get d_L in terms of z, a_0 and H_0. But the integration has to be done numerically

Note that now we need a_0 in addition to H_0 and z. But if we know Ω_{0k} we can get a_0 (since $\rho_{0k} = -\frac{3}{8\pi G}\frac{k}{a_0^2}$) except for the case $k=0$. However, for $k=0$ $S_k(\chi) = \chi^2$ and a_0 drops out of d_L. For $k \neq 0$ we can use $\Omega_{0k} = 1-\Omega_0$ to infer Ω_{k0} and use it above. So, ~~since using~~ recalling that

$$\cancel{a_0}\ \Omega_{0k} = \frac{8\pi G}{3H_0^2}\rho_{0k} = -\frac{k}{H_0^2 a_0^2} \Rightarrow$$

then

$$a_0^2 = -\frac{k}{\Omega_{0k}H_0^2} \quad \text{or} \quad a_0 = \frac{1}{H_0\sqrt{|\Omega_{0k}|}} = \frac{1}{H_0\sqrt{|1-\Omega_0|}}$$

(provided $k \neq 0$).

So, finally

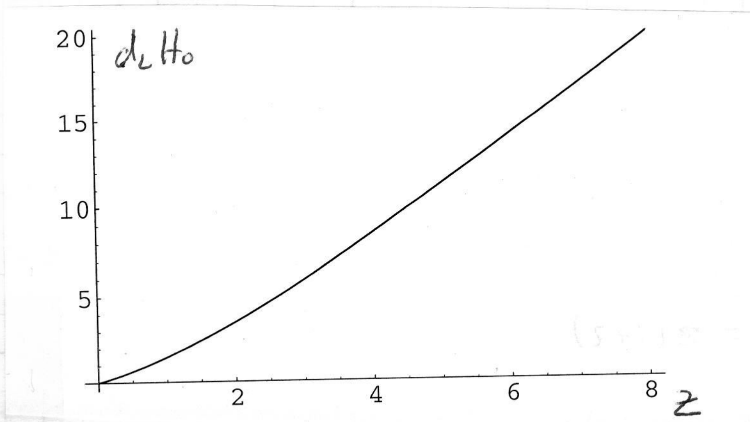
$$\boxed{d_L = \frac{(1+z)}{H_0\sqrt{|1-\Omega_0|}}\, S_k\left[\sqrt{|1-\Omega_0|}\int_0^z \frac{dz'}{E(z')}\right]}$$

Exercise: do the integral $\int_0^z \frac{dz'}{E(z')}$ (numerically?) → Ellyptic integral... need numerics to plot anyway.

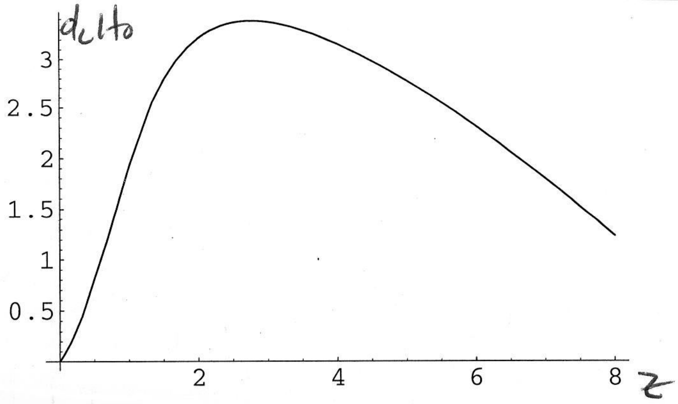
for the case that we have only Λ and matter (and the three cases $k=0,\pm 1$).

$$E^2(z) = \Omega_m (1+z)^3 + \Omega_\Lambda + \Omega_k (1+z)^2$$

where $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$.



$\Omega_m = 0.3$
$\Omega_\Lambda = 0.5$
$\Omega_k = 0.2$ $k=-1$



$\Omega_m = 0.3$
$\Omega_\Lambda = 1.5$
$\Omega_k = -0.8$ $k=+1$

Note the maximum from $S_k[\chi] = \sin\chi$ (eventually has a zero).

There are other measures of distance:

(i) Proper motion distance, d_M.

In flat space

~~d_M~~ $\delta r_\perp = d_M \delta\theta$ so, divide by δt

So define: $$d_M = \frac{\dot{r}_\perp}{\dot{\theta}}$$

ii) Angular diameter distance, d_A:

In flat space, 

, so $d_A = \frac{D}{\theta}$.

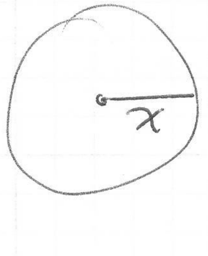
Exercise: Show $d_A = (1+z)^{-2} d_L$ and $d_M = (1+z)^{-1} d_L$.

Ans: For d_A let the observer be at $\chi = 0$ and the light emitted from χ, with θ ranging from 0 to θ.

Null lines still have $\dot{\chi} = \frac{1}{a}$. But doing this way is problematic since comparing the tangent vectors at the observer (the origin) is bad (coordinate singularity).

Avoiding coordinate singularity is messy.

Easier: (observer at $\chi=0$).



with $\theta = 2\pi$ in $d_A = \frac{D}{\theta}$. But now by geometry, at emission:

$$D = 2\pi a_{em} S_k(\chi)$$

so $$d_A = \frac{2\pi a_{em} S_k(\chi)}{2\pi} = a_{em} S_k(\chi) = \frac{1}{1+z} a_0 S_k(\chi) = \frac{d_L}{(1+z)^2}$$

Similarly $$d_M = \frac{\delta r_\perp / \delta t_{emiss}}{\delta\theta / \delta t_{obs}} = \frac{\delta r_\perp}{\delta\theta} \cdot \frac{\delta t_{obs}}{\delta t_{em}}$$

But $\frac{\delta r_\perp}{\delta\theta} = d_A = \frac{1}{(1+z)} a_0 S_k(\chi)$ and $\frac{\delta t_{obs}}{\delta t_{em}} = (1+z) \Rightarrow d_M = a_0 S_k(\chi) = \frac{d_L}{1+z}$

Lookback Time:

If today's time is t_0 and the time when a photon was emitted by a comoving observer (or at an event coinciding with a comoving observer) with coordinate x is t_{em}, then

$$\Delta t = t_0 - t_{em} = \int_{t_{em}}^{t_0} dt = \int_{a_{em}}^{a_0} \frac{da}{\dot{a}} = \int_{a_{em}}^{a_0} \frac{da}{aH}$$

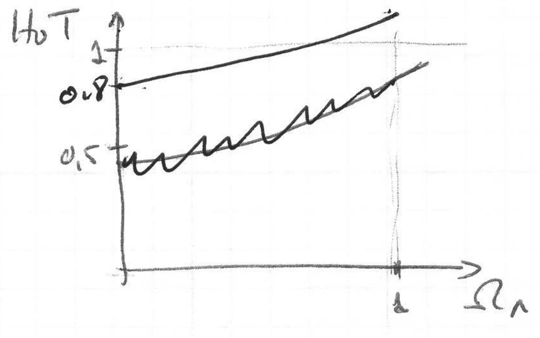
Using $H = H_0 E(z)$ and $a = \frac{a_0}{1+z}$ we have

$$\boxed{\Delta t = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')E(z')}} \quad \text{Lookback time}$$

The integral is dimensionless, the units are set by $H_0^{-1} \sim 10^{10}$ yrs. In particular, as $z \to \infty$ the integral goes to a fixed finite number (that depends on the details of $E(z')$), of order 1. So we are tempted to say

$$T = \text{age of universe} = \frac{1}{H_0} \int_0^\infty \frac{dz'}{(1+z')E(z')} \approx \frac{1}{H_0}$$

In fact, I get (from mathematica) that for $\Omega_m = 0.3$ $\Omega_{radiation} = 0$ ~~$\Omega_{rad} = 0$~~



So, for fixed Ω_m, T increases with Ω_Λ (albeit slowly).

This is not the whole story because there is also radiation! But adding $\Omega_{rad} = 10^{-3} \Omega_m$ changes the result a negligible amount.

Black Holes

Start with Schwarzschild:
As seen briefly in 1st quarter

$$\boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2}$$

is a solution of Einstein's equations in empty space

$$R_{\mu\nu} = 0$$

~~that~~ It has spherical symmetry and it is static.

Birkhoff's theorem asserts that the Schwarzschild metric is the unique static, spherically symmetric solution of Einstein's equations in empty space.

We won't go over the proof here. But the ingredients are
(1) Define spherical symmetry as having corresponding isometries: there are three killing vectors that generate the symmetries of the sphere. These are the generators of the Lie Algebra of the group of rotations, SO(3), that leave the sphere (S^2) invariant.
You are familiar with the algebra, it is just the same as angular momentum in Q.M.:

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k$$

or, since we use anti-hermitean generators, let $X_i = iL_i \Rightarrow$

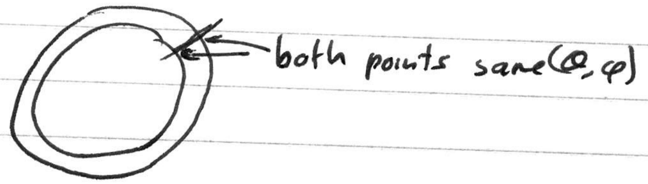
$$[X_i, X_j] = \epsilon_{ijk} X_k$$

Exercise: transforming to spherical coordinates $X_1 = y\partial_z - z\partial_y$, $X_2 = z\partial_x - x\partial_z$
$X_3 = x\partial_y - y\partial_x$ show that these killing vectors are

$$R = \partial_\phi \qquad S = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \qquad T = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi$$

(ii) Frobenius theorem then allows one to show the space is folliated by 2-spheres. Basically the theorem says that if you have a set of vector fields that closes under commutation, $[X_i, X_j] =$ lin. combination of X_i's, then the integral curves form a submanifold of the manifold on which they are defined.

(iii) Put spherical coordinates θ, ϕ on one sphere. Extend do other neighboring spheres using orthogonal geodesics



and characterize the other spheres by two coordinates, say p & q, (the space of orthogonal geodesics through one point on a sphere is $4-2=2$ dimensional). Then ~~by suitable~~ one has

$$ds^2 = g_{pp}(p,q)\,dp^2 + 2g_{pq}(p,q)\,dp\,dq + g_{qq}(p,q)\,dq^2 + r^2(p,q)\,d\Omega_2^2$$

and by changing variables one can write

$$ds^2 = T(t,r)\,dt^2 + R(t,r)\,dr^2 + r^2 d\Omega_2^2$$

(iv) Plug this into Einstein's equations and solve. Impose the condition that the metric is static. This too has to be defined with some care. A metric is stationary if it has a timelike Killing vector near infinity, and a stationary metric is static if in addition the timelike Killing vector is orthogonal to a family of hypersurfaces.

Singularities in Schwarzschild.

It is difficult to define in general what is meant by a singularity. One common means of determining whether there is a singularity is to look for infinities in geometric quantities (coordinate independent), such as R, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$, etc.

In the case at hand the metric

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

is singular at $r=2GM$ and at $r=0$. But are these real singularities or artifacts of the metric.

In this case $R=0$ and $R_{\mu\nu}=0$. But $R_{\mu\nu\lambda\sigma}\neq 0$ and computing explicitly one finds

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = \frac{48G^2M^2}{r^6}$$

$\Rightarrow$ there is no singularity at $r=2GM$ as far as this invariant can show, but there certainly is one at $r=0$.

In fact we will introduce coordinates that have a perfectly regular metric at $r=2GM$.

Another way of defining singularities is by finding inextendible geodesics that terminate at finite affine parameter. Let's study geodesics.

Geodesics

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\Gamma^{\mu}{}_{\nu\lambda} = g^{\mu\rho}\,\Gamma_{\rho\nu\lambda} \qquad \Gamma_{\rho\nu\lambda} = \frac{1}{2}\left(g_{\rho\nu,\lambda} + g_{\lambda\rho,\nu} - g_{\nu\lambda,\rho}\right)$$

$$\Gamma_{rtt} = -\frac{1}{2}g_{tt,r} = \frac{GM}{r^2}$$

$$\Gamma_{ttr} = \Gamma_{trt} = -\frac{GM}{r^2}$$

$$\Gamma_{rrr} = \frac{1}{2}g_{rr,r} = -\left(1-\frac{2GM}{r}\right)^{-2}\frac{GM}{r^2}$$

$$\Gamma_{r\theta\theta} = -\frac{1}{2}g_{\theta\theta,r} = -r$$

$$\Gamma_{r\phi\phi} = -r\sin^2\theta$$

$$\Gamma_{\theta r\theta} = \Gamma_{\theta\theta r} = r$$

$$\Gamma_{\phi r\phi} = \Gamma_{\phi\phi r} = r\sin^2\theta$$

$$\Gamma_{\theta\phi\phi} = -\frac{1}{2}g_{\phi\phi,\theta} = -r^2\sin\theta\cos\theta$$

$$\Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = r^2\sin\theta\cos\theta$$

$$\Gamma^{r}{}_{tt} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{t}{}_{rt} = \Gamma^{t}{}_{tr} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right]^{-1}$$

$$\Gamma^{r}{}_{rr} = -\left(1-\frac{2GM}{r}\right)^{-1}\frac{GM}{r^2}$$

$$\Gamma^{r}{}_{\theta\theta} = -r\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

Geodesic Eqn!

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 - r\left(1-\frac{2GM}{r}\right)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\,\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

To solve these, use constants of the motion (1^{st} integrals). We have four Killing vectors, three from SO(3) symmetry and one timelike Killing vector. For each

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

along the geodesic. Moreover, for massive particles we can take $\lambda = \tau$ so that

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu} = -1 \qquad \text{timelike geodesic}$$

and for massless particles

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} g_{\mu\nu} = 0 \qquad \text{null geodesic.}$$

The Killing vectors associated with SO(3) are like angular momentum, $\vec{L}$. Just as in flat space, $\vec{L}$ = constant implies motion in a plane orthogonal to $\vec{L}$ and with fixed magnitude of $\vec{L}$. So we can fix the plane of motion choosing

$$\theta = \frac{\pi}{2}.$$

The magnitude of L corresponds to the Killing vector ∂_ϕ

$$(\partial_\phi)^\mu = (0,0,0,1)$$

In addition, the timelike Killing vector is

$$(\partial_t)^\mu = (1,0,0,0)$$

The conserved quantities are

$$E = -g_{\mu\nu}(\partial_t)^\mu \frac{dx^\nu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}$$

and

$$L = (\partial_\phi)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} = r^2 \sin^2\theta \frac{d\phi}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \qquad (\text{since } \theta = \tfrac{\pi}{2}).$$

The constants are named E & L, suggestively. But these are just labels. We can discuss energy and angular momentum later.

For timelike geodesics we have ($U^\mu U_\mu = -1$):

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

or multiplying by $\left(1-\frac{2GM}{r}\right)$ and using E & L

$$-E^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(1+\frac{L^2}{r^2}\right) = 0$$

This is like a particle in a central potential

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \mathcal{E}$$

with $V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(1+\frac{L^2}{r^2}\right)$ $\qquad \mathcal{E} = \frac{1}{2}E^2$ $\qquad$ (I)

The null geodesic is similar, but the LHS -1 is replaced by 0:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_n(r) = \mathcal{E}$$

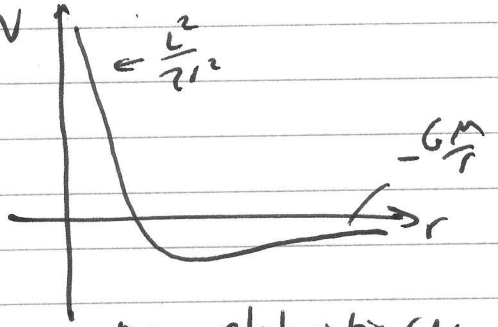
$$V_n(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\frac{L^2}{r^2} \qquad \mathcal{E} = \frac{1}{2}E^2 \qquad \text{(II)}$$

(or, together

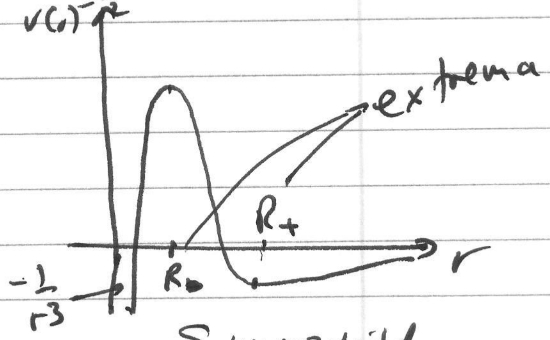
$$V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\kappa + \frac{L^2}{r^2}\right) \qquad \kappa = \begin{cases} 0 & \text{null-like} \\ 1 & \text{time-like} \end{cases}$$

Expanding (I):

$$V(r) = \underset{\substack{\uparrow \\ \text{irrelevant}}}{\frac{1}{2}} - \underset{\substack{\uparrow \\ \text{Newtonian}}}{\frac{GM}{r}} + \underset{\substack{\uparrow \\ \text{std. } L}}{\frac{L^2}{2r^2}} - \underset{\substack{\uparrow \\ \underline{\underline{\text{new}}}}}{\frac{GML^2}{r^3}}$$



non-relativistic (Newtonian)

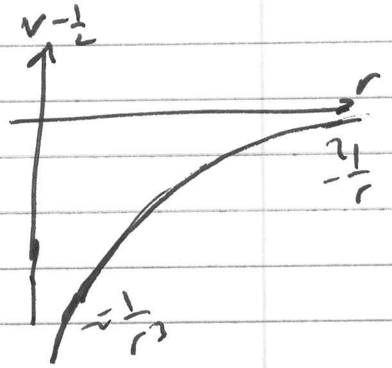


Schwarzschild

find extrema $\frac{\partial V}{\partial r} = 0 = GMr^2 - L^2 r + 3GML^2$

or $$R_\pm = \frac{1}{2GM}\left[L^2 \pm \sqrt{L^4 - 12(GML)^2}\right]$$

For $12G^2M^2 > L^2$ → no extrema of V



For $L^2 > 12G^2M^2$ R_+ a minimum, R_- a maximum as in figure.
⇒ stable circular orbit at $r = R_+$ (unstable at $r = R_-$).
Now ~~R_+~~ for $L^2 \gg 12(GM)^2$ $R_+ \simeq \frac{L^2}{GM}$ the Newtonian formula.

Now, as we vary L^2, R_+ is smallest at $L^2 = 12(GM)^2$, where

$$R_+ = \frac{1}{2GM} 12(GM)^2 = 6GM \quad \text{so}$$

$$R_+ > 6GM$$

Here is a smallest stable circular orbit!

(Similarly unstable circular orbits radii are restricted to by $R_- < 6GM$ – same calculation – and on low end take $L \to \infty$

$$R_- \to \frac{1}{2GM}\left[L^2 - L^2\left[1 - \frac{1}{2}\frac{12(GM)^2}{L^2} + \cdots\right]\right\} = 3GM$$

so $$3GM < R_- < 6GM).$$

(~~Note that this calculation also gives~~

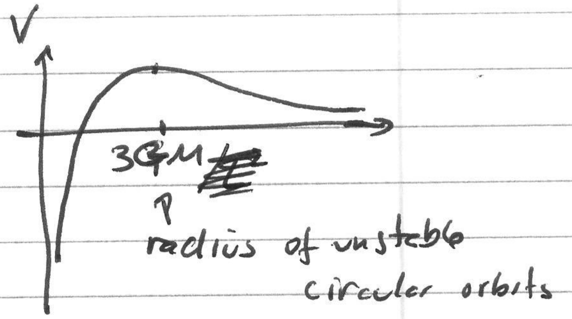
~~$L \to \infty \Rightarrow R_+ \to \frac{L^2}{GM}$~~

(Note: at this point comparison of ω_r (from small perturbations about circular orbit, with $\omega_r^2 = \frac{d^2V}{dr^2}$) and $\omega_\phi = \dot\phi$ gives precession of perihelion → mercury → classical test. This must have been covered?).
(No time here).

Null geodesics:

$$V_n(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$\left(\frac{\partial V}{\partial r} = 0 = L^2 r - 3GML^2 \right)$$



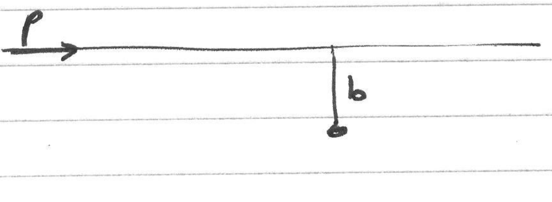
Now recall E & L are (in arbitrary units) the energy and angular momentum of the particle (photon?), ~~and~~ the energy necessary for the particle to go over the potential barrier is the height of the barrier:

$$\frac{1}{2}E^2 = V_n(3GM) = \frac{L^2}{27(GM)^3}\left(\tfrac{1}{2}3GM - GM\right) = \frac{L^2}{54G^2M^2}$$

$$\Rightarrow \quad \frac{L}{E} = 3\sqrt{3}\,GM$$

But L/E has a simple interpretation. In the asymptotically flat region $(r \gg GM)$ it corresponds to the impact parameter

$$b = \frac{L}{E}$$



$L = bp$
and $E = p$ (massless)

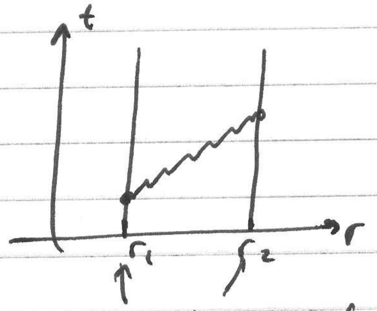
For $b < 3\sqrt{3}\,GM$ the photon is captured
For $b > 3\sqrt{3}\,GM$ it is scattered

Capture cross section

$$\sigma_c = \pi b_{crit}^2 = 27\pi(GM)^2$$

Red-Shift

Similar to what we did before:



two observers (not on geodesics)

both with $U^\mu = (U^0, 0, 0, 0)$

(Note, all we need is that U be of this form instantaneously)

so $$U\cdot U = -1 \Rightarrow U^0 = 1/\sqrt{-g_{tt}} = \frac{1}{\sqrt{1-\frac{2GM}{r}}}$$

Now, if $k^\mu = \frac{dx^\mu}{d\lambda}$ on the null geodesic for the photon then the observers measure $\omega = -U\cdot k = +\sqrt{1-\frac{2GM}{r}}\,\frac{dt}{d\lambda}$

and $\frac{dt}{d\lambda} = E/(1-\frac{2GM}{r}) \Rightarrow \omega = \frac{E}{\sqrt{1-\frac{2GM}{r}}}$

Since E is constant we have

$$\omega_1\sqrt{1-\frac{2GM}{r_1}} = \omega_2\sqrt{1-\frac{2GM}{r_2}}$$

or $$\boxed{\frac{\omega_2}{\omega_1} = \sqrt{\frac{1-2GM/r_1}{1-2GM/r_2}}}$$ Gravitational redshift.

For weak fields,

$$\frac{\omega_2}{\omega_1} \simeq 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \Phi_1 - \Phi_2 = 1 - \Delta\Phi$$

which is the formula obtained last quarter (see Schwartz) on general grounds (principle of equivalence) for weak fields.

Kruskal Coordinates and extension

Coordinate singularities vs real singularities: toy models first (warm-up):

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2$$

defined for $x \in (-\infty, \infty)$ and $t \in (0, \infty)$. Seems singular but defining $t' = \frac{1}{t}$, we have

$$ds^2 = -dt'^2 + dx^2$$

$\Rightarrow$ original spacetime is a portion of Minkowski space with $t' > 0$. Note that the original spacetime is not geodesically complete: geodesics approaching $t \to \infty$ take finite affine parameter to get there (even though approaching $t = 0$ take infinite affine parameter)

(Check:

$$\Gamma_{ttt} = \tfrac{1}{2} g_{tt,t} = 2t^{-5} \qquad \Gamma^t_{tt} = -\frac{2}{t}$$

$$\frac{d^2t}{d\lambda^2} + \left(-\frac{2}{t}\right)\left(\frac{dt}{d\lambda}\right)^2 = 0 \qquad \frac{d^2x}{d\lambda^2} = 0 \Rightarrow \frac{dx}{d\tau} = v = \text{const}$$

Let ~~$v = \frac{dt}{d\lambda} \Rightarrow \frac{\dot{v}}{v^2} = \frac{2}{t} \quad \int \frac{dv}{v^2} = 2\int \frac{dt}{t} = -\frac{1}{v} = 2\ln t$~~

~~$v = \frac{dt}{d\lambda} = \frac{1}{-2\ln t} \qquad \int dt \ln t = -\frac{1}{2}\int d\lambda$~~

~~$t(\ln t - 1) = -\frac{1}{2}\lambda$~~

But $U \cdot U = -1 \Rightarrow -\frac{1}{t^4}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -1$

$$\frac{dt}{d\tau} = t^2\sqrt{1+v^2} \qquad \int \frac{dt}{t^2} = \sqrt{1+v^2}\int d\tau \Rightarrow \frac{1}{t_0} - \frac{1}{t} = \sqrt{1+v^2}\,\tau$$

So $t \to \infty$ as $\tau \to 1/t_0\sqrt{1+v^2}$. However $t \to 0$ as $\tau \to \infty$).

2nd example: Rindler spacetime

$$ds^2 = -x^2 dt^2 + dx^2$$

$t \in (-\infty, \infty)$ $x \in (0, \infty)$

Singularity at $x = 0$?

Geodesics? $\begin{pmatrix} \Gamma_{xtt} = -\frac{1}{2} g_{tt,x} = x & \Gamma^x_{tt} = x \\ \Gamma_{txt} = \Gamma_{ttx} = \frac{1}{2} g_{tt,x} = -x & \Gamma^t_{tx} = \Gamma^t_{xt} = \frac{1}{x} \end{pmatrix}$

$p_t = \text{cost} \Rightarrow g_{tt} \frac{dt}{d\tau} = E = \text{cost.}$ $\frac{dt}{d\tau} = -\frac{E}{x^2}$

$$-1 = -x^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -\frac{v^2}{x^2} + \left(\frac{dx}{d\tau}\right)^2$$

$$\frac{dx}{d\tau} = \sqrt{\frac{E^2}{x^2} - 1} \qquad \int \frac{x\,dx}{\sqrt{E^2 - x^2}} = \int d\tau$$

let $E^2 \varsigma = x^2$ $E^2 d\varsigma = 2x\,dx$ $\tau = \frac{E}{2} \int \frac{d\varsigma}{\sqrt{1-\varsigma}} = E\sqrt{1-\varsigma} = \frac{E}{V^3}\sqrt{1 - V^2 x^2}$
$= \sqrt{E^2 - x^2}$

so $\tau = \sqrt{E^2 - x^2}$ $x^2 = E^2 - \tau^2$ $\frac{dt}{d\tau} = -\frac{E}{E^2 - \tau^2}$

$$t = E \int \frac{d\tau}{-\tau^2 + V^2} \qquad ~~\tau = V\sin\theta \quad \frac{V}{V^2}\int \frac{V\cos\theta\, d\theta}{\cos^2\theta}~~ = \int \frac{d\theta}{\cos\theta} =$$

$\frac{1}{E - \tau} + \frac{1}{E + \tau} = \frac{2E}{E^2 - \tau^2}$ $\rightarrow t = \frac{1}{2} \ln \frac{E - \tau}{E + \tau}$ $t \to \infty$ is finite τ ($\tau \to E$)

Geodesically incomplete. How about curvature?

$$R^t{}_{xtx} = \partial_t \Gamma^t_{xx} - \partial_x \Gamma^t_{tx} + \Gamma^t_{t\lambda} \Gamma^\lambda_{xx} - \Gamma^t_{x\lambda} \Gamma^\lambda_{tx}$$
$$= 0 - \partial_x \frac{1}{x} + 0 - \frac{1}{x^2} = 0$$

so this is a portion of Minkowski space, again.

Q: How to find coordinates that are non-singular starting from this, not using the fact that this is Minkowski?

Use a family of geodesics that head towards the singularity, with affine parameter as one coordinate. Must avoid crossing of geodesics because this would give new coordinate singularities. In 2-DIM we can take null ingoing and outgoing geodesics (they never cross because if null geodesics have same tangent they agree everywhere) null geodesics:

$$0 = -x^2 \left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2$$

$$\Rightarrow \qquad x\frac{dt}{d\lambda} = \pm\frac{dx}{d\lambda}$$

$$\Rightarrow \qquad \pm\frac{dx}{x} = dt$$

$$\Rightarrow \qquad t = \pm \ln x + \text{constant}$$

Define
$$u = t - \ln x \qquad\qquad t = \tfrac{1}{2}(u+v)$$
$$\Leftrightarrow$$
$$v = t + \ln x \qquad\qquad x = e^{\frac{1}{2}(v-u)}$$

So geodesics are $u = \text{constant}$ or $v = \text{constant}$. Then

$$ds^2 = -x^2dt^2 + dx^2 = -e^{(v-u)}\frac{1}{4}(du+dv)^2 + e^{(v-u)}\frac{1}{4}(dv-du)^2$$

or
$$ds^2 = -e^{v-u}\,du\,dv$$

We want to analyze the singularity at $x=0$. Can't do that yet since u, v in $(-\infty,\infty)$ still has $x>0$. But now we can extend the space beyond $x=0$, i.e., beyond u, v infinite by ~~rep~~ introducing new coordinates $U(u)$ and $V(v)$. Calculate affine parameter along null geodesics. Since

$$g_{tt}\frac{dt}{d\lambda} = -E = \text{constant} \qquad \Rightarrow \qquad \frac{dt}{d\lambda} = \frac{E}{x^2}$$

so, along $u = \text{constant}$ we have $\frac{dt}{d\lambda} = \frac{1}{2}\frac{dv}{d\lambda} = E e^{-(v-u)}$

or
$$\lambda = \frac{1}{2E}\int e^{v-u}\,dv = C + \frac{e^{v-u}}{2E} \qquad (u = \text{constant})$$

Along outgoing null geodesics $\lambda_{out} = e^{v}$ is an affine parameter while $\lambda_{in} = -e^{-u}$

So use $U = -e^{-u}$ $\quad V = e^{v}$ $\qquad ds^2 = -dU\,dV$

Now $ds^2 = -dU dV$ has $U < 0$ $V > 0$

but there is no obstruction to extending these to $(-\infty, \infty)$. and we get Minkowski space after

$$T = \frac{1}{2}(U+V) \qquad \Leftrightarrow \qquad U = T - X$$

$$X = \frac{1}{2}(V-U) \qquad\qquad V = T + X$$

$$ds^2 = -dT^2 + dX^2$$

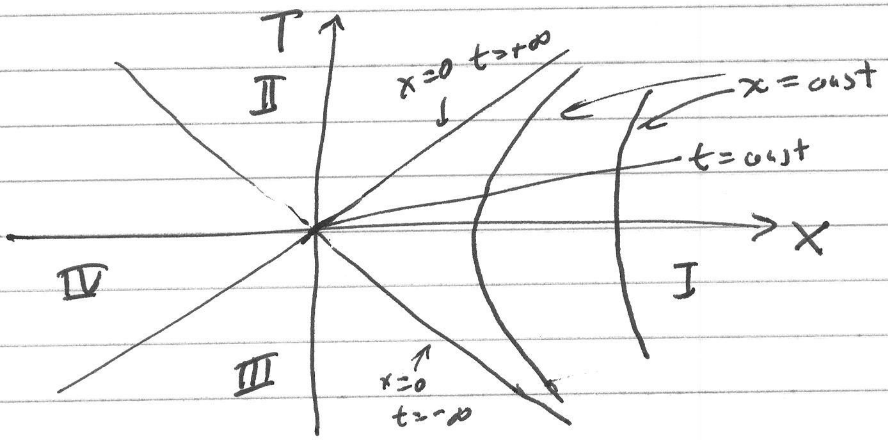
The original coordinates are given, in terms of these, by

$$t = \frac{1}{2}(u+v) = \frac{1}{2}(-\ln(-U) + \ln V)$$

$$= \frac{1}{2}(-\ln(X-T) + \ln(X+T))$$

$$= \frac{1}{2}\ln\frac{X+T}{X-T} = \tanh^{-1}\left(\frac{T}{X}\right)$$

$$x = e^{\frac{1}{2}(v-u)} = \sqrt{-VU} = \sqrt{X^2-T^2}$$



original space is the wedge I in Minkowski space time. $(X > |T|)$.

Now do the same for Schwarzschild. We can ignore angular coordinates for most of the discussion. Consider

$$ds^2 = -(1-2GM/r)dt^2 + (1-\frac{2GM}{r})^{-1}dr^2$$

Null geodesics:

$$-(1-\frac{2GM}{r})\left(\frac{dt}{d\lambda}\right)^2 + (1-\frac{2GM}{r})^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

$$\Rightarrow \quad \left(\frac{dt}{dr}\right)^2 = (1-\frac{2GM}{r})^{-2}$$

$$t = \pm r_* + \text{constant}$$

r_* is the "Regge-Wheeler tortoise coordinate" given by

$$r_* = \int \frac{dr}{1-\frac{2GM}{r}} = \int dr\left[\frac{r-2GM+2GM}{r-2GM}\right] = r+2GM\ln(\frac{r}{2GM}-1)$$

Then define null coordinates

$$u = t - r_*$$

$$v = t + r_*$$

Calculate metric: $\cancel{du}$ $du = dt - dr_* = dt - (1-\frac{2GM}{r})^{-1}dr$

$$dv = dt + (1-\frac{2GM}{r})^{-1}dr$$

$$du\,dv = dt^2 - (1-\frac{2GM}{r})^{-2}dr^2$$

or

$$ds^2 = -(1-\frac{2GM}{r})\,du\,dv$$

with r understood as $r = r(u,v)$.

with $v-u = 2r_* = 2r + 4GM\ln(\frac{r}{2GM}-1)$

we have

$$e^{\frac{v-u}{4GM}} = e^{\frac{r}{2GM}}(\frac{r}{2GM}-1) = e^{\frac{r}{2GM}}(1-\frac{2GM}{r})\frac{r}{2GM}$$

so

$$ds^2 = -\frac{2GM}{r}e^{-\frac{r}{2GM}}e^{\frac{v-u}{4GM}}\,du\,dv$$

This is useful because the factor $\frac{2GM}{r} e^{-\frac{r}{2GM}}$ is not singular as $r \to 2GM$.

Now, as in Rindler case, we introduce

$$U = -e^{-u/4GM}$$

$$V = e^{v/4GM}$$

$$\Rightarrow \cancel{ds^2} = dU = \frac{1}{4GM} e^{-u/4GM} du \quad dV = \frac{1}{4GM} e^{v/4GM} dv$$

and

$$ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r} dU\, dV$$

While this is defined for $V > 0$ and $U < 0$, we can now extend to $(-\infty, \infty)$ and define as before

$$T = \tfrac{1}{2}(U + V)$$

$$X = \tfrac{1}{2}(V - U)$$

The full metric is now

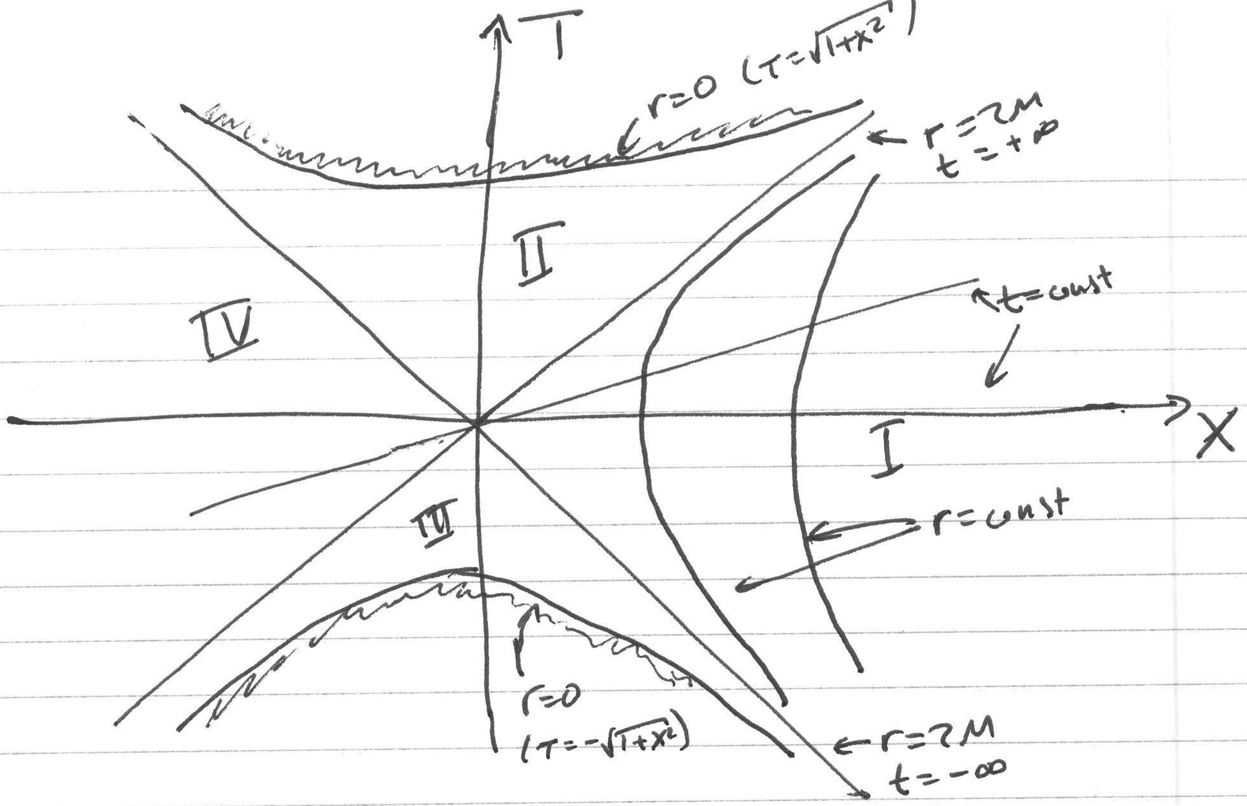
$$\boxed{ds^2 = \frac{32(GM)^3 e^{-r/2GM}}{r} (-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

Relating to original coordinates:

$$X^2 - T^2 = -UV = e^{\frac{v-u}{4GM}} = e^{\frac{2r_*}{4GM}} = e^{\frac{r}{2GM}} \left(\frac{r}{2GM} - 1\right) \quad (*)$$

$$\tanh^{-1}\frac{T}{X} = \frac{1}{2}\ln\left(\frac{T+X}{X-T}\right) = \frac{1}{2}\ln\frac{V}{-U} = \frac{1}{2}\ln e^{\frac{v+u}{4GM}} = \frac{t}{4GM}$$

which would have been hard to guess. Eq $(*)$ also gives $r = r(T, X)$ for the metric. Note that $r > 0$ in $(*)$ gives the allowed range for X, T: $X^2 - T^2 > -1$.



Keep in mind each point is a S^2 with radius r.

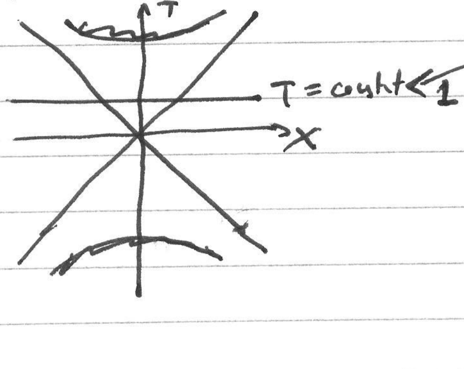
Causal structure: null geodesics are 45° lines.

- Singularities at $r=0$ are spacelike. Two of them
  - Future of region II
  - Past of region III

NOT a timelike line at origin, as suggested by original coordinates.

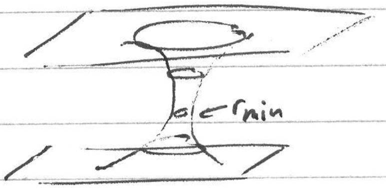
- Region I corresponds to original $r>2M$, exterior gravitational field of ~~body~~ a spherical body. Radially infalling observer that crosses $X=T$ can never escape back to region I AND will eventually hit singularity ergo "black hole"
- Region III has the time reversed properties of I ⇒ "white hole".
- Region IV has identical properties to I, asymptotically flat.

To see what's going on, consider hypersurfaces of T=constant, restore one angular variable (θ):



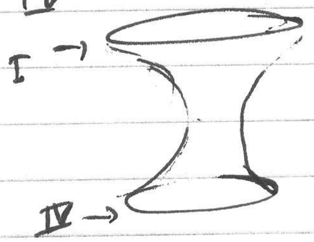
$$e^{\frac{r}{2GM}}\left(\frac{r}{2GM}-1\right)=X^2-T^2$$

as X goes from $-\infty$ to $+\infty$ r goes from ∞ to a minimum and back to ∞.



for $T=1$
$r_{min}=0$

There is another space, on the other side of the blackhole. Can we communicate with our brothers there? No, as is clear from causality diagram. What happens in this picture is that as an observer wants to go from I to IV



the radius of the throat is shrinking and it necessarily pinches off before the observer makes it to the other side.

Penrose Diagram

Recall
$$ds^2 = -\frac{32(GM)^3}{r}e^{-\frac{r}{2GM}}\,dU\,dV + r^2 d\Omega_2^2$$

Now let
$$\hat{U} = \arctan\left(\frac{U}{\cancel{\sqrt{2GM}}}\right) \qquad \hat{V} = \arctan\left(\frac{V}{\cancel{\sqrt{2GM}}}\right)$$

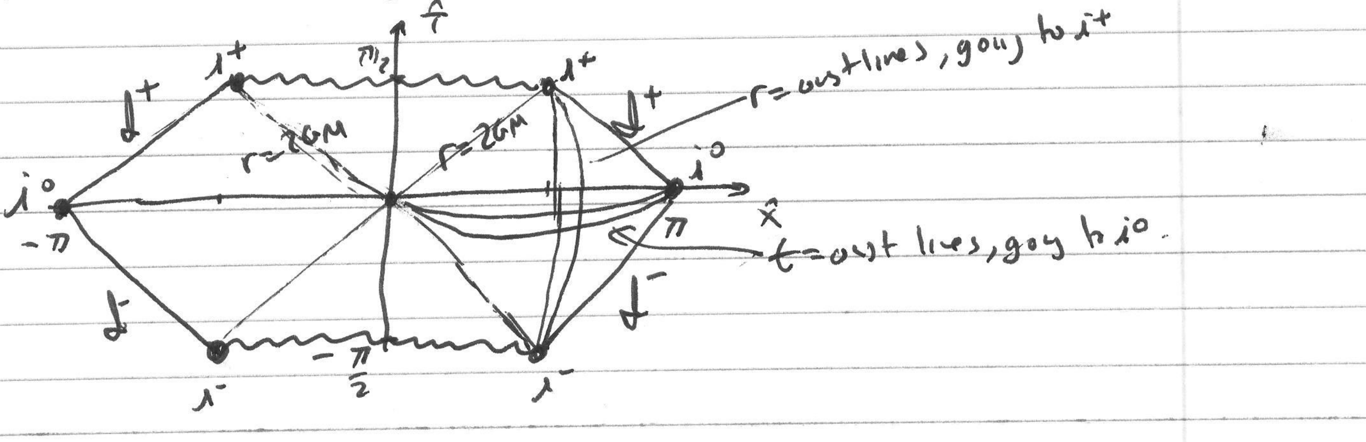
So $\hat{U} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\hat{V} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(It is quite irrelevant what the metric is in $\hat{U}, \hat{V}$, just matters that it has finite range). Also, since $-UV = e^{r/2GM}(\frac{r}{2GM}-1) > -1 \Rightarrow UV < 1$

$\Rightarrow \tan\hat{U}\tan\hat{V} < 1 \Rightarrow \cos(\hat{U}+\hat{V}) > 0$ $|\hat{U}+\hat{V}| < \frac{\pi}{2}$

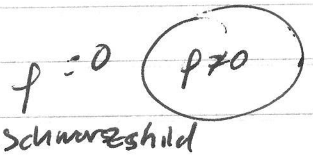
Also $r=0$ is $UV=1$ $\hat{U}+\hat{V} = \pm\frac{\pi}{2}$. Now let $\hat{T} = \frac{1}{2}(\hat{U}+\hat{V})$ $\hat{X} = \frac{1}{2}(\hat{V}-\hat{U})$

So $r=0$ is $\hat{T} = \pm\frac{\pi}{2}$ and $\hat{T} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\hat{X} \in (-\pi, \pi)$

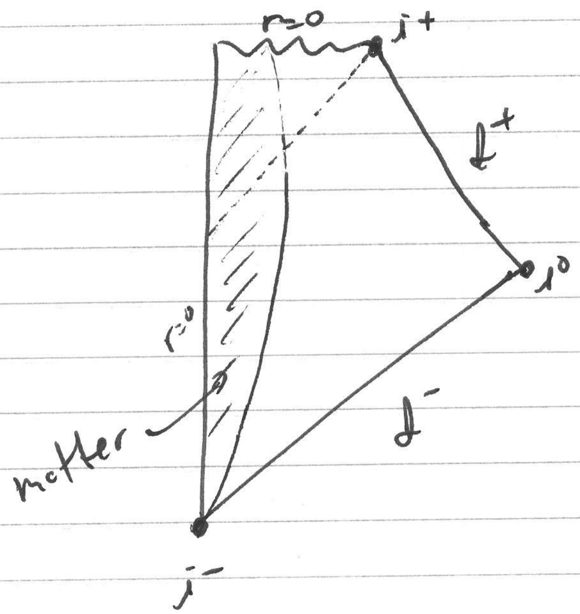


Black holes in nature arise from gravitational collapse of stars. ~~Even~~ This is ~~at~~ unlikely to be exactly spherically symmetric, but we can consider the idealized case. ~~Since at some time.~~

The metric is spherically symmetric with some mass/energy density distributed with spherical symmetry. Birkhoff's theorem gives that outside the region of mass the metric is Schwarzschild (well, not quite; that would be true if the situation were static}, but let's imagine it is slow, so the metric is approximately Schwarzschild). Yet, inside the metric is not Schwarzschild and in fact is regular at $r=0$.



~~Moreover, at~~ As we let the star collapse (which <u>will</u> happen if the object is dense enough, in fact if the mass $GM > \frac{4R}{9}$) once the mass falls within $R=2GM$, it will continue falling irreversibly towards the singularity. A picture (conformal diagram caricature) of the process is

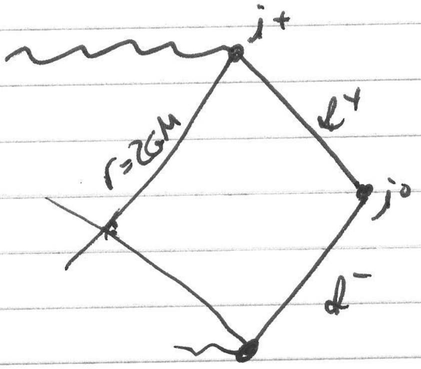


and we see such a spacetime has no~~t~~ white-hole nor a region <u>IV</u>.

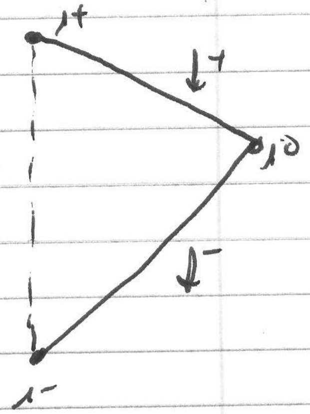
More General Black Holes

What characterizes black holes?

Recall Schwarzschild



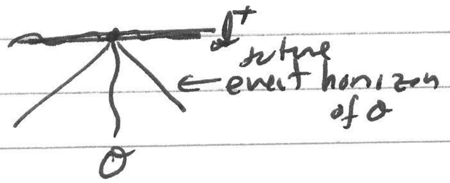
Flat Minkowski



Two important ingredients:

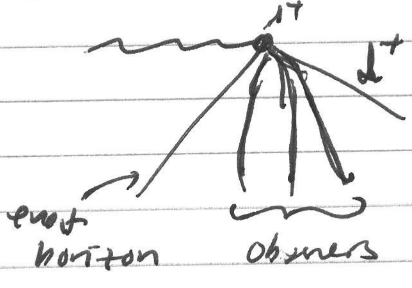
(i) Asymptotically flat, so it looks like Minkowski on the "outside".

(ii) Has an event horizon (future) (at $r=2GM$). Recall we had

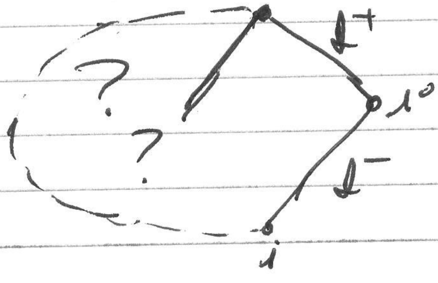


in de-Sitter.

So in Schwarzschild all observers that remain in I go to i^+ at infinity, and they all share $r=2GM$ as a future event horizon



So, more general black hole



Reissner-Nordstrom: Charged Black Hole.

Look for spherically symmetric and static (or just stationary?)
(so $ds^2 = -T(r,t)\,dt^2 + R(r,t)\,dr^2 + r^2 d\Omega_2^2$)
solutions to Einstein's Equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

with matter given by electromagnetic field.

Recall $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

(For contact with conventional non-relativistic notation in Minkowski space the
$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$, Φ is A^0 and $\vec{\nabla} = \partial_i$. With low indices,

$$E_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}$$

Similarly $\vec{B} = (\vec{\nabla} \times \vec{A})$ or $B_i = \epsilon_{ijk}\partial_j A_k = \frac{1}{2}\epsilon_{ijk} F_{jk}$

Note that $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$ as it should).

Then, as we saw earlier

$$T_m^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad S = \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

$$\Rightarrow \quad T_m^{\mu\nu} = \int d^4x \left[\frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g_{\mu\nu}} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \right]$$

The first term requires $\delta(\det A) = \underset{\lambda \text{ of } A}{\underset{\text{eigenvalues}}{\delta \Pi \lambda}} = \delta e^{\sum_\lambda \ln\lambda} = e^{\sum \ln \lambda} \sum \frac{1}{\lambda}\delta\lambda$
$= \det A \,\mathrm{Tr}\, A^{-1}\delta A$

or $\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}$ so $\frac{1}{\sqrt{-g}}\delta\sqrt{-g} = \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu}$

For the second term use

$$\mathcal{L} = -\frac{1}{4} F_{\mu\lambda} F_{\nu\rho}\, g^{\mu\nu} g^{\lambda\rho}$$

so that there is no implicit dependence on $g_{\mu\nu}$ in F, and then

$$\frac{\delta g^{\lambda\rho}}{\delta g_{\mu\nu}} = - g^{\lambda\mu} g^{\rho\nu} - g^{\lambda\nu} g^{\rho\mu} \quad , \text{so}$$

$$\int d^4x \, \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = F^{\mu}{}_{\lambda} F^{\nu}{}_{\rho}\, g^{\lambda\rho}$$

or

$$T^{\mu\nu} = F^{\mu}{}_{\lambda} F^{\nu}{}_{\rho}\, g^{\lambda\rho} - \frac{1}{4} g^{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}$$

or

$$\boxed{T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}}$$

For spherical symmetry need radial $\vec{E}$ (and possibly $\vec{B}$), so in radial coordinates we have $F_{t\theta} = F_{t\phi} = 0$ and

$$F_{tr} = -F_{rt} = f(t,r)$$

For $\vec{B}$ to be radial we need to generalize $B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$: "B_μ" $= \frac{1}{2} \epsilon_{0\mu\nu\lambda} F^{\nu\lambda}$ and use $\epsilon^{\mu\nu\lambda\rho} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\mu\nu\lambda\rho}$. Or, more directly, but pedestrianly, go back to cartesian $x\,y\,z$. Then

$F_{\theta\phi} = F_{ij} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi}$ and $F_{ij} = \epsilon_{ijk} B^k_k \propto \epsilon_{ijk} x^k$ for radial (times some g(

But then $F_{\theta\phi} = g(r)\, \epsilon_{ijk} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi} x^k = \sin\theta\, g(r)$.

(The factor $\epsilon_{ijk} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi} x^k$ is just the determinant $\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} & x \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} & y \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} & z \end{vmatrix}$

which is the measure for the volume integral at $r=1$, $\sin\theta$).

So we take

$$ds^2 = -T(\overset{\text{static}}{\cancel{t}}, r)\,dt^2 + R(r)\,dr^2 + r^2\,d\Omega_2^2$$

$$F_{tr} = f(r)$$

$$F_{\theta\phi} = g(r)\sin\theta$$

and plug into Einstein's.

Quick computation $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\nu\lambda,\mu})$

NOT FOR CLASS

$$\begin{aligned}
\Gamma_{rtt} &= +\tfrac{1}{2}T' & \Gamma^r_{tt} &= \tfrac{1}{2}T'/R \\
\Gamma_{trt} &= -\tfrac{1}{2}T' & \Gamma^t_{rt} &= \tfrac{1}{2}T'/T \\
\Gamma_{rrr} &= \tfrac{1}{2}R' & \Gamma^r_{rr} &= \tfrac{1}{2}R'/R \\
\Gamma_{r\theta\theta} &= -r & \Gamma^r_{\theta\theta} &= -r/R \\
\Gamma_{\theta r\theta} &= r & \Gamma^\theta_{r\theta} &= \tfrac{1}{r} \\
\Gamma_{r\phi\phi} &= -r\sin^2\theta & \Gamma^r_{\phi\phi} &= -r\sin^2\theta/R \\
\Gamma_{\phi r\phi} &= r\sin^2\theta & \Gamma^\phi_{r\phi} &= \tfrac{1}{r} \\
\Gamma_{\theta\phi\phi} &= -r^2\sin\theta\cos\theta & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma_{\phi\theta\phi} &= r^2\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} &= \cos\theta/\sin\theta
\end{aligned}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$R_{tt} = \frac{1}{2}T''/R - \frac{1}{2}\frac{T'R'}{R^2} + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(\frac{1}{2}T'/R\right) - 2\left(\frac{1}{2}\frac{T'}{R}\right)\left(\frac{1}{2}\frac{T'}{T}\right)$$

$$= \frac{1}{2}\frac{T''}{R} - \frac{1}{4}\frac{T'R'}{R^2} - \frac{1}{4}\frac{T'^2}{RT} + \frac{1}{r}\frac{T'}{R}$$

$$R_{rr} = \frac{1}{2}\frac{R''}{R} - \frac{1}{2}\frac{R'^2}{R^2} - \partial_r\left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right) + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\frac{1}{2}\frac{R'}{R}$$

$$- \left(\frac{1}{2}\frac{T'}{T}\right)^2 - \left(\frac{1}{2}\frac{R'}{R}\right)^2 - 2\frac{1}{r^2}$$

$$= -\frac{1}{2}\frac{T''}{T} + \frac{1}{4}\frac{T'^2}{T^2} + \frac{1}{4}\frac{T'R'}{TR} + \frac{R'}{rR}$$

$$R_{rt} = R_{tr} = \cancel{\partial_t\left(\frac{1}{2}\frac{T'}{T}\right.}\Big) + (\cdot) - 0 = 0$$

$$R_{\theta\theta} = \partial_r\left(-\frac{r}{R}\right)\overset{+\partial_\theta \cot\theta}{\wedge} + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(-\frac{r}{R}\right) - 2\left(-\frac{r}{R}\right)\left(\frac{1}{r}\right) - \left(\frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{1}{2}\frac{rR'}{R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1$$

$$R_{\phi\phi} = \partial_r\left(-\frac{r\sin^2\theta}{R}\right)\overset{+\partial_\theta(-\sin\theta\cos\theta)}{\vee} + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(-\frac{r\sin^2\theta}{R}\right) + \left(\frac{\cos\theta}{\sin\theta}\right)(-\sin\theta\cos\theta)$$

$$- 2\left(\frac{1}{r}\right)\left(-\frac{r\sin^2\theta}{R}\right) - 2(-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin^2\theta\left[\frac{rR'}{2R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1\right] = \sin^2\theta\, R_{\theta\theta}$$

Ricci scalar

$R = g^{\mu\nu}R_{\mu\nu} = \cdots$ better use trace of T

So write

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

$$\rightarrow \quad -R = 8\pi G T$$

$$\Rightarrow R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(-8\pi G T) = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Now, compute $T_{\mu\nu}$:

$$T_{tt} = F_{t\lambda}F_t{}^\lambda - \frac{1}{4}g_{tt}F_{\lambda\rho}F^{\lambda\rho} = \frac{1}{R}f^2 + \frac{1}{2}T\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$

$$= \frac{1}{2R}f^2 + \frac{1}{2}\frac{Tg^2}{r^4}$$

$$T_{rr} = F_{r\lambda}F_r{}^\lambda - \frac{1}{4}g_{rr}F_{\lambda\rho}F^{\lambda\rho} = -\frac{f^2}{T} - \frac{1}{2}R\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$

$$= -\frac{1}{2}\frac{f^2}{T} - \frac{g^2R}{2r^4}$$

$$T_{\theta\theta} = F_{\theta\lambda}F_\theta{}^\lambda - \frac{1}{4}g_{\theta\theta}F_{\lambda\rho}F^{\lambda\rho} = \frac{1}{r^2\sin^2\theta}\sin^2\theta\, g^2 - \frac{1}{2}r^2\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$

$$= \frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2f^2}{RT}$$

$$T_{\phi\phi} = \sin^2\theta\, T_{\theta\theta}$$

So $T = g^{\mu\nu} T_{\mu\nu} = -\frac{1}{T}\left(\frac{1}{2R}f^2 + \frac{1}{2}\frac{Tg^2}{r^4}\right) + \frac{1}{R}\left(-\frac{1}{2}\frac{f^2}{T} - \frac{g^2R}{2r^4}\right)$

$$+ \frac{2}{r^2}\left(\frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2f^2}{RT}\right) = 0$$

And we have

$$tt: \quad \frac{1}{2}\frac{T''}{R} - \frac{1}{4}\frac{T'R'}{R^2} - \frac{1}{4}\frac{T'^2}{RT} + \frac{1}{r}\frac{T'}{R} = \left(\frac{f^2}{2R} + \frac{1}{2}\frac{Tg^2}{r^4}\right)8\pi G$$

$$rr: \quad -\frac{1}{2}\frac{T''}{T} + \frac{1}{4}\frac{T'^2}{T^2} + \frac{1}{4}\frac{T'}{T}\frac{R'}{R} \overset{+\frac{R'}{rR}}{\wedge} = \left(-\frac{1}{2}\frac{f^2}{T} - \frac{g^2R}{2r^4}\right)8\pi G$$

Same eqn.

$$\theta\theta: \quad \frac{1}{2}\frac{rR'}{R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1 = 8\pi G\left(\frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2f^2}{RT}\right)$$

2 eqs, 4 unknowns: need more: Maxwell's Equations

$$g^{\mu\nu}\nabla_\mu F_{\nu\lambda} = 0 \qquad \text{and} \qquad \nabla_{[\mu} F_{\nu\lambda]} = 0$$

Recall $\nabla_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - \Gamma^\rho_{\mu\nu} F_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} F_{\nu\rho}$

So, in components

$$g^{\mu\nu}\nabla_\mu F_{\nu t} = -\frac{1}{T}\left[\cancel{-\frac{1}{2}\frac{T'}{R}(-f) + \ldots}\right] + \frac{1}{R}\left(-f' - \frac{1}{2}\frac{R'}{R}(-f) - \frac{1}{2}\frac{T'}{T}(-f)\right)$$

$$+ \frac{1}{r^2}\left[-\left(-\frac{r}{R}\right)(-f)\right] + \frac{1}{r^2\sin^2\theta}\left[-\left(-\frac{r\sin^2\theta}{R}\right)(-f)\right]$$

$$= -\frac{f'}{R} + \frac{1}{2}\frac{R'}{R}f + \frac{1}{2}\frac{T'}{T}f - \frac{2f}{rR} = \cancel{0}$$

$g^{\mu\nu}\nabla_\mu F_{\nu r} = -\frac{1}{T}[0] + \frac{1}{R}(0) + \frac{1}{r^2}(0) = 0$ automatic

$$g^{\mu\nu}\nabla_\mu F_{\nu\theta} = -\frac{1}{T}(0) + \frac{1}{R}(0) + \frac{1}{r^2}[0] + \frac{1}{r^2\sin^2\theta}[0] = 0$$

$$g^{\mu\nu}\nabla_\mu F_{\nu\phi} = -\frac{1}{T}(0) + \frac{1}{R}(0) + \frac{1}{r^2}\left[\cos\theta\, g - \frac{\cos\theta}{\sin\theta}\sin\theta\, g\right] + \frac{1}{r^2\sin\theta}(0) = 0$$

and for the 2nd equation take

$$\nabla_r F_{\theta\phi} + \nabla_\theta F_{\phi r} + \nabla_\phi F_{r\theta}$$

$$= \left(\sin\theta\, g' - \frac{2}{r}\sin\theta\, g\right) + \left(-\frac{1}{r}(-g\sin\theta)\right) + \left(-\frac{1}{r}(-g\sin\theta)\right)$$

$$= \sin\theta\, g' \qquad ?$$

So $\nabla_{[r} F_{\theta\phi]} = 0 \Leftrightarrow \sin\theta\, g' = 0 \quad \Rightarrow \quad g = \text{constant.}$

And Maxwell's equation gives

$$\frac{f'}{f} = \left[\frac{1}{2}\frac{R'}{R} + \frac{1}{2}\frac{T'}{T} - \frac{2}{rR}\right]R$$

Look for a solution with $TR = 1$ $\quad T'/T = -R'/R$

then $\quad f'/f = -\frac{2}{r} \qquad \frac{df}{f} = -2\frac{dr}{r} \qquad f = \frac{K}{r^2}$

and

$$\frac{rR'}{R^2} - \frac{1}{R} + 1 = 4\pi G \frac{(g^2+K^2)}{r^2} \qquad (ⓐ)$$

and

$$-\frac{1}{2}\frac{T''}{T} - \frac{T'}{rT} = -\frac{4\pi G}{T}\frac{(K^2+g^2)}{r^4}$$

or $$T'' + \frac{2}{r}T' = \frac{8\pi G(K^2+g^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^2}(r^2T')' = \frac{8\pi G(K^2+g^2)}{r^4} \Rightarrow r^2T' = -\frac{8\pi G(K^2+g^2)}{r} + 2GM$$

$$T = 1 - \frac{2GM}{r} + \frac{4\pi G(g^2+K^2)}{r^2}$$

Check (ⓐ): $-r\frac{T'}{TR} - T + 1 = -rT' - T + 1 = \left[\frac{8\pi G(K^2+g^2)}{r^2} - \frac{2GM}{r}\right] + \frac{2GM}{r} - \frac{4\pi G}{r^2}$

$$= \frac{4\pi G(K^2+g^2)}{r^2} \quad ✓ \quad \text{Yes!}$$

The solution is $ds^2 = -T dt^2 + R dr^2 + r^2 d\Omega_2^2$

$$T = \frac{1}{R} = 1 - \frac{2GM}{r} + \frac{4\pi G(p^2+q^2)}{r^2}$$

and $F_{tr} = \frac{\overset{q}{\cancel{\pi}}}{r^2}$ $F_{\theta\phi} = p\sin\theta$

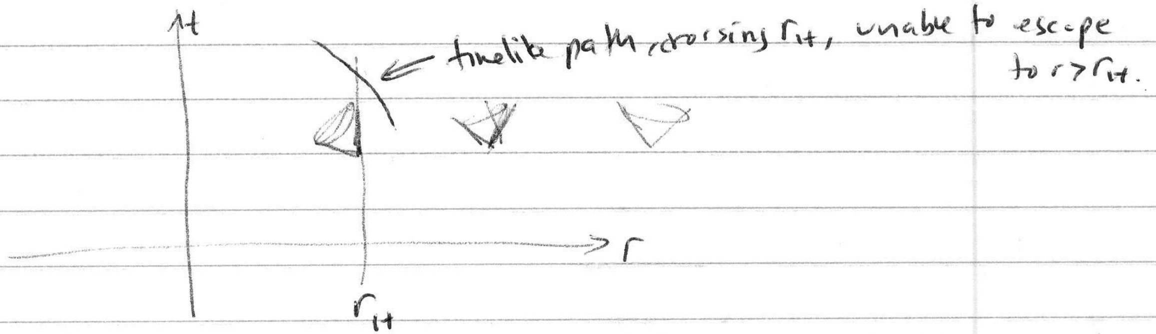
(Note that $E_r = F_{tr} = \frac{q}{r^2}$ $B_r = \frac{F_{\theta\phi}}{r^2\sin\theta} = \frac{p}{r^2}$

so q & p are electric & magnetic charges, and the notation (q,p) is standard for "dyons").

Singularity at $r=0$. Horizon singularities (see below) are coordinate effects:

Event horizons? In a static space-time (Killing vector ∂_t, asymptotically time-like, $\partial_t g_{\mu\nu} = 0$) choose coordinates (r,θ,ϕ) so metric looks Minkowski as $r \to \infty$.

Hypersurface $r = \text{const}$: timelike "cylinder" (topology $S^2 \times \mathbb{R}$) as $r \to \infty$. Now decrease r from infinity to some r_H where the surface becomes null → an event horizon



timelike path, crossing r_H, unable to escape to $r > r_H$.

How to determine r_H? $\partial_\mu r$ is a 1-form normal to $r = \text{const}$ hypersurface, with norm

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}$$

We want this to vanish, so $\boxed{g^{rr}(r_H) = 0}$

This method is very restrictive (to spaces that are static and with coordinates (r,θ,ϕ) as above, found).

Method applies for RN-metric. So

r_H is at $g^{rr} = 1 - \frac{2GM}{r} + \frac{4\pi G(q^2+p^2)}{r^2} = 0$

There are two solutions (at most):

$$r_{\pm} = \frac{2GM \pm \sqrt{(2GM)^2 - 4^2\pi G(p^2+q^2)}}{2}$$

$$= GM \pm \sqrt{(GM)^2 - 4\pi G(p^2+q^2)}$$

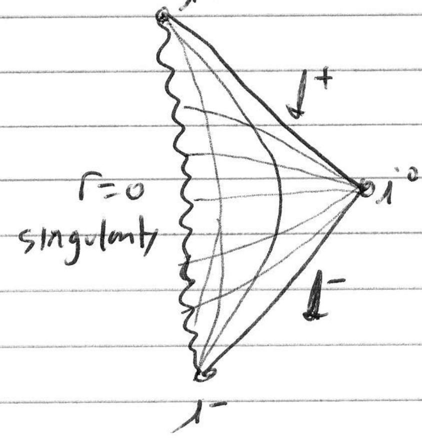
Cases:

(1) $4\pi(p^2+q^2) > GM^2$

No solutions ⇒ no event horizon.

"Naked singularity"

Penrose:



Cosmic Censorship Conjecture (Penrose): Nature abhors a naked singularity, or

Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition:

for all timelike vectors t^μ, $T_{\mu\nu}t^\mu t^\nu \geq 0$ (so far, "weak energy condition")

and $T^{\mu\nu}t_\nu$ is a non-spacelike vector (Basically, $\rho > |p|$).

$(T_{\mu\nu}T^{\nu}{}_{\lambda}t^\mu t^\lambda \leq 0)$.

(iii) $4\pi(p^2+q^2) > GM^2$

Two distinct solutions, with $r_- < r_+$

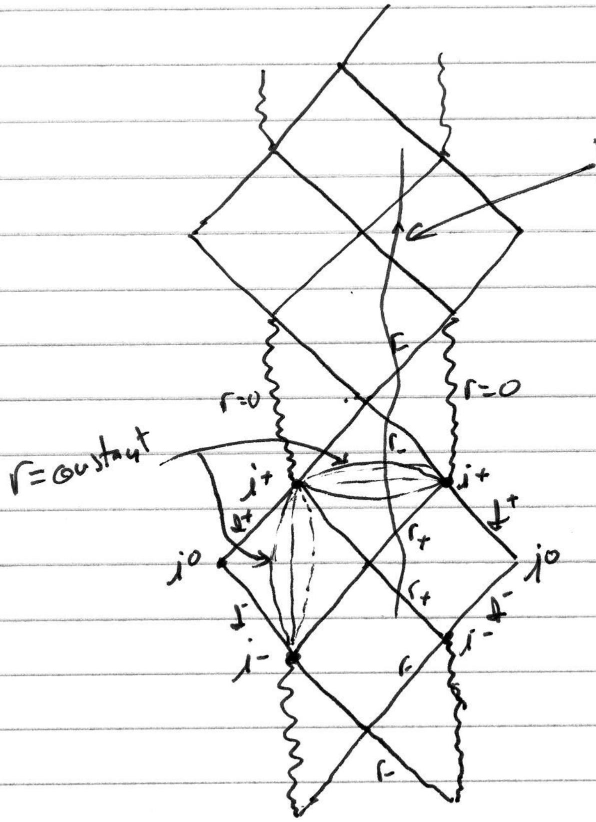
In $r_- < r < r_+$ ∂_r is timelike and ∂_t is spacelike.

But both for $r > r_+$ AND $r < r_-$ ∂_r is spacelike & ∂_t is timelike.

$\Rightarrow$ ① If you fall into this black hole within a spaceship full of gas, once you get to $r = r_+$ you must continue falling towards lesser r, but once you come out to $r < r_-$ you can turn on your thrust engines, turn around before you hit $r=0$, go back to $r = r_-$. Then you must continue, ~~with~~ until you come out to $r = r_+$. You can then decide to continue out to $r = \infty$ or turn around and "re-enter" the black hole?

② $r=0$ singularity is timelike (recall, for Schwarzschild, spacelike)

Conformal diagram



trajectory going in and out of black hole

MTW has a step by step on how to derive this.

(iii) $GM^2 = 4\pi(q^2+p^2)$

"Extreme" RN-solution. ~~Here~~

In this case $r_+ = r_- = GM$, and

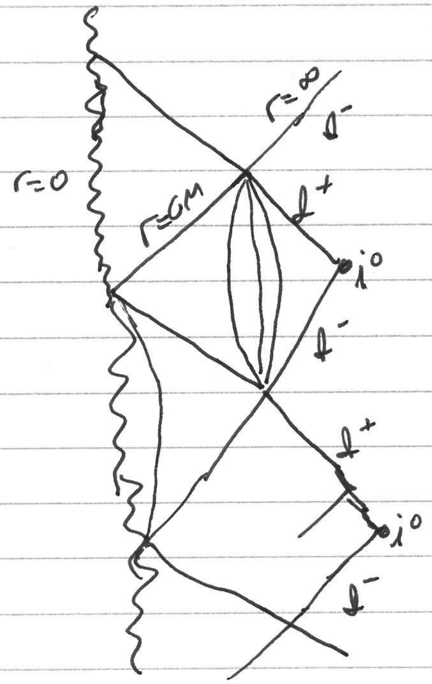
$$g^{rr} = \left(1-\frac{GM}{r}\right)^2$$

In fact

$$ds^2 = -\left(1-\frac{GM}{r}\right)^2 dt^2 + \left(1-\frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

So, there is a horizon at $r=GM$, but r is never timelike.
The singularity is at $r=0$ and it is timelike.

Penrose diagram



Solutions with many extreme RN black holes: remarkably, we can produce metrics which are exact solutions of Einstein's equations in empty space with as many RN black holes as we want.

In $$ds^2 = -\left(1-\frac{2GM}{r}\right)^2 dt^2 + \left(1-\frac{2GM}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

let
$$\rho = r - GM$$

so
$$r^2 = (\rho + GM)^2 = \rho^2 H^2(\rho)$$

where $H(\rho) = 1 + \frac{GM}{\rho}$

Also $$1 - \frac{2GM}{r} = 1 - \frac{2GM}{\rho + GM} = \frac{\rho}{\rho + GM} = \frac{1}{1+\frac{GM}{\rho}} = H^{-1}$$

so
$$ds^2 = -H^{-2}(\rho)\, dt^2 + H^2(\rho)\left[d\rho^2 + \rho^2 d\Omega_2^2 \right]$$

Now, the term in [ ] is just the metric of flat Euclidean 3-space in spherical coordinates, so we can write

$$ds^2 = -H^{-2}(|\vec{x}|)\, dt^2 + H^2(|\vec{x}|)\left[dx^2 + dy^2 + dz^2\right] \qquad (\star)$$

where $|\vec{x}|^2 = x^2 + y^2 + z^2$

If we take the metric $(\star)$ as an ansatz and plug into Einstein's equations, with $H = H(\vec{x})$, we find it is a solution provided $\nabla^2 H = 0$. To be precise, we need an EM field too. To motivate it, we have, in the single extremal RN solution

$$F_{tr} = \frac{q}{r^2} = -\partial_r A_t + \partial_t A_r \qquad \text{(sign may be wrong)}$$
$$= -\partial_r A_t$$

so $$A_t = \frac{q}{r} = \sqrt{\frac{GM^2}{4\pi}}\frac{1}{\rho + GM} \quad \text{but} \quad \frac{1 - H^{-1}}{GM} = \frac{1}{r} = \frac{1}{\rho + GM}$$

So $$A_t = \sqrt{\frac{GM^2}{4\pi}}\,\frac{1-H^{-1}}{GM} = \sqrt{\frac{1}{4\pi G}}\,(1-H^{-1}) \qquad (**)$$

So one looks for solutions with (*) and (**). Then H must satisfy

$$\vec{\nabla}^2 H = 0 \qquad \text{where} \qquad \vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

The most general solution with $H \to 1$ as $|x| \to \infty$ is

$$H = 1 + \sum_{k=1}^{N} \frac{GM_k}{|\vec{x} - \vec{x}_k|}$$

(Actually, I guess

$$H = 1 + \int d^3x' \, \frac{G\rho(x')}{|\vec{x} - \vec{x}'|}$$

works too, provided ρ has support in a finite region, $|\vec{x}| < R$).
(But $\nabla^2 H = 4\pi G \rho(\vec{x}) \neq 0$ ... so works outside that region).

~~I think, however, these solutions are inconsistent with the electrical forces between the charges! I don't know what the proper resolution of this is?~~

Note that the electric repulsion between holes cancels the gravitational attraction:

$$F_{12} = -\frac{GM_1M_2}{r^2} + \frac{q_1 q_2}{r^2} = 0 \quad \text{if} \quad q_1 q_2 = GM_1M_2$$

or $q_1 = \sqrt{G}M_1 \quad q_2 = \sqrt{G}M_2$

and we are off by a $\sqrt{4\pi}$.

?
is there a $\frac{1}{4\pi}$ here?

Note: Verify solution:

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2+dy^2+dz^2)$$

$$\Gamma_{itt} = -\tfrac{1}{2}g_{tt,i} = -\tfrac{1}{2}(-1)(-2H^{-3})\partial_i H = -H^{-3}\partial_i H \qquad \Gamma^i_{tt} = -H^{-5}\partial_i H$$

$$\Gamma_{tit} = H^{-3}\partial_i H \qquad \Gamma^t_{it} = -H^{-1}\partial_i H$$

$$\Gamma_{ijk} = \tfrac{1}{2}\left((H^2\delta_{ij})_{,k} + (H^2\delta_{ik})_{,j} - (H^2\delta_{jk})_{,i}\right) = H\left(\delta_{ij}H_k + \delta_{ik}H_j - \delta_{jk}H\right.$$

$$\Gamma^i_{jk} = H^{-1}(\delta_{ij}H_{,k} + \delta_{ik}H_{,j} - \delta_{jk}H_{,i})$$

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$R_{tt} = \partial_i(-H^{-5}\partial_i H) + (-H^{-1}\partial_i H + 3H^{-1}\partial_i H)(-H^{-5}\partial_i H) - 2(H^{-1}\partial_i H)(H^{-5}\partial_i H)$$

$$= 5H^{-6}(\partial_i H)^2 - H^{-5}\nabla^2 H - 4H^{-6}(\partial_i H)^2 = H^{-6}(\partial_i H)^2 - H^{-5}\nabla^2 H$$

$$R_{ij} = \partial_k\left[H^{-1}(\delta_{kj}H_{,i} + \delta_{ik}H_{,j} - \delta_{ji}H_{,k})\right] - \partial_j\left[2H^{-1}\partial_i H\right]$$

$$+ \left[2H^{-1}\partial_k H\right]\left[H^{-1}(\delta_{kj}H_{,i} + \delta_{ik}H_{,j} - \delta_{ji}H_{,k})\right]$$

$$- (H^{-1}\partial_i H)(H^{-1}\partial_j H) - H^{-2}(\delta_{kj}H_{,l} + \delta_{kl}H_{,j} - \delta_{jl}H_{,k}) \times$$

$$(\delta_{li}H_{,k} + \delta_{kl}H_{,i} - \delta_{ik}H_{,l})$$

$$= -H^{-2}(\cancel{2H_jH_i} - \delta_{ij}H_{,k}^2) + H^{-1}(\cancel{2H_{,ij}} - \delta_{ij}\nabla^2 H) + \cancel{2H^{-2}H_jH_i} - \cancel{2H^{-1}H_{,ij}}$$

$$+ 2H^{-2}(2H_jH_i - \delta_{ij}H_{,k}^2) - H^{-2}H_iH_j - H^{-2}(5H_iH_j - 2\delta_{ij}H_{,k}^2)$$

$$= -\delta_{ij}H^{-1}\nabla^2 H + H^{-2}\delta_{ij}H_{,k}^2 - 2H^{-2}H_{,i}H_{,j}$$

The em part: $F_{ti} = -\partial_i A_t = \sqrt{1/4\pi G}\,(-H^{-2}H_{,i})$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \cancel{\sqrt{1/4\pi G}\,H^{-2}}\ 0$$

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}$$

$$T_{tt} = F_{ti}F_t{}^i - \tfrac{1}{4}g_{tt}2F_{ti}F^{ti} = H^{-2}\frac{1}{4\pi G}(-H^{-2}H_i)^2 - \tfrac{1}{2}(-H^{-2})(-1)\left(-\frac{H^{-2}H_{,i}}{\sqrt{4\pi G}}\right)^2$$

$$= \frac{1}{8\pi G}H^{-6}H_{,i}^2$$

$$R_{tt} = 8\pi G\,T_{tt} \Rightarrow H^{-6}H_{,i}^2 - H^{-5}\nabla^2 H = 8\pi G\left\{\frac{1}{8\pi G}H^{-2}H_{,i}^2\right)$$

$$\Rightarrow \nabla^2 H = 0$$

(assuming $T_{\lambda\sigma}g^{\lambda\sigma} = 0$).

Kerr Metric: Rotating Black Holes

No hair "theorem": stationary, asymptotically flat solutions to Einstein's + Maxwell's are fully characterized by M, Q (+P) and $J = aM$ [illegible])

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dt\,d\phi + d\phi\,dt)$$

$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where

$$\Delta(r) = r^2 - 2GMr + a^2$$

$$\rho^2(r,\theta) = r^2 + a^2 os^2\theta$$

Note: Include charge by changing $2GMr \rightarrow 2GMr - 4\pi G(Q^2+P^2)$.

And $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with

$$A_t = \frac{Qr - Pa\cos\theta}{\rho^2} \qquad A_\phi = \frac{-Qar\sin^2\theta + P(r^2+a^2)\cos\theta}{\rho^2}$$

(Kerr-Newman)

The novel feature is $J = aM \neq 0$, so let's simply set $Q = P = 0$ and study Kerr's solution.

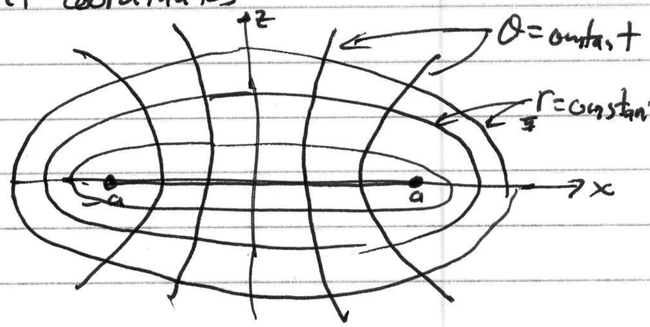
Note that for $M = 0$ we have flat space but in weird coordinates (called Boyer-Lindquist coordinates):

$$ds^2 = -dt^2 + \underbrace{\frac{r^2+a^2\cos^2\theta}{r^2+a^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2}_{\text{ellipsoidal coordinates}}$$

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$

$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$

$$z = r\cos\theta$$



Killing vectors: ∂_t and ∂_ϕ

Also a killing tensor $K_{\mu\nu} = \cancel{2}\rho^2(l_\mu n_\nu + l_\nu n_\mu) + r^2 g_{\mu\nu}$

with $l^\mu = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a)$ and $n^\mu = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a)$

(they have $l^2 = n^2 = 0$ $\quad l \cdot n = -1$).

So geodesics are easy to find: three constants. (use actual energy/ang. mom not just per unit mass)

$$\frac{E}{\boxed{m}} = -(\partial_t)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} \qquad \frac{L}{\boxed{m}} = +(\partial_\phi)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} \qquad S = K_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

included for massive particles only.

plus

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -1 \text{ or } 0 \quad \text{for timelike or null}$$

Note that

$$E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} + \frac{2GMar\sin^2\theta}{\rho^2}\frac{d\phi}{d\lambda} = -g_{tt}\frac{dt}{d\lambda} - g_{t\phi}\frac{d\phi}{d\lambda}$$

$$L = g_{t\phi}\frac{dt}{d\lambda} + g_{\phi\phi}\frac{d\phi}{d\lambda}$$

and

$$\frac{L}{E} = \frac{g_{t\phi} + g_{\phi\phi}\, d\phi/dt}{-g_{tt} - g_{t\phi}\, d\phi/dt}$$

so if $\omega \equiv d\phi/dt$ we have

$$g_{t\phi} + \omega g_{\phi\phi} + \frac{L}{E}(g_{tt} + g_{t\phi}\omega) = 0$$

$$\omega = \frac{-\frac{L}{E}g_{tt} - g_{t\phi}}{g_{\phi\phi} + \frac{L}{E}g_{t\phi}}$$

~~so~~ In particular, even if $L=0$ we can have $\omega \neq 0$ $(\omega = -g_{t\phi}/g_{\phi\phi})$ or with $\omega = 0$ we can have $L \neq 0$.

Horizons: $g^{rr}=0 \Leftrightarrow \frac{\Delta}{\rho^2}=0$. Since $\rho^2 \geq 0$ this is $\Delta=0$ or

$$r^2-2GMr+a^2=0$$

or

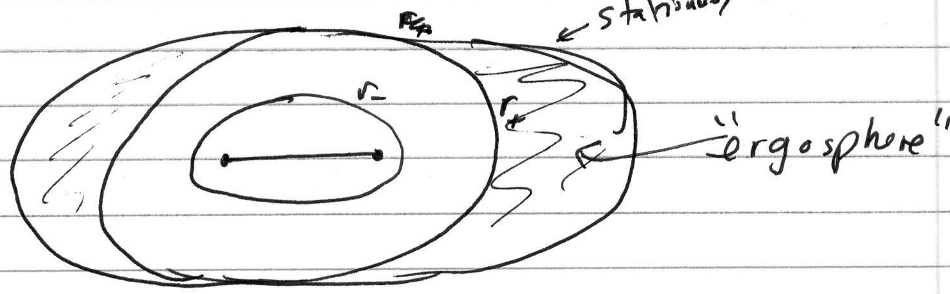
$$r=R_\pm = GM \pm \sqrt{(GM)^2-a^2} \qquad (GM>|a|).$$

Stationary Limit Surface: by definition this is a surface where ∂_t becomes null:

$$g_{\mu\nu}(\partial_t)^\mu(\partial_t)^\nu=0 \Leftrightarrow 1-\frac{2GMr}{\rho^2}=0$$

or

$$r^2+a^2\cos^2\theta-2GMr=0 \qquad (\text{~~i.e. $\Delta(r)=0$~~}$$



∂_t is spacelike outside the outer horizon, i.e. the "ergosphere".
Moreover, at $r=r_+$ $\quad g_{\mu\nu}(\partial_t)^\mu(\partial_t)^\nu = \frac{r_+^2+a^2\cos^2\theta-2GMr_+}{r_+^2+a^2\cos^2\theta} = \frac{a^2\sin^2\theta}{r_+^2+a^2\cos^2\theta} \geq 0$

and the equality holds only at $\theta=0$ where the stationary limit surface and $r=r_+$ coincide.

The ergosphere is quite peculiar. Consider for simplicity the equator, $\theta=\frac{\pi}{2}$. Null lines have (with $r=$ constant, i.e. look at tangential emission).

$$0 = g_{tt}dt^2 + 2g_{t\phi}\,dt\,d\phi + g_{\phi\phi}d\phi^2$$

or

$$\omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\left(\frac{g_{tt}}{g_{\phi\phi}}\right)}$$

Ergosphere: $g_{tt}>0$ so $\sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\frac{g_{tt}}{g_{\phi\phi}}} \cancel{<} \left|\frac{g_{t\phi}}{g_{\phi\phi}}\right|$, so both solutions ~~ha~~ $\pm$ have same sign!

and at stationary limit surface one solution has $\omega=0$

In fact $$-\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2GMar\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}\left[(r^2+a^2)^2-a^2\Delta\sin^2\theta\right]} = \omega$$

has the sign determined by by $a = J/M$.

$\Rightarrow$ photons emitted tangentially (with $\dot{r}=0$ and $\dot{\theta}=0$) from the ergosphere move in same direction as rotation of black hole.

Null geodesics in more detail:

We had

$$E = -g_{tt}\frac{dt}{d\lambda} - g_{t\phi}\frac{d\phi}{d\lambda} \qquad L = g_{t\phi}\frac{dt}{d\lambda} + g_{\phi\phi}\frac{d\phi}{d\lambda} \qquad (1)$$

Instead of doing most general, using S, we limit ourselves to $\theta = \frac{\pi}{2}$ trajectories. Then

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \Rightarrow$$

$$g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

Solve (1) above for $\frac{dt}{d\lambda}$ and $\frac{d\phi}{d\lambda}$, write

$$M\begin{pmatrix} dt/d\lambda \\ d\phi/d\lambda \end{pmatrix} = \begin{pmatrix} -E \\ L \end{pmatrix} \quad \text{where } M = \begin{pmatrix} +g_{tt} & +g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

we need M^{-1} which is just the inverse of the metric:

$$M^{-1} = \frac{1}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}$$

and

$$\begin{pmatrix} dt/d\lambda \\ d\phi/d\lambda \end{pmatrix} = M^{-1}\begin{pmatrix} -E \\ L \end{pmatrix} = \underbrace{\frac{1}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}}_{D}\begin{pmatrix} -g_{\phi\phi}E - g_{t\phi}L \\ g_{t\phi}E + g_{tt}L \end{pmatrix}$$

So

$$g_{tt}\frac{(g_{\phi\phi}E + g_{t\phi}L)^2}{D^2} - 2g_{t\phi}\frac{1}{D^2}(g_{\phi\phi}E + g_{t\phi}L)(g_{t\phi}E + g_{tt}L)$$

$$+ \frac{1}{D^2}g_{\phi\phi}(g_{t\phi}E + g_{tt}L)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

This is of the form

$$\left(\frac{dr}{d\lambda}\right)^2 + V_{eff} = 0$$

Compute (at $\theta = \frac{\pi}{2}$) $(\rho^2 = r^2)$ $(\Delta = r^2 + a^2 - 2GMr = \rho^2 - 2GMr)$

$$D = g_{tt} g_{\phi\phi} - g_{t\phi}^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)\left(\frac{(r^2+a^2)^2}{\rho^2} - \frac{a^2\Delta}{\rho^2}\right) - \left(\frac{2GMar}{\rho^2}\right]^2$$

$$= -\frac{\Delta}{\rho^4}\left(\rho^4 - a^2\Delta\right) - \frac{(2GMar)^2}{\rho^4}$$

Better yet, since $\omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$ $\quad D = g_{tt} g_{\phi\phi} - \omega^2 g_{\phi\phi}^2 = g_{\phi\phi}(g_{tt} - \omega^2 g_{\phi\phi})$

$$V_{eff} \cdot D^2 g_{rr} = g_{tt}\left(g_{\phi\phi}^2 E^2 + 2EL g_{\phi\phi} g_{t\phi} + L^2 g_{t\phi}^2\right)$$

$$- 2 g_{t\phi}\left(g_{\phi\phi} g_{t\phi} E^2 + EL\underset{g_{\phi\phi} g_{tt} + g_{t\phi}^2}{\cancel{g_{t\phi}(g_{tt} + g_{\phi\phi})}}) + L^2 g_{tt}^2 g_{t\phi}\right)$$

$$+ g_{\phi\phi}\left(g_{t\phi}^2 E^2 + 2 g_{t\phi} g_{tt} EL + g_{tt}^2 L^2\right)$$

$$= E^2\left(g_{tt} g_{\phi\phi}^2 - 2 g_{\phi\phi} g_{t\phi}^2 + g_{\phi\phi} g_{t\phi}^2\right)$$

$$+ 2EL\left(g_{tt} g_{t\phi} g_{\phi\phi} - g_{t\phi} g_{\phi\phi} g_{tt} - g_{t\phi}^3 + g_{\phi\phi} g_{t\phi} g_{tt}\right)$$

$$+ L^2\left(g_{tt} g_{t\phi}^2 - 2 g_{tt} g_{t\phi}^2 + g_{\phi\phi} g_{tt}^2\right)$$

$$= E^2\left(g_{tt} g_{\phi\phi}^2 - \omega^2 g_{\phi\phi}^3\right) + 2EL\left(\omega^3 g_{\phi\phi}^3 - \omega g_{tt} g_{\phi\phi}^2\right)$$

$$+ L^2\left(g_{\phi\phi} g_{tt}^2 - \omega^2 g_{tt} g_{\phi\phi}^2\right)$$

$$V_{eff} = \frac{g_{\phi\phi}}{g_{rr} D}\left[E^2 - 2EL\omega + L^2 \frac{g_{tt}}{g_{\phi\phi}}\right]$$

~~$= \frac{\Delta}{\rho^2} \frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]\left[E^2 - 2EL\omega + L^2 \frac{(1 - 2GMr/\rho^2)}{\frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]}\right]$~~

prefactor ~~$\frac{r^2 + a^2 - 2GMr}{r^4}\left[(r^2+a^2)^2 - a^2(r^2 + a^2 - 2GMr)\right]$~~

$r^2(r^2+a^2) + 2GMra^2$

$$V_{eff} = \frac{1}{g_{rr}(g_{tt} - \omega^2 g_{\phi\phi})}\left[E^2 - 2EL\omega + L^2 \frac{g_{tt}}{g_{\phi\phi}}\right]$$

Since $\left(\frac{dr}{d\lambda}\right)^2 \geq 0$ solutions only exist for

$(E-V_+)(E-V_-) > 0$ that is both $V_\pm > E$ or both $V_\pm < E$

So study $V_\pm$. As $r \to \infty$ $V_\pm \sim \pm\frac{L}{r}$ Let's take $L>0$ so that $V_+ > V_-$ (but we should also investigate $L<0$, since presumably the relative sign of L and $a = \frac{J}{M}$ matter, and we are assuming $a \geq 0$).

Clearly $V_+ = V_-$ at $\Delta = 0 \Rightarrow$ the event horizon $r = R_+$.

There

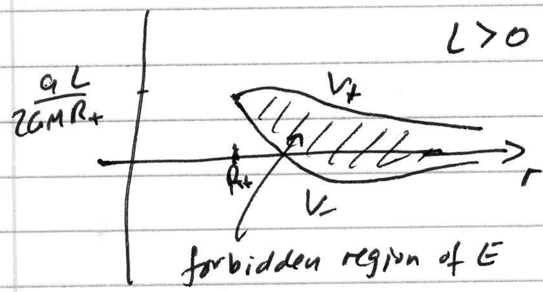
$$V_+ = V_- = \frac{2GMR_+aL}{(R_+^2+a^2)^2} = \frac{aL}{2GMR_+}$$

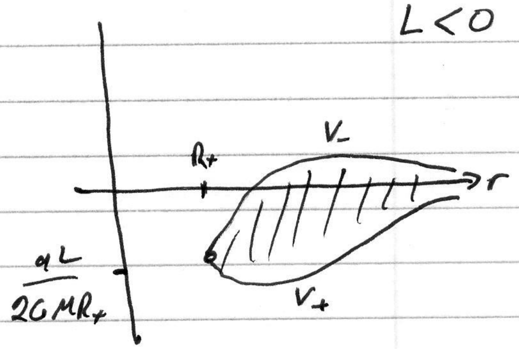
Note that V_+ has no zeroes, while V_- has a zero at

$$2GMr_0a = r_0^2\sqrt{\Delta(r_0)}$$

$$\rightarrow (2GMa)^2 \cancel{r_0} = r_0^2(r_0^2+a^2-2GMr_0)$$

In principle four zeroes, but note that at $a=0$ three zeroes are at $r_0=0$ while one is at $r_0 \to 2GM$. So we suspect only one zero is in the region $r > R_+$.





Now, writing $-g_{tt} = 1 - \frac{2GMr}{\rho^2} = \frac{\Delta - a^2\sin^2\theta}{\rho^2}$

we have at $\theta = \frac{\pi}{2}$

$$\frac{g_{tt}}{g_{\phi\phi}} = -\frac{\Delta - a^2}{(r^2+a^2)^2 - a^2\Delta} \qquad \omega = \frac{2GMra}{(r^2+a^2)^2 - a^2\Delta}$$

and $$\omega^2 - \frac{g_{tt}}{g_{\phi\phi}} = \frac{(2GMra)^2 + (\Delta - a^2)\left[(r^2+a^2)^2 - a^2\Delta\right]}{\left[(r^2+a^2)^2 - a^2\Delta\right]^2}$$

$$\text{numerator} = (2GMra)^2 + (r^2 - 2GMr)\left[(r^2+a^2)^2 - a^2(r^2+a^2) + 2GMra^2\right]$$

$$= r^2\left[(r^2+a^2)r^2 + 2GMra^2\right] - 2GMr\left[r^2+a^2\right]r^2$$

$$= (r^2+a^2)r^4 - 2GMr^5$$

$$= r^4(r^2+a^2-2GMr) = r^4\Delta$$

so

$$\boxed{V_\pm = L\left[\frac{2GMra \pm r^2\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}\right]}$$

and we have

$$\left(\frac{dr}{d\lambda}\right)^2 = -\frac{(E-V_+)(E-V_-)}{g_{rr}(g_{tt} - \omega^2 g_{\phi\phi})}$$

$$\text{den} = -g_{rr}\,g_{\phi\phi}\left(\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}\right)$$

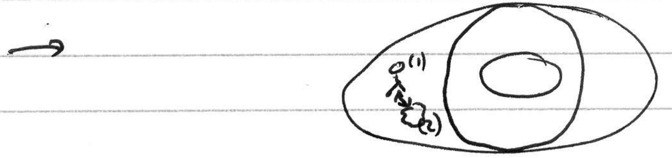
$$= -\frac{\rho^2}{\Delta}\,\frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]\frac{r^4\Delta}{\left[(r^2+a^2)^2 - a^2\Delta\right]^2} = -\frac{r^4}{\left[(r^2+a^2)^2 - a^2\Delta\right]}$$

and

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E-V_+)(E-V_-)}$$

Penrose process

jump into ergosphere



push rock away inside ergosphere

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}$$ local conservation of p^μ

or, contracting with $(\partial_t)^\mu$

$$E^{(0)} = E^{(1)} + E^{(2)}$$

Clearly $E^{(0)}>0$, but if you push $E^{(2)}$ hard enough you can arrange $E^{(2)}<0$ so

$$E^{(1)} > E^{(0)}$$

$\Rightarrow$ come out of ergosphere with more than the original total energy.

Energy comes from black hole $\Rightarrow$ reduce b.h's angular momentum (rock must be thrown against rotation of b.h.).

To see this lets figure out the condition that the rock (2) crosses the event horizon R_+. We must be slightly careful since $r=R_+$ is a null surface.

Killing Horizons: if a Killing vector χ^μ is null on a null hypersurface Σ, then we say Σ is a Killing Horizon.

For Kerr, $\vec{\partial}_t$ is not null on the event horizon: it is null on the SLS (stationary limit surface) by def'n.

The event horizon is null, and

$$\vec{\chi} = \vec{\partial}_t + \Omega_H \vec{\partial}_\phi$$

is null for some constant Ω_H. Exercise: show $\Omega_H = \frac{a}{a^2+R_+^2}$

[Calculate: $\chi^2 = 0 = \partial_t^2 + 2\Omega_H \vec{\partial}_t \cdot \partial_\phi + \Omega_H^2 \partial_\phi^2$

$$= g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}$$

so $$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

Now on $r = R_+$ $\Delta = r^2 - 2GMr + a^2 = 0$ and

$$g_{tt} = -\left(1 - \frac{2GMr}{\rho^2}\right) = -\frac{1}{\rho^2}\left(\rho^2 - 2GMr\right) = -\frac{1}{\rho^2}\left(r^2 + a^2\cos^2\theta - r^2 - a^2\right)$$

$$= +\frac{1}{\rho^2} a^2 \sin^2\theta$$

$$g_{\phi\phi} = \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] = \frac{\sin^2\theta}{\rho^2}(r^2+a^2)^2$$

$$g_{t\phi} = -\frac{2GMra\sin^2\theta}{\rho^2} = -\frac{a(a^2+r^2)\sin^2\theta}{\rho^2}$$

so $$\Omega_H = \frac{a}{a^2+R_+^2} \pm \sqrt{\frac{a^2}{(a^2+R^2)^2} - \frac{a^2}{(a^2+R^2)^2}} = \frac{a}{a^2+R_+^2}$$]

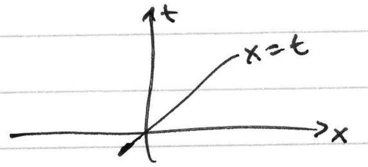
Exercise: show $r = R_+$ is null.

Note: if Σ is defined through $f(x^\mu) = \text{constant}$, then Σ is null $\Leftrightarrow$ $\nabla_\mu f$ is null.

[Calculate: $r = R_+$ is the same as $f(t, r, \theta, \varphi) = r$. Then

$\partial_\mu r$ has $g^{\mu\nu} \partial_\mu r \partial_\nu r = 0$ since $g^{rr} = 0$ at $r = R_\pm$.]

To see what condition we want to impose on $p^{(2)}$ (that signals $p^{(2)}$ crosses R_+), look at a null cone in Minkowski space first



The surface is defined by $x - t = \text{constant}$, and $\nabla_\mu (x-t)$ is a null vector both normal and tangent to it.

$$n_\mu = \nabla_\mu (x-t) \Rightarrow n^\mu = (1,1)$$

Now, if we have a particle moving along $x^\mu(\lambda)$, then

$$n \cdot \frac{dx}{d\lambda} = -\frac{dt}{d\lambda} + \frac{dx}{d\lambda}$$

So, if $n \cdot \frac{dx}{d\lambda} < 0 \Rightarrow \frac{dx}{d\lambda} < \frac{dt}{d\lambda}$ or $\frac{dx}{dt} < 0$

while if $n \cdot \frac{dx}{d\lambda} > 0$ then $\frac{dx}{dt} > 0$

Going back to our problem, if χ^μ is a null tangent killing vector to $r = R_+$ then

$$\chi \cdot p^{(2)} < 0 \quad \text{at } r = R_+$$ signals motion inwards,

but moreover since χ is a killing vector $\chi \cdot p^{(2)} = \text{constant}$.

Now $\vec{\chi} = \vec{\partial}_t + \Omega_H \vec{\partial}_\phi$ with $\Omega_H = \frac{a}{a^2 + R_+^2}$ is a null killing vector on $r = R_+$. Ω_H can be interpreted as the angular velocity of the black hole at the event horizon R_+, ~~as can be a~~, since it corresponds to the minimum ω for a massive particle at $r = R_+$.

Then the condition is $\chi \cdot p^{(2)} < 0$

But $\chi \cdot p^{(2)} = \vec{\partial}_t \cdot p^{(2)} + \Omega_H \vec{\partial}_\phi \cdot p^{(2)} = -E^{(2)} + \Omega_H L^{(2)} < 0$

$$\Rightarrow \quad L^{(2)} < \frac{E^{(2)}}{\Omega_H} < 0$$

So the angular momentum of the black hole decreases by $L^{(2)}$.

$$\delta M = E^{(2)}$$
$$\delta J = L^{(2)}$$

and $$\delta J < \frac{\delta M}{\Omega_H}$$

Conclusion: Energy is extracted from the black hole. As a result the black hole looses ~~energy~~ mass and spin.

However, the process cannot violate the <u>area theorem</u>, that the area of the event horizon is non-decreasing.

The area of the event horizon is

$$A = 4\pi (R_+^2 + a^2)$$

[Calculate from $ds_2^2 = \gamma_{ij}\, dx^i dx^j = ds^2$ at $dt = dr = 0$, $r = R_+$.

then $A = \int |\det\gamma|^{1/2} d\theta\, d\phi$

$$= \int \sqrt{\rho^2 \cdot \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2 \Delta \sin^2\theta\right]}\, d\theta\, d\phi$$

at $r = R_+$, we have $\Delta = 0$ so

$$A = (R_+^2 + a^2) \int \sin\theta\, d\theta\, d\phi \quad] .$$

To show that A is non-decreasing, define the "irreducible mass" throug

$$M_{irr}^2 = \frac{A}{16\pi G^2} = \frac{1}{4G^2}(R_+^2 + a^2)$$

$\Delta = 0 \Rightarrow R_+^2 + a^2 = 2GMR_+$

$R_+ = GM + \sqrt{(GM)^2 - a^2}$

$$= \frac{1}{4G^2} 2GM\left[GM + \sqrt{(GM)^2 - a^2}\right]$$

or $$M_{irr}^2 = \frac{1}{2}\left[M^2 + \sqrt{M^4 - (J/G)^2}\right]$$ $(J = Ma)$

Now

$$2\cdot 2M_{irr}\,\delta M_{irr} = 2M\delta M + \frac{1}{2}\frac{4M^3\delta M - 2J\delta J/G^2}{\sqrt{M^4-(J/G)^2}}$$

$$= \frac{2\left(M\sqrt{M^4-(J/G)^2}+M^3\right)\delta M - J\delta J/G^2}{\sqrt{M^4-(J/G)^2}}$$

we recognize

$$M^3 + M\sqrt{M^4-(J/G)^2} = M\left(M^2+\sqrt{M^4-(J/G)^2}\right)$$

$$= 2M\,M_{irr}^2$$

$$= 2M\,\frac{1}{4G^2}(R_+^2+a^2)$$

$$= \frac{2Ma}{4G^2}\,\frac{1}{\Omega_H}$$

$$= \frac{J}{2G^2}\,\frac{1}{\Omega_H}$$

so

$$\delta M_{irr} = \frac{J/G^2\left[\,\delta M/\Omega_H - \delta J\,\right]}{4M_{irr}\sqrt{M^4-(J/G)^2}}$$

so our bound that $\delta J < \delta M/\Omega_H$ implies $\delta M_{irr} > 0$

Now $\delta A = 16\pi G^2\,\delta M_{irr}^2 = \dfrac{8\pi J\left[\,\delta M/\Omega_H - \delta J\,\right]}{\sqrt{M^4 - J^2/G^2}}$

or

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H\,\delta J$$

where $\kappa = \dfrac{\sqrt{G^2M^2 - J^2/M^2}\;\Omega_H}{J/M} = \dfrac{\sqrt{G^2M^2-a^2}}{R_+^2+a^2}$

or $$\kappa = \frac{\sqrt{G^2M^2-a^2}}{2GM\left(GM+\sqrt{(GM)^2-a^2}\right)}$$

[Note: κ is the surface gravity of the Kerr metric. For a Killing horizon with Killing (null) vector $\vec{\chi}$, the surface gravity is

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)$$

study this later]

Now

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$$

is just like

$$dE = T\,dS - p\,dV$$

for a thermodynamic system, with the association

$$E \leftrightarrow M$$

$$\frac{A}{4G} \leftrightarrow S$$

$$T \leftrightarrow \frac{\kappa}{2\pi}$$

The ambiguity in the association of A & T (where do we put the $8\pi G$) is settled by Hawking's black hole evaporation.

	Thermodynamics	Black Holes
0th Law:	T is constant in thermal equilibrium	Stationary black holes have constant κ.
1st Law!		Energy conservation
2nd Law:	$\delta S > 0$	$\delta A > 0$

Generalized 2nd Law $\delta\left(S + \frac{A}{4G}\right) > 0$.

Note: To make sense of units, S is dimensionless ($k_B = 1$) but $\frac{A}{4G}$ has units of mass × length, same as $\hbar$. So, it really should be

$S \leftrightarrow \frac{A}{4\hbar G}$ and $T \leftrightarrow \frac{\kappa\hbar}{2\pi}$ (or $\hbar \to h$).

Stationary axisymmetric space: general observations.

(i) General case: require $g_{\mu\nu}=g_{\mu\nu}(r,\theta)$ (not of t,ϕ),

and symmetry $t\to -t$, $\phi \to -\phi$ (so $g_{tr}=g_{t\theta}=0=g_{\phi r}=g_{\phi\theta}$).

$$\Rightarrow \quad ds^2 = -\tilde{A}\,dt^2 + B\,d\phi^2 - 2B\omega\,dt\,d\phi + C\,dr^2 + D\,d\theta^2$$

$$= -A\,dt^2 + B(d\phi - \omega\,dt)^2 + C\,dr^2 + D\,d\theta^2 \qquad \tilde{A} = A - B\omega^2$$

Note that

$$g^{rr} = \frac{1}{C}, \; g^{\theta\theta} = \frac{1}{D} \quad \text{if } G = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \Rightarrow G^{-1} = \frac{1}{\det G}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}, \quad \det G = g_{tt}g_{\phi\phi} - g_{t\phi}^2$$

$$= -(A - B\omega^2)B - (B\omega)^2$$

$$= -AB$$

$$\Rightarrow \quad g^{tt} = -\frac{1}{A} \qquad g^{\phi\phi} = \frac{A - B\omega^2}{AB} \qquad g^{t\phi} = -\frac{\omega}{A}$$

For Kerr, plug into $R_{\mu\nu}=0$. Huge mess!

(ii) Killing vectors $\vec{\partial}_t$, $\vec{\partial}_\phi$ $\Rightarrow$ conserved quantities

$$L = p_\phi = m\,g_{\phi\mu}\frac{dx^\mu}{d\tau} \quad \text{and} \quad E = p_t = m\,g_{t\mu}\frac{dx^\mu}{d\tau}$$

or, replace $\tau/m \to \lambda$, for massless.

More explicitly

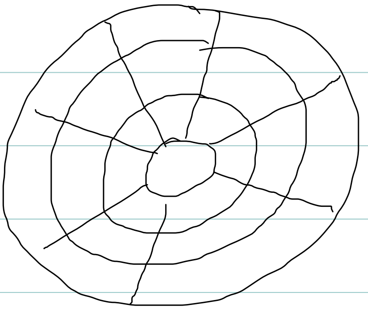
$$L = g_{\phi\phi}\frac{d\phi}{d\lambda} + g_{\phi t}\frac{dt}{d\lambda}$$

$$E = g_{t\phi}\frac{d\phi}{d\lambda} + g_{tt}\frac{dt}{d\lambda}$$

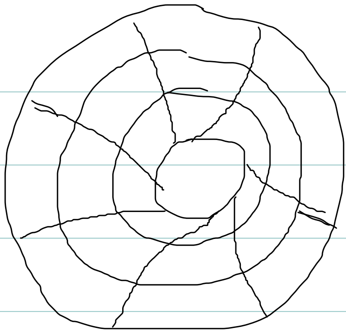
$\underline{L=0}$: $\dfrac{d\phi}{dt} = -\dfrac{g_{\phi t}}{g_{\phi\phi}} = \omega(r,\theta)$ ANGULAR VELOCITY WITHOUT ANGULAR MOMENTUM!

Suppose metric is asymptotically flat (as in Kerr). Then "drop" body from ∞ towards center (from $r=\infty$ towards $r=0$) with $\frac{d\phi}{dt}=0$ originally (since $\omega(r,\theta)\big|_{r=\infty}=0$). Then $\frac{d\phi}{dt}$ will change as body drops.

"Dragging of inertial frames": our test body is free falling, so locally it is moving in straight line → interpret $\frac{d\phi}{dt}\neq 0$ as moving/rotating inertial frames.



no dragging



dragging

iii) Stationary limit surface

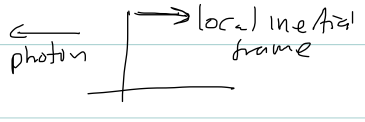
Consider photon emitted in ϕ- direction (from (r, θ, ϕ)).

At emission $d\theta = 0 = dr \Rightarrow ds^2 = 0 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2$

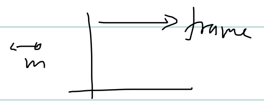
$$\Rightarrow \quad \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \sqrt{\frac{A}{B}}$$

- While $g_{tt}/g_{\phi\phi} < 0$ get $< \begin{matrix} d\phi/dt > 0 \rightarrow \gamma \text{ emitted in + dir} \\ d\phi/dt < 0 \rightarrow \gamma \text{ emitted in − dir} \end{matrix}$

- On a $g_{tt} = 0$ surface $\left(\frac{A}{B} = \omega^2\right)$ $\frac{d\phi}{dt} = \begin{cases} 2\omega & \rightarrow \text{twice as fast?} \\ 0 & \rightarrow \text{going nowhere?!?} \end{cases}$



so for massive particles



Massive particles all dragged in same direction on $g_{tt} = 0$ surface.

"stationary limit surface" = any surface with $g_{tt} = 0$

Q: Schwarszchild? (leave for student to ponder)

Inside stationary limit surface all bodies and radiation are forced to move in same direction, cannot remain fixed.

To see this

Suppose U^μ is 4-vel of body, $U^2=-1$. If we take $U^\mu=(U^t,0,0,0)$
$\Rightarrow$ $g_{tt}=-1/(U^t)^2<0$, incompatible with interior of lim. surf.
But nothing wrong with $g_{tt}(U^t)^2+2g_{t\phi}U^tU^\phi+g_{\phi\phi}U^{\phi 2}=-1$ since $g_{t\phi}=-\omega g_{\phi\phi}$
and the relative sign of U^t & U^ϕ not fixed. Just off $g_{tt}=0$ we neglect $g_{tt}(U^t)^2$
and get $U^\phi(U^\phi-2\omega U^t)=-1/g^2_{\phi\phi}<0$ which is easily satisfied.

(iv) Redshift: recall for comoving observers (fixed coordinates)
emitting/receiving light,

$$\frac{\lambda_{rec}}{\lambda_{emit}}=\sqrt{\frac{g_{tt}(rec)}{g_{tt}(emit)}}$$

For an observer at stationary limit surface, $g_{tt}(emit)\to 0$ $\Rightarrow$ $\lambda_{rec}\to\infty$.
This is just as with Schwarzschild

(v) Event Horizons. Again we look for null 3-surfaces.

$f(x^\mu)=0$ defines surface

$\partial_\mu f$ = gradient = normal to surface = n_μ

Tangent $f(x^\mu(\lambda))=0 \Rightarrow 0=\frac{d}{d\lambda}f(x^\mu(\lambda))=\frac{dx^\mu}{d\lambda}\partial_\mu f \Rightarrow \frac{dx^\mu}{d\lambda}n_\mu=0 \Rightarrow$ vectors "$\perp$" to n_μ.

In particular, if n_μ is null then $n^\mu=g^{\mu\nu}n_\nu$ is $\perp$ to n_μ ($g^{\mu\nu}n_\mu n_\nu=0$).
$\Rightarrow$ Look for $g^{\mu\nu}\partial_\mu f\,\partial_\nu f=0$

Recall in spherically symmetric case we take $f=f(r)$ $\Rightarrow$

$$g^{\mu\nu}\partial_r f\,\partial_r f=0 \Rightarrow g^{rr}(\partial_r f)^2=0 \Rightarrow g^{rr}=0$$

Now, with axial symmetry, $f=f(r,\theta)$, $g^{rr}(\partial_r f)^2+2g^{r\theta}\partial_r f\,\partial_r\theta+g^{\theta\theta}(\partial_\theta f)^2=0$
We can still look for solutions with $f=f(r)$ and $g^{rr}=0$. Let's look at this (and previous) in Kerr metric

Back to Kerr

(i) Singularities. From $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ one finds $\rho = 0$ is a singularity.

Now $\rho^2 = r^2 + a^2\cos^2\theta = 0 \Rightarrow r = 0 \quad \theta = \frac{\pi}{2}$

Recall Boyer-Lindquist coordinates in Cartesian:

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$
$$z = r\cos\theta$$

so $r=0$ is $z=0$, $x^2+y^2 \le a^2$ disk:

$$x = (a\sin\theta)\cos\phi$$
$$y = (a\sin\theta)\sin\phi$$

$\Rightarrow \theta = \frac{\pi}{2}$ is $x^2+y^2=a^2$ circle ("equator?").

The singularity is not a point but S^1!

(ii) Event Horizons: look for $g^{rr} = 0$. Now $g^{rr} = \frac{\Delta}{\rho^2}$, so need $\Delta = 0$.

$\Rightarrow r^2 - 2GMr + a^2 = 0 \Rightarrow r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$

Note, if $|a| < GM \Rightarrow r_- < a < r_+$ and the singularity is behind the horizon (singularity at $r=0$, less than r_-).

For $|a| > GM$, no horizons, naked singularity.

$|a| = GM = r_\pm$ is "extreme Kerr B.H." [It is belived, through calculations, that realistic BH's are near extremal Kerr BH's, since aereation increases $a = J/M$. Limited only by accreting matter radiating away some angular momentum. Calculations give $a \gtrapprox 0.998GM$ - see text by Hobson, Efstathiou & Lasenby, p. 324).

Geometry: take $r = \text{const}\ (=r_\pm)$ $t = \text{const}$ 2-dim surface. Line element is

$$ds^2 = \rho_\pm^2\, d\theta^2 + \left(\frac{2GMr_\pm}{\rho_\pm}\right)^2 \sin^2\theta\, d\phi^2$$

Not the geometry of S^2 embedded in R^3. Rather a pancake, or more techically, an axisymmetric ellipsoid (embedded in R^3).

(iii) Stationary Limit Surface: $g_{tt}=0$

$$1 - \frac{2GMr}{\rho^2} = 0 \Rightarrow r^2 + a^2\cos^2\theta - 2GMr = 0$$

$$r_\pm(\theta) = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}$$

(Not the same $r_\pm$ as before, excuse the notation)

Geometry (as previous)

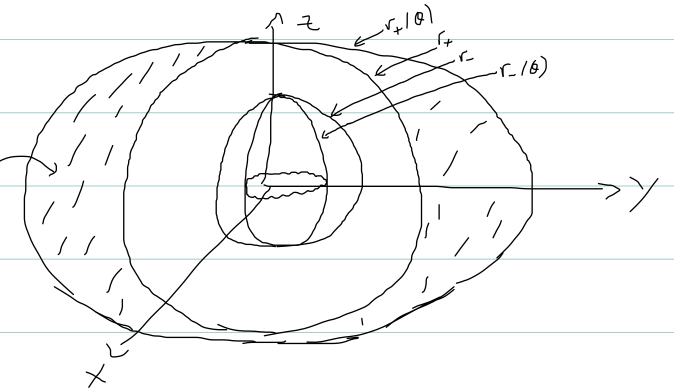
$$ds^2 = \rho_\pm^2\, d\theta^2 + \frac{2GMr_\pm(2GMr_\pm + 2a^2\sin^2\theta)}{\rho_\pm}\sin^2\theta\, d\phi^2$$

with $r_\pm = r_\pm(\theta)$
and $\rho_\pm = \rho_\pm(\theta)$
& $\rho_\pm^2 = 2GMr_\pm$.

Which is bigger? $r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$ or $r_\pm(\theta) = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}$

$\Rightarrow r_+(\theta) > r_+ > r_- > r_-(\theta)$, with $r_+(0,\pi) = r_+$ and $r_-(0,\pi) = r_-$ (equal at poles)

Also $r_-(\frac{\pi}{2}) = 0 \equiv$ singularity.



"ergosphere"
region with $g_{tt} > 0$
not behind event horizon.

In ergosphere everything is forced to move. As before $U^2 = -1$ with $U^\mu = (U^t, 0, 0, U^\phi)$

$\Rightarrow (U^t)^2 (g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2) = -1$ where $\Omega = \frac{U^\phi}{U^t} = \frac{d\phi}{dt}$

U^t real $\Rightarrow g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 < 0 \Rightarrow \Omega \in (\Omega_-, \Omega_+)$ with

$$\Omega_\pm = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \sqrt{\frac{A}{B}}$$

Special cases

(i) $A = 0$ ($g_{tt} = 0$) $\Rightarrow \Omega_- = 0$, $\Omega_+ = 2\omega$. This occurs at $r = r_+(\theta)$ (outer stat. lim. surf)
(already discussed)

(ii) $\omega^2 = g_{tt}/g_{\phi\phi}$ $\Rightarrow \Omega_- = \Omega_+ = \omega$. This occurs at $r = r_+$, so call this Ω_H.

We have $\Omega_H = \omega(r_+, \theta) = \frac{a}{2GMr_+}$ (from Kerr metric).

At the horizon the angular velocity is limited to the one value Ω_H.

Hawking radiation: baby version

In QFT, vacuum fluctuations: vac → $\gamma + \gamma$ → vac are virtual processes $E^{(1)} + E^{(2)} = 0$ so $E^{(1)} > 0$ $E^{(2)} < 0$ and photon's cannot propagate freely. But in ~~curved background~~ vicinity of horizon, imagine 2nd γ crosses horizon. Then even if $E^{(2)} < 0$, it must fall into singularity. The 1st photon escapes, taking energy away to asymptotically flat region.

Take a freely falling observer with 4-velocity $\vec{U}$. ~~In his locally flat system~~ {He is inertial, so he sees normal laws of quantum electrodynamics. He sees vacuum fluctuations. A fluctuation of energy E lasts no longer that

$$\Delta t \sim \frac{\hbar}{E}$$

{This is a bit backwards. To measure energy E need at least time Δt, $\Delta t \cdot E \gtrsim \hbar$}. If he is close to horizon, then he has limited time to measure and this sets the scale for energy of photon's he sees are pair created (virtual photons).

Say he starts free falling from $r = R + \epsilon$ $(R = 2GM)$. then,

$$-1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

and

$$\hat{E} = -\vec{\partial}_t \cdot \vec{U} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}$$

so

$$\left(\frac{dr}{d\tau}\right) = -\sqrt{\hat{E}^2 - \left(1 - \frac{2GM}{r}\right)}$$

Since $\frac{dr}{d\tau} = 0$ at $r = R + \epsilon$, we have

$$\hat{E}^2 = 1 - \frac{2GM}{R+\epsilon} = 1 - \frac{2GM}{2GM+\epsilon} = \frac{\epsilon}{2GM+\epsilon} \approx \frac{\epsilon}{2GM}$$

How much time does he have to observe photon creation before crossing $r = R$?

$$\Delta\tau = \int_0^{\Delta\tau} d\tau = \int_{R+\epsilon}^{R} dr \frac{1}{\frac{dr}{d\tau}} = + \int_{R+\epsilon}^{R} \frac{dr}{\sqrt{E^2 - (1 - \frac{2GM}{r})}}$$

$$= + \int_{2GM+\epsilon}^{2GM} \frac{dr}{\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R+\epsilon}}}$$

Mathematica can do full integral. We are happy with the approximation: with $\epsilon << 2GM$, change variables to $\xi = r - 2GM$

$$\frac{1}{2GM+\xi} - \frac{1}{2GM+\epsilon} = \frac{\epsilon - \xi}{(2GM+\xi)(2GM+\epsilon)} \simeq \frac{\epsilon - \xi}{(2GM)^2}$$

$$\Delta\tau \approx \sqrt{2GM}\int_\epsilon^0 \frac{d\xi}{\sqrt{\epsilon - \xi}} = 2\sqrt{2GM\epsilon}$$

So the energy of the photon created which escapes to ∞ is

$$\mathcal{E} = \frac{\hbar}{\Delta\tau} = \frac{\hbar}{2\sqrt{2GM\epsilon}} \quad \text{as observed in his frame.}$$

Now

$$\mathcal{E} = -\vec{p}\cdot\vec{U}$$

where $\vec{U}$ is for our falling observer. The energy E of the photon as observed at ∞ is

$$E = -\vec{p}\cdot\vec{\partial}_t$$

Or ~~$\frac{E}{\mathcal{E}} = \frac{\vec{p}\cdot\vec{U}}{\vec{p}\cdot\vec{\partial}_t}\Big|_{\text{at } r=R+\epsilon} = \frac{g_{tt}p^tU^t}{g_{tt}p^t} = \frac{\hat{E}}{1-\frac{2GM}{R+\epsilon}} = \frac{1}{\hat{E}} = \sqrt{\frac{2GM}{\epsilon}}$~~

~~$\Rightarrow E = \sqrt{\frac{2GM}{\epsilon}}.$~~

Computing the ratio at $r = R + \epsilon$

$$\frac{E}{\mathcal{E}} = \frac{\vec{p} \cdot \vec{\partial}_t}{\vec{p} \cdot \vec{U}} = \frac{g_{tt} p^t}{g_{tt} p^t U^t} = \frac{1}{U^t} = \frac{1 - \frac{2GM}{R+\epsilon}}{\hat{E}} = \hat{E} = \sqrt{\frac{\epsilon}{2GM}}$$

$$\Rightarrow \quad E = \sqrt{\frac{\epsilon}{2GM}} \cdot \frac{\hbar}{2\sqrt{2GM\epsilon}} = \frac{\hbar}{4GM}$$

An observer at ∞ sees a~~n energ~~ photon with energy

$$E \approx \frac{\hbar}{4GM}$$

This is independent of ϵ in the argument above! So we don't know exactly where the photon was emitted, but it does not matter.

A complete calculation shows the spectrum of photons is thermal with temperature $T = E/k_B$ with E as above.

Recall we had before that for black hole thermodynamics

$$T = \frac{\kappa}{\cancel{2\pi}} \quad \text{where } \kappa \text{ for Schwarzschild is } \kappa = \frac{1}{4GM}$$

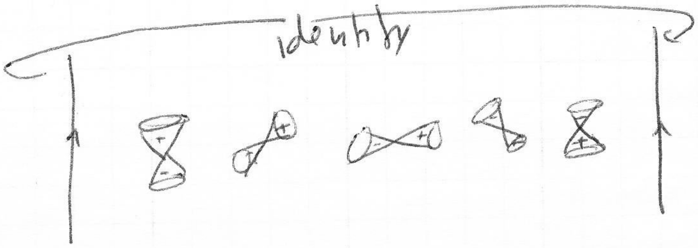
In units of $\hbar = 1 =$ [illegible] we see that this agrees with the previous!

Causal Structure

The followins definitions are for any spacetime (M, g_{ab})

(M, g_{ab}) is time orientable if as you vary continuously $p \in M$ the future lightcone at p can be continuously defined

Example: non-time-orientable

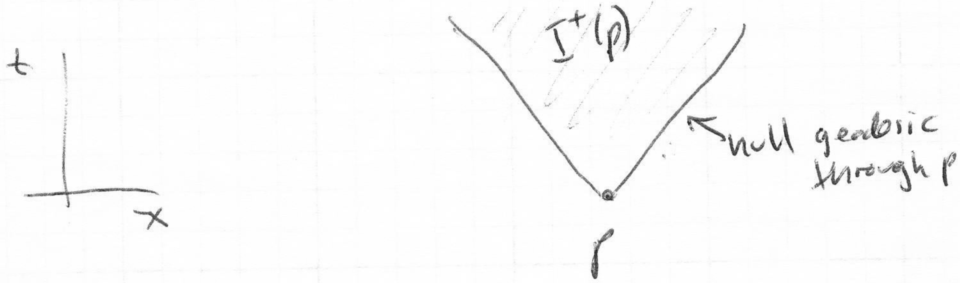


We assume (M, g_{ab}) is time orientable from here on.

time orientable $\Leftrightarrow$ there is a continuous timelike vector field

$I^+(p)$: Chronological future of p

set of points that can be reached from p by a future directed timelike curve

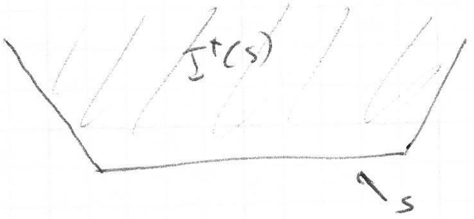


Notes

- $I^+(p)$ is open (the null geodesic cone is not in $I^+(p)$).
- Generally $p \notin I^+(p)$, but p may be in $I^+(p)$ if there is a closed timelike cur from p to p.

For any set S define $I^+(S) = \bigcup_{p\in S} I^+(p)$

~~Particularly it~~ $I^+(S)$ is also open. ~~and I^+I^+(s)~~ We'll be interested in surfaces S



Define also $I^-(p)$ & $I^-(S)$ (replace "future" by "past")

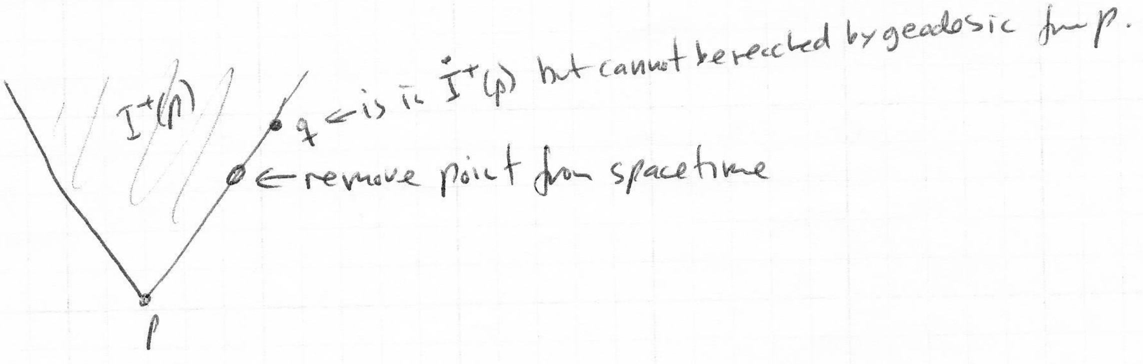
and $J^\pm(p)$ & $J^\pm(S)$ (replace "timelike curve" by "causal curve").
(ie, non-spacelike

$\dot{I}^+(p) \equiv$ boundary of $I^+(p)$

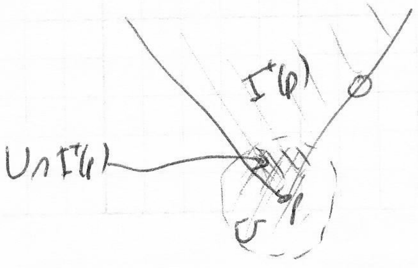
In Minkowski spacetime $I^+(p)$ is the set of points that can be reached by future directed timelike geodesics starting at p.

and $\dot{I}^+(p)$ is ~~geo~~ future light cone (generated by future null geodesics).

This is locally true in any spacetime, but not necessarily globally. An artificial example



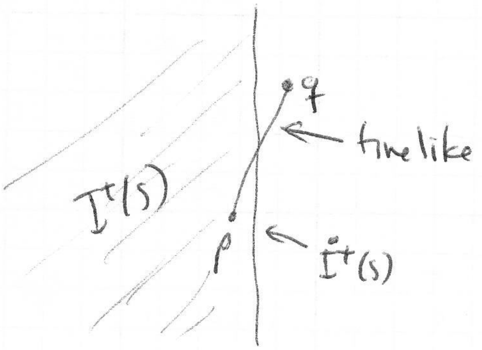
Still for $U \subset M$ small enough, $p \in U$



still holds there.

Some fun "theorems"

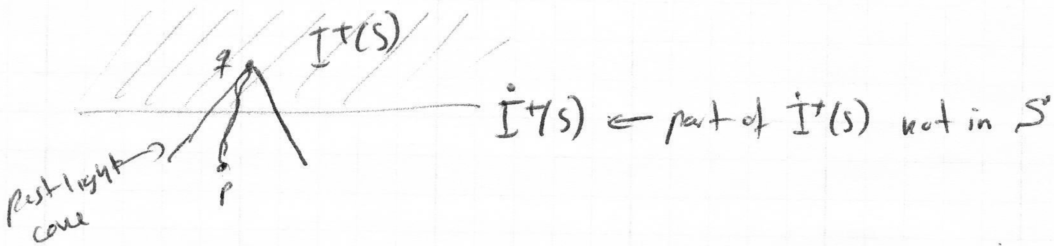
☆ $\dot{I}^+(S)$ cannot be timelike:



so $q \in I^+(S)$

but $q \notin I^+(p)$, a contradiction

☆ $\dot{I}^+(S)$ cannot be spacelike, except for the set S itself



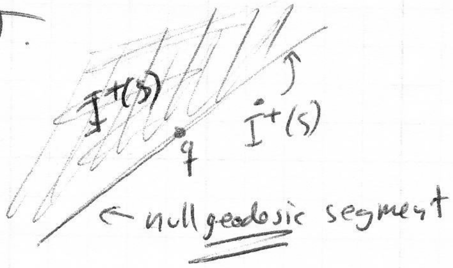
no point p in q's past light cone is in ~~I^+~~ S
⇒ a contradiction.

☆ Therefore $\dot{I}^+(S)$ is null, apart from S itself.

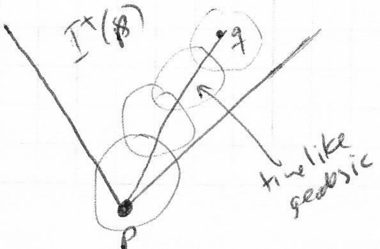
~~Moreover, to be precise~~

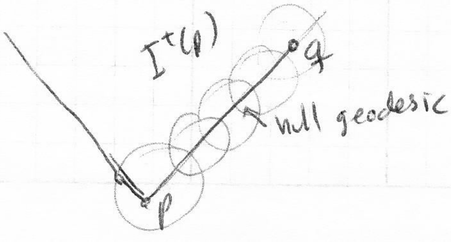
~~(If $q \in \dot{I}^+(S)$ but $q \notin \bar{S}$ ⇒ $\exists$ past directed null geodesic through q lying on $\dot{I}^+(S)$).~~

picture



Extra information: $\dot{I}^+(S)$ is generated by null ~~t~~ geodesic segments (we did not get this above, we just got that $\dot{I}^+(S)$ is null (up to S)). The proof uses the fact that it is locally true (see previous page) and that one can show one can find a finite cover with p, q





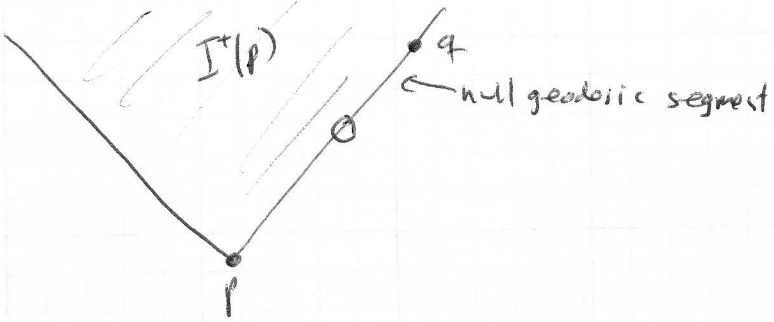
This does not conflict with taking points out of space if one phrases it carefully, for example:

- If $q \in J^+(p) - I^+(p) \Rightarrow$ any causal curve from p to q is a null geodesic

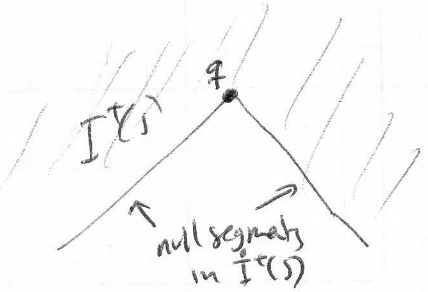
or

- If $q \in \dot{I}^+(S)$ but $q \notin S \Rightarrow$ there is a past directed null geodesic segment through q lying on $\dot{I}^+(S)$.

So in our example



If there is more than one past directed null geodesic segment through q lying on $\dot{I}^+(S) \Rightarrow q$ is the endpoint of the segments



So the structure of $\dot{I}^+(S)$ is ~~as fo~~llows:

- it is generated by null geodesic segments that
  - have past endpoints only on S
  - have future endpoints in the boundary (and would then pass into the interior of $I^+(S)$) if they intersect another generator.
  - may have no endpoints

Example: Minkowski space with a horizontal line segment (in 1+1 dim) removed