225B - Winter 2014

REVIEW

## Maps of Manifolds

$\phi : M \to M'$ is a map between manifolds

(is a $C^r$ map if the corresponding map of coordinates is $C^r$

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle Y} \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

so $Y \circ \phi \circ X^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is $C^r$)

Notes:

- In grnl not one-to-one
- Even if one-to-one may not have an inverse

So it goes one way.

Let $f : M' \to \mathbb{R}$ a fction on $M'$

(a "scalar field")

then $\phi$ defnes a fction on $M$, $\phi^* f$

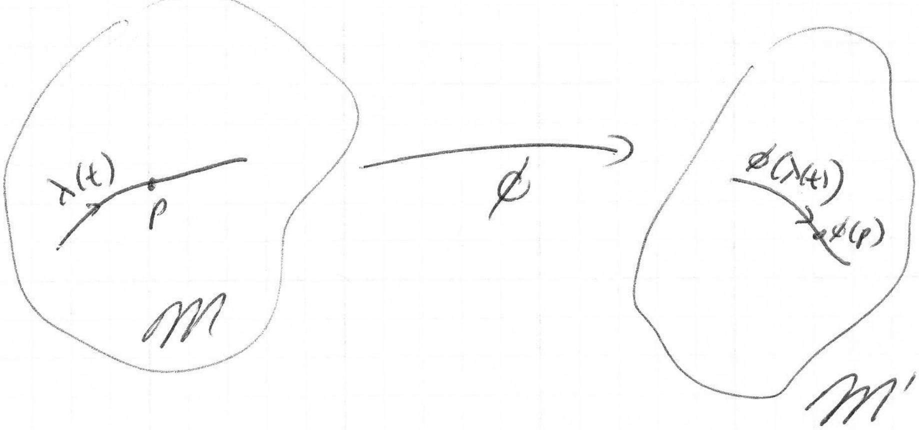
$$\phi^* f : M \to \mathbb{R}$$

defined by $M \xrightarrow{\phi} M' \to \mathbb{R}$

ie if $p \in M$ then $\phi(p) \in M'$ and $f(\phi(p))$ is defined.

(This is a "pull-back" of a zero form)

Go the other direction




By mapping curves $\lambda(t)$ in $\mathcal{M}$ into $\mathcal{M}'$ we can get maps of tangent vectors.

If $T_p(\mathcal{M})$ is the tangent space to $\mathcal{M}$ at $p$ then

$$\phi_* : T_p(\mathcal{M}) \rightarrow T_{\phi(p)}(\mathcal{M}') \qquad \text{push-forward}$$

defined by mapping $\left(\frac{\partial}{\partial t}\right)_\lambda \longrightarrow \left(\frac{\partial}{\partial t}\right)_{\phi(\lambda)}$ $\leftarrow$ denote this by $\phi_*\left(\frac{\partial}{\partial t}\right)_\lambda\Big|_{\phi(p)}$

This is a linear transformation between the vector spaces: if $x^\mu$ and $y^a$ are local coordinates on patches of $\mathcal{M}$ & $\mathcal{M}'$, then the curve is $x^\mu(t)$, mapped into $y^a(x^\mu(t))$ and

$$\frac{dy^a}{dt}\Big|_0 = \frac{\partial y^a}{\partial x^\mu}\Big|_p \frac{dx^\mu}{dt}\Big|_0$$

or
$$N^a = \frac{\partial y^a}{\partial x^\mu}\Big|_p M^\mu \qquad \text{where } \vec{N} \in T_{\phi(p)}(\mathcal{M}'), \ \vec{M} \in T_p(\mathcal{M})$$

so $\phi_*$ is just the matrix $\frac{\partial y^a}{\partial x^\mu}\Big|_p$. and we write $\vec{N} = \phi_* \vec{M}$

Since a vector $\vec{M}$ is a directional derivative, we have $\vec{M}(f)$ defined.

(A vector gives a map of any function $f$ at $p$ into a number. If $\vec{M} = \frac{\partial}{\partial t}$ then $\vec{M}(f) = \frac{\partial f}{\partial t}\Big|_{p=\lambda(t_0)}$ ie the derivative along $\lambda(t)$

Explicitly $\frac{df(x^\gamma(t))}{dt} = \frac{\partial f}{\partial x^\mu}\dot{x}^\mu$ so the action of the vector $\vec{A}$ with coordinates $a^\mu$ on $f$ is $\vec{A}(f) = a^\mu \partial_\mu f|_p$ ).

$$\begin{tikzcd} T_p(M) \arrow[r, "\phi_*"] & T_{\phi(p)}(M') \end{tikzcd} \quad \text{defines } \vec{N}(f) = \phi_* \vec{M}(f)$$

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[rr, bend right, "\phi^* f"'] & M' \arrow[r, "f"] & R \end{tikzcd}$$

defines $\vec{M}(\phi^* f)$

$$\vec{M}(\phi^* f)\Big|_p = \phi_* \vec{M}(f)\Big|_{\phi(p)}$$

check:

$$m^\mu \partial_\mu (f(y(x)))\Big|_p = m^\mu \frac{\partial f}{\partial y^a}\Big|_{\phi(p)} \frac{\partial y^a}{\partial x^\mu}\Big|_p = \underbrace{\left(m^\mu \frac{\partial y^a}{\partial x^\mu}\right)}_{n^a \text{ components of } \phi_* \vec{M}} \frac{\partial f}{\partial y^a}\Big|_{\phi(p)}$$

Students should always flesh out relations in terms of coordinate patches, to make sure they understand.

Go on to 1-forms: define pull-back

$$\phi^*: \; T^*_{\phi(p)}(M') \to T^*_p(M)$$

by requiring the contraction is mapped properly: $\phi^*: \tilde{\omega} \to \phi^*\tilde{\omega}$

$$\left( \phi^*: \quad \tilde{\omega} \in T^*_{\phi(p)}(M') \longrightarrow \phi^*\tilde{\omega} \in T^*_p(M) \right)$$

with $\boxed{\tilde{\omega}(\phi_* \vec{M}) = \phi^*\tilde{\omega}(\vec{M})}$ ~~defines $\phi^*\tilde{\omega}$~~

~~Recall~~

$$T_p \xrightarrow{\phi_*} T_{\phi(p)}$$

$$M \xrightarrow{\phi} M'$$

$$T^*_p \xleftarrow{\phi^*} T^*_{\phi(p)}$$

Recall $\tilde{\omega}(\vec{N})$ is a number, ie $\tilde{\omega}$ is a map from $T_p \to \mathbb{R}$.
(In components $\tilde{\omega}(\vec{N})\big|_p = \omega_\mu\big|_p N^\mu\big|_p$, the index contraction.
Some texts write $\langle \tilde{\omega}, \vec{N} \rangle$).

So the ~~def~~ above gives the action of $\phi^*\tilde{\omega}$ on vectors $\vec{M} \in T(M)$ in terms of the action of $\tilde{\omega}$ on vectors $\vec{N} \in T(M')$. ~~which is~~

(In components $(\phi^*\tilde{\omega})_\mu M^\mu = \omega_a N^a = \omega_a \dfrac{\partial y^a}{\partial x^\mu} M^\mu$

that is $(\phi^*\tilde{\omega})_\mu = \omega_a \dfrac{\partial y^a}{\partial x^\mu}$).

In particular $\phi^*(df) = d(\phi^* f)$

(In components $df = f_{,a}\, dy^a \qquad \phi^*(df) \cancel{=} f_{,a} \dfrac{\partial y^a}{\partial x^\mu} dx^\mu$

while $d(\phi^* f) = df(y(x)) = \left( \dfrac{\partial f}{\partial y^a} \dfrac{\partial y^a}{\partial x^\mu} \right) dx^\mu$ ✓ ).

Clearly this can be extended to tensors of type $T^r_0$ and $T^0_r$

$$\phi_* : T^r_0(p) \to T^r_0(\phi(p))$$

recall $T \in T^r_0(p)$ ~~is~~ acts on $r$ 1-forms $T(\tilde{\omega}^1, \dots, \tilde{\omega}^r) \in \mathbb{R}$

so $T \to \phi_* T$ by $T(\phi^*\tilde{\omega}^1, \dots, \phi^*\tilde{\omega}^r) = \phi_* T(\tilde{\omega}^1, \dots, \tilde{\omega}^r)$

And $T \in T^0_r$ acts on $r$ vectors so $\phi^* : T^0_r(\phi(p)) \to T^0_r(p)$

$$\phi^* T(\vec{M}_1, \dots, \vec{M}_r) = T(\phi_*\vec{M}_1, \dots, \phi_*\vec{M}_r)$$

---

In components $\phi_*$: $T^{a_1 \dots a_r} = \dfrac{\partial y^{a_1}}{\partial x^{\mu_1}} \dots \dfrac{\partial y^{a_r}}{\partial x^{\mu_r}} T^{\mu_1 \dots \mu_r}$

$$\phi^* : \quad T_{\mu_1 \dots \mu_r} = \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} T_{a_1 \dots a_r}$$

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<u>Defs</u> <u>Rank</u> : $\phi$ is $\mathcal{M} \to \mathcal{M}'$ is rank $k$ (at $p$) if the dimension of the tangent space at $\phi(p)$ $(\phi_*(T_p(\mathcal{M}))$ is $k$.

<u>Injective</u> $\phi$ above is injective (at $p$) if rank = dimension of $\mathcal{M}$

$$k = n.$$

(In this case $n \le n'$).

Exercise: If $\phi$ is injective then no non-zero vectors in $T_p(\mathcal{M})$ are mapped to zero by $\phi_*$

<u>Surjective</u>: $\phi$ is surjective at $p$ if rank of $\phi$ = dimension of $\mathcal{M}'$

$k = n'$

(So that $n \geq n'$)

(~~<u>Immersion</u>: $\phi$ is an immersion if it has an inverse $\phi^{-1}$, (with same differentiability as $\phi$) such that for each $p \in \mathcal{M}$ there is $\mathcal{U} \subset \mathcal{M}$ with $p \in \mathcal{U}$ $\ni$ $\phi^{-1}: \phi(\mathcal{U}) \to \mathcal{U}$~~

(Skip immersion: it ~~depe~~ is subtle only when $C^r$ properties matter)

If $\phi$ is injective $\forall p \in \mathcal{M}$ we say $\phi$ is an immersion (actually, defn of immersion is ~~of~~ given in terms of existance of differentiable inverse of $\phi$, and then equivalence of statements is proved) $\Longrightarrow$ $\phi_*: T_p \to \phi_*(T_p) \subset T_{\phi(p)}$ is an isomorphism.

Then $\phi(\mathcal{M}) \subset \mathcal{M}'$ is an $n$-dimensional <u>immersed</u> submanifold in $\mathcal{M}'$.

This is one-one locally, but may not be so globally.
An imbedding is, basically, an immersion that is one-one (actully a homeomorphism onto its image).

<u>Diffeomorphism:</u> one-to-one map $\phi : \mathcal{M} \to \mathcal{M}'$ with inverse $\phi^{-1} : \mathcal{M}' \to \mathcal{M}$.

Then $n = n' = k$, $\phi$ is injective and surjective.

Thm: If $\phi_*$ is injective and surjective at $p$ then there is an open $U \in \mathcal{M}$, $p \in U$ + $\phi : U \to \phi(U)$ is a diffeomorphism.

That is if $\phi_* : T_p \to T_{\phi(p)}$ is an isomorphism then $\phi$ is a local diffeomorphism.

With a diffeomorphism we can go with $\phi_* : T_p(\mathcal{M}) \to T_{\phi(p)}(\mathcal{M}')$ and with $(\phi^{-1})^* : T_p^*(\mathcal{M}) \to T^*_{\phi(p)}(\mathcal{M}')$

So for any tensor in $T^r_s$

$$T(\tilde{\omega}^1, \ldots, \tilde{\omega}^s, \vec{M}_1, \ldots, \vec{M}_r)\Big|_p = \phi_* T\left((\phi^{-1})^*\tilde{\omega}^1, \ldots, (\phi^{-1})^*\tilde{\omega}^s, \phi_* M_1, \ldots \phi_* M_r\right)\Big|_{\phi(p)}$$

# Differentiation without a connection

Two types arise naturally:

- Exterior derivative
- Lie Derivative

Exterior derivative $d : \Omega_s \to \Omega_{s+1}$

$\Omega_s$ : linear space of s-forms $\tilde{a} = a_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots dx^{\mu_n}$

($\Omega_s \subset T^0_s$ , is the totally antisymmetric $T^0_s$ tensors).

Recall if $\tilde{a}$ & $\tilde{b}$ are p & q forms , $\tilde{a} \wedge \tilde{b} = (-1)^{pq}\, \tilde{b} \wedge \tilde{a}$ .

d acts by

$$d\tilde{a} = da_{\mu_1 \cdots \mu_s} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_s}$$

$$= \frac{\partial a_{\mu_1 \cdots \mu_s}}{\partial x^{\sigma}} dx^{\sigma} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_s}$$

Exercise: show

- this is indeed a $T^0_{s+1}$ (tensor) (obvious from first line).
- $d(a \wedge b) = d\tilde{a} \wedge \tilde{b} + (-1)^s a \wedge db$ if a is an s-form
- $d(d\tilde{a}) = 0$
- $d(\phi^* \tilde{a}) = \phi^*(d\tilde{a})$

Useful integration results (reminder)

if $\phi$ is a diffeomorphism and $\tilde{a}$ is an n-form ($n = \dim M$)

$$\int_M \tilde{a} = \int_{M' = \phi(M)} \phi_* \tilde{a}$$

If $\tilde{b}$ is an n-1 form

$$\int_{\partial M} \tilde{b} = \int_M d\tilde{b}$$

Stoke's theorem.

# Lie derivative

~~Let~~ $\vec{M}$ vector field on $\mathcal{M}$

Thm. (Fund. th. of diff. eqs) $\Longleftrightarrow$ unique curve $\lambda(t)$ ~~point~~ through $p$ with $\lambda(0) = p$ and $\vec{M} = \frac{\partial}{\partial t}$

~~With~~ Locally, with coordinates $x^\mu$, $\lambda(t)$ is $x^\mu(t)$ ~~and~~ with tangent $\frac{dx^\mu}{dt}$; so the theorem above is the statement of uniqueness of solution of

$$\frac{dx^\mu(t)}{dt} = M^\mu(x(t))$$


$\lambda(t)$ is the "integral curve of $\vec{M}$"

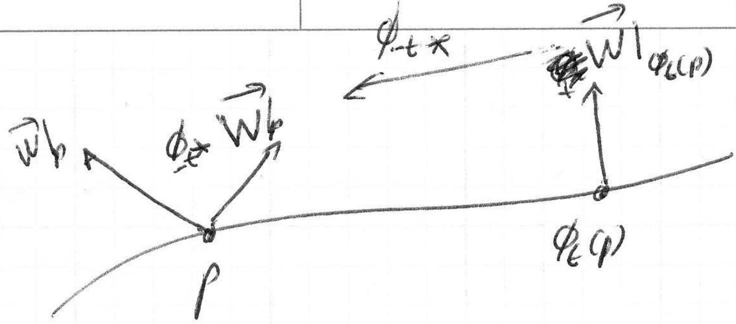
Given $\vec{M}$ we can construct a diffeomorphism $\phi_t$ of $\mathcal{M}$ into itself (actually from small open neighborhoods $U \ni p$ into $\mathcal{M}$), that maps $p$ into the point along the curve a ~~distance t~~ parameter $t$ away


(Note $\phi_t$ forms a one parameter local group of diffeomorphisms.

$$\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t \qquad \phi_{-t} = (\phi_t)^{-1} \qquad \phi_0 = e \text{ identity})$$

From $\phi_t$ construct $\phi_{t*} : T^r_s(p) \to T^r_s(\phi(p))$

$$T|_p \mapsto \phi_{t*} T|_{\phi(p)}$$




Since $\phi_{t*}$ is a diffeomorphism, $\phi_{t*}$ is an isomorphism, we can directly compare $\phi_{t*}\vec{W}$ with $T$. Define Lie derivative at $p$ by

$$\mathcal{L}_{\vec{M}} T = \lim_{t \to 0} \frac{1}{t}\left[ \phi_{-t*} T_p - T_p \right]$$

Note: both at $p$.

Properties:

(i) If $T \in T^r_s(p) \Rightarrow \mathcal{L}_M T \in T^r_s(p)$

(ii) $\mathcal{L}_{\vec{M}}$ is linear

(iii) $\mathcal{L}_{\vec{M}}$ preserves contractions

(iv) $\mathcal{L}_{\vec{M}}(T \otimes S) = \mathcal{L}_{\vec{M}} T \otimes S + T \otimes \mathcal{L}_{\vec{M}} S$

(v) $\mathcal{L}_{\vec{M}} f = \vec{M}(f)$ ($f$ a function $f: M \to \mathbb{R}$).

## $\mathcal{L}_V W$

Start from
$$\begin{array}{ccc} M & \xrightarrow[\phi_t]{} & M \\ x\downarrow & & y\downarrow \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

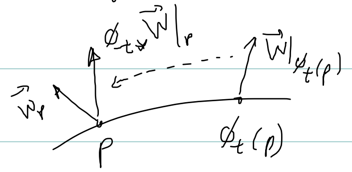
with $\phi_t$ the integral curve of $\vec{V}$, $\dfrac{dx^\mu}{dt} = V^\mu(x(t))$

For small $t$ this is $x^\mu(t) = x^\mu(0) + t V^\mu(x^\nu(0))$.

Starting from $x^\mu$, the coordinate for $p$, this is $x^\mu(t) = x^\mu + t V^\mu(x^\nu)$.

So $y^\mu = x^\mu + t V^\mu(x^\nu)$ (to order $t$).

Now for $\mathcal{L}_V W$ we need:




$$\begin{array}{ccc} M & \xrightarrow{\phi_{-t}} & M \\ y\downarrow & & x\downarrow \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

$x^\mu = y^\mu - t V^\mu(x)$ (or $y$, difference is higher order in $t$).

So we take the vector field $\vec{W}$ at $\phi_t(p)$, $W^\mu(x^\mu + tV^\mu)$ and push forward by $\phi_{-t}$: $(\phi_{-t*}W)^a\big|_p = \dfrac{\partial x^a}{\partial y^\mu}\Big|_{\phi_t(p)} W^\mu\Big|_{\phi_t(p)}$, where $x^a = y^a - t V^a(x)$

with $y$ the coordinate of $\phi_t(p)$.

That is, $\dfrac{\partial x^a}{\partial y^\mu}\Big|_{\phi_t(p)} = \delta^a_\mu - t V^a{}_{,\mu}\Big|_{\phi_t(p)}$

So in terms of coordinates at $p$,
$$(\phi_{-t*}W)^a\big|_p = \left(\delta^a_\mu - t V^a{}_{,\mu}(x+tV)\right) W^\mu(x+tV)$$

Now $\mathcal{L}_V W = \lim_{t\to 0} \frac{1}{t}\left[(\phi_{-t*}W^a)\big|_p - W^a\big|_p\right] = V^\mu \partial_\mu W^a - \partial_\mu V^a W^\mu$

Note, with $\vec{V} = V^\mu \partial_\mu$ and $\vec{W} = W^\mu \partial_\mu$ then

$[V^\lambda\partial_\lambda, W^\nu\partial_\nu] = V^\lambda\partial_\lambda W^\nu\partial_\nu - W^\lambda\partial_\lambda V^\nu\partial_\nu = (V^\lambda\partial_\lambda W^\nu - W^\lambda\partial_\lambda V^\nu)\partial_\nu$

$\Rightarrow \mathcal{L}_{\vec{V}}\vec{W} = [\vec{V}, \vec{W}]$

Look closely at $\mathcal{L}_V(f)$

Recall $\quad M \xrightarrow[\phi]{} M' \xrightarrow[f]{} \mathbb{R} \Rightarrow \phi^* f : M \to \mathbb{R}$ is $\phi^* f = f \circ \phi$

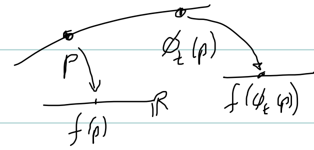
or $\phi^* f(p) = f(\phi(p))$.

Also if $\phi$ is a diffeomorphism then $M \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} M'$

The push forward of the inverse is just the pull-back:

$(\phi^{-1})_* f : M \to \mathbb{R}$. is $\phi^* f : M \to \mathbb{R}$

For $\mathcal{L}_V f$ we need $(\phi_{-t*} f)\big|_p$ :




$(\phi_{-t*} f\big|_p$ just says it's the function that maps p to the image under f of $\phi(p)$

or $f(\phi_t(p)) \Rightarrow \mathcal{L}_V(f) = \frac{1}{t}\left(f(\phi_t(p)) - f(p)\right) = \frac{1}{t}\left(f(x+Vt) - f(x)\right) = V^\mu \partial_\mu f$

Formally $\quad M \xrightarrow[\phi_t]{} M \xrightarrow{f} \mathbb{R} \quad$ defines $\phi_t^* f(p) = f(\phi_t(p))$

The def of $\mathcal{L}_V$: $\quad \mathcal{L}_V T = \lim_{t \to 0} \frac{1}{t}\left[\phi_{-t*} T\big|_p - T\big|_p\right]$

Just recall that $(\phi_{-t})_*$ is a push-forward from the neighborhood of $\phi_t(p)$ to p.

Which is the same as a pull-back from p to $\phi(p)$. Which is the pic above: $\mathcal{L}_V(f) = V^\mu \partial_\mu f$

Finally $\mathcal{L}_V(\omega_\mu W^\mu) = V^\mu \partial_\mu(\omega_\nu W^\nu) = \mathcal{L}_V(\omega_\mu) W^\mu + \omega_\mu \mathcal{L}_V(W^\mu)$

$\mathcal{L}_V(\omega_\mu) W^\mu = V^\mu \partial_\mu(\omega_\nu W^\nu) - \omega_\mu\left(V^\nu \partial_\nu W^\mu - \partial_\nu V^\mu W^\nu\right) = \left(V^\mu \partial_\mu \omega_\nu + \partial_\nu V^\mu \omega_\mu\right) W^\nu$

Get $\mathcal{L}_{\vec{M}} \vec{W}$ explicitly:

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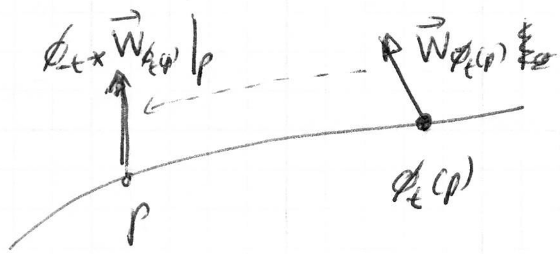


Recall $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$

then $\vec{W}_p \rightarrow \phi_* \vec{W}|_{\phi(p)}$

means $W^\mu \rightarrow (\phi_* W)^a|_{\phi(p)} = \frac{\partial y^a}{\partial x^\mu}\Big|_p W^\mu|_p$

Moreover, for our case $\phi_{-t*}$ at $p$ is what? Take

$$\mathcal{M} \xrightarrow{\phi_{-t}} \mathcal{M}'$$

$$\vec{W}_{\phi_t(p)} \rightarrow \phi_{-t*}\vec{W}|_p$$

$$(\phi_{-t*}W)^a|_p = \frac{\partial x^a}{\partial x^\mu}\Big|_{\phi_t(p)} W^\mu\Big|_{\phi_t(p)}$$

But $y^a(x^\mu)$ is just the shift in coordinates along the curve: $x^a$ are the coordinates of $p$, in terms of $x^\mu$, the coordinates of $\phi_t(p)$.

If the curve is the integral of $\frac{dx^\mu}{dt} = M^\mu$ ($\vec{M}$ a vector field).

then, $x^\mu(t) = x^\mu(0) + tM^\mu$ to order $t$, and $y^a(x^\mu)$ is just

$x^\mu(0) = x^\mu(t) - tM^\mu$ so $\frac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi_t(p)} = \delta^\nu_\mu - M^\nu_{\ ,\mu}t$

$W^\mu|_{\phi_t(p)}$ is just $W^\mu(x^\mu(t)) = W^\mu(x^\mu(0) + tM^\mu) = W^\mu|_p + tM^\nu W^\mu_{\ ,\nu}|$

so $\phi_{-t*}W|_p - W|_p = (\delta^\nu_\mu - M^\nu_{\ ,\mu}t)(W^\mu + tM^\nu W^\mu_{\ ,\nu}) - W^\nu$

and $(\mathcal{L}_{\vec{M}}\vec{W})^\mu = M^\nu W^\mu_{\ ,\nu} - W^\nu M^\mu_{\ ,\nu} = [M, W]^\mu$

"Lie bracket"

"commutator"

In particular, this shows $\mathcal{L}_{\vec{M}} \vec{W} = -\mathcal{L}_{\vec{W}} \vec{M}$

From this, ~~then~~ one can obtain ~~the res~~ the action of $\mathcal{L}_{\vec{M}}$ on other tensors:

$$\mathcal{L}_{\vec{M}} (\tilde{\omega} \otimes \vec{W}) = \mathcal{L}_{\vec{M}} \tilde{\omega} \otimes \vec{W} + \tilde{\omega} \otimes \mathcal{L}_{\vec{M}} \vec{W}$$

now, contracting $\Rightarrow$

The rest of this page has been done above, albeit a little differently... ignore

$$\mathcal{L}_{\vec{M}} ( \tilde{\omega}(\vec{W})) = \mathcal{L}_{\vec{M}} \tilde{\omega} (\vec{W}) + \tilde{\omega} (\mathcal{L}_{\vec{M}}(\vec{W}))$$

Now if we use $\vec{W} = \vec{E}_\mu$, a basis vector we can get $\mathcal{L}_{\vec{M}} \tilde{\omega}$.

In particular, if $\vec{E}_\mu = \frac{\partial}{\partial x^\mu}$, the coordinate basis, then

$$\mathcal{L}_{\vec{M}} (\tilde{\omega}) (\vec{E}_\mu) = (\mathcal{L}_{\vec{M}} (\tilde{\omega}))_\mu$$ the components we are looking for.

$$\mathcal{L}_{\vec{M}} ( \tilde{\omega}(\vec{E}_\mu)) = \mathcal{L}_{\vec{M}} (\omega_\mu) = \vec{M}(\omega_\mu) \quad \text{(property (v))}$$

$$= \frac{\partial \omega_\mu}{\partial x^\nu} M^\nu = \omega_{\mu,\nu} M^\nu$$

and $$\tilde{\omega} ( \mathcal{L}_{\vec{M}} (\vec{E}_\mu))^\nu = \frac{\partial (\vec{E}_\mu)^\nu}{\partial x^\rho} M^\rho - \frac{\partial M^\nu}{\partial x^\rho} (\vec{E}_\mu)^\rho = -\frac{\partial M^\nu}{\partial x^\mu}$$

so $$\tilde{\omega} ( \mathcal{L}_{\vec{M}} (\vec{E}_\mu)) = -\omega_\nu M^\nu{}_{,\mu}$$

$$\Rightarrow \quad (\mathcal{L}_{\vec{M}} (\tilde{\omega}))_\mu = \omega_{\mu,\nu} M^\nu + M^\nu{}_{,\mu} \omega_\nu$$

Exercise: Show

$$\mathcal{L}_{\vec{M}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = M^\sigma \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$- (\partial_\lambda M^{\mu_1}) T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$- (\partial_\lambda M^{\mu_2}) T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$\vdots$$

$$+ (\partial_{\nu_1} M^\lambda) T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_\ell}$$

$$+ (\partial_{\nu_2} M^\lambda) T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_\ell}$$

In particular

$$\mathcal{L}_{\vec{M}} g_{\mu\nu} = M^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu M^\lambda g_{\lambda\nu} + \partial_\nu M^\lambda g_{\mu\lambda}$$

Since these are <u>tensors</u> equations, we can replace $\partial$ by $\nabla$.

$$\Rightarrow \quad \mathcal{L}_M g_{\mu\nu} = \cancel{M}\; M^\lambda{}_{;\mu} g_{\lambda\nu} + M^\lambda{}_{;\nu} g_{\mu\lambda} = M_{\nu;\mu} + M_{\mu;\nu}$$

or

$$\boxed{\mathcal{L}_M g_{\mu\nu} = 2 M_{(\mu;\nu)}}$$

~~Easy~~ This is useful stuff. We will use it for symmetries later, but ~~the~~ here is a simple application. Assume the action for GR breaks down into

$$S = S_G(g_{\mu\nu}) + S_M(g_{\mu\nu}, \psi) \qquad (\star)$$

$\psi$ = matter fields

$S_G$ = "Hilbert" action (gives Einstein's eqs ... we'll use this later it came).

~~Consider~~ This theory is "diffeomorphism invariant": ~~the~~ given $\phi: M \to M$

$(M, g_{\mu\nu}, \psi)$ and $(M, \phi^* g_{\mu\nu}, \phi^* \psi)$

represent the same physics. The change in $S_M$ under a diffeomorphism

$$\delta S_M = \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int d^4x \, \frac{\delta S_M}{\delta \psi} \delta \psi$$

Since we could have set $\psi = 0$, $\delta S_G$ can be considered separately (it is invariant by itself; here is where the separation assumption in $(\star)$ comes in).

But $\frac{\delta S_M}{\delta \psi} = 0$ for <u>any</u> variation. So while here we look only at variations from diffeomorphisms, that term vanishes separately for any variation. Left with first term, we consider diffeomorphisms generated by a vector field $U^\mu$:

$$\delta g_{\mu\nu} = \mathcal{L}_U g_{\mu\nu} = 2 U_{(\mu;\nu)}$$

$$\Rightarrow \quad \delta S_M = 0 = \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} 2 U_{(\mu;\nu)} = 4 \int d^4x \, \frac{\delta S_M}{\delta g_{\mu\nu}} U_{\mu;\nu}$$

or

$$= 4 \int d^4x \left( \left[ \frac{\delta S_M}{\delta g_{\mu\nu}} U_\mu \right]_{;\nu} - U_\mu \left( \frac{\delta S_M}{\delta g_{\mu\nu}} \right)_{;\nu} \right)$$

Dropping the surface term and multiplying by $\frac{\sqrt{-g}}{\sqrt{-g}}$ we have

$$\int d\mathcal{V} \; U_\mu \nabla_\nu \left[ \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right] = 0$$

~~If~~ Since this holds for arbitrary $U_\mu$ (diffeomorphisms generated by arbitrary vector fields) it must be that

$$\nabla_\nu \left( \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right) = 0$$

But $\qquad T^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \qquad$ is the energy-momentum tensor.

# Symmetries, Isometry, Killing Vectors

$\phi : \mathcal{M} \to \mathcal{M}$ a diffeomorphism, $T$ a tensor.

$\phi$ is a symmetry of $T$ if

$$\boxed{\phi^* T = T}$$ $T$ symmetric

Some symmetries are discrete. But for continuous symmetries there is a one parameter set of diffeomorphism $\phi_t$, and then $T$ is symmetric iff

$$\boxed{\mathcal{L}_U T = 0}$$ $T$ symmetric, continuous symmetry.

(Clearly $U$ generates the curve, $U = \frac{\partial}{\partial t}$).

Note that one can choose coordinates locally so that $t$ itself is one of the coordinates. In such coordinates

$$\mathcal{L}_U T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \partial_t T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}$$

so $\mathcal{L}_U T = 0 \Rightarrow$ all components of $T$ are independent of $t$.
(Converse is obviously true!)

(a diffeomorphism $\phi$ ...)

An isometry is a symmetry of the metric tensor,

$$\phi^* g_{\mu\nu} = g_{\mu\nu}$$

~~Ef~~ A vector field $\vec{K}$ that generates an isometry is called a Killing vector field:

$$\boxed{\mathcal{L}_K g_{\mu\nu} = 0}$$

continuous isometry, Killing vector

then $\vec{K}$ satisfies

$$K_{(\mu;\nu)} = 0$$

One can show the ~~op~~ reverse: if $K_{(\mu;\nu)} = 0$ then $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$ where $\phi_t$ is generated by ~~ma~~ $K = \frac{\partial}{\partial t}$. This is done by integration (see Hawking & Ellis).

Again, one can choose local coordinates that include $t$, and then $g_{\mu\nu}$ is independent of $t$.

Now, in first quarter (Schutz, 7.4) we saw that the geodesic equation can be written in terms of $\vec{p} = m\vec{U}$ as

$$m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$$

so if $g_{\nu\alpha}$ is independent of one coordinate (say "$t$"), then the corresponding $p_\beta$ is conserved, $\frac{dp_t}{d\tau} = 0$

This can be done in a covariant language as follows:
assume $p_\mu$ satisfies geodesic equation:

$$p^\mu p^\nu{}_{;\mu} = 0 \qquad (\nabla_p p = 0)$$

Then

$$p^\mu \nabla_\mu (p^\nu K_\nu) = p^\mu p^\nu \nabla_\mu K_\nu + K_\nu p^\mu \nabla_\mu p^\nu = 0$$

But LHS is just $\frac{d}{d\tau}(p^\nu K_\nu)$ so $\boxed{p^\nu K_\nu}$ is constant along particle path → a conserved quantity, as before.

<u>Exercise</u>: If $K_{\mu_1\cdots\mu_r}$ is a Killing tensor, i.e., it satisfies

$$\nabla_{(\mu}K_{\mu_1\cdots\mu_r)} = 0,$$

show that $K_{\mu_1\cdots\mu_r}p^{\mu_1}\cdots p^{\mu_r}$ is conserved.

We can see this more generally with our Killing field technology:

let
$$P^\mu = T^{\mu\nu} K_\nu$$
then
$$P^\mu{}_{;\mu} = T^{\mu\nu}{}_{;\mu} K_\nu + T^{\mu\nu} K_{\nu;\mu}$$
$$= T^{\nu\mu}{}_{;\mu} K_\nu + \tfrac{1}{2} T^{\mu\nu} K_{(\nu;\mu)} = 0$$

So the vector $P^\mu$ is a "conserved current". ~~By Gauss's the~~

Example: In flat space (which is highly symmetric):

Killing vectors
$$\vec{p}_{(\alpha)} = \frac{\partial}{\partial x^\alpha} \qquad \text{(a vector for each } \alpha = 0,1,2,3\text{)}$$

and
$$\vec{m}_{(i)} = x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0} \qquad i = 1,2,3$$
$$\vec{l}_{(ij)} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \qquad i,j = 1,2,3$$

10 - isometries generate 10 parameter Lie-Group of isometries of flat spacetime, the "inhomogeneous Lorentz group"

To see how this works ~~choose obvious coordinates~~ write components out: $\vec{p}_{(0)}{}^\mu = (1,0,0,0)$ $\;p^\mu_{(1)} = (0,1,0,0)$ -- ($p^\mu_{(\alpha)} = \delta^\mu_\alpha$)

So $p^{(\alpha)}_{\mu,\nu} = 0$ trivially for all $(\alpha)$.

Less trivial: $m^\mu_{(1)} = (x^1, x^0, 0, 0) \rightarrow m_{(1)\mu} = (-x^1, x^0, 0, 0)$

$$m_{(1)\mu,\nu} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{so} \qquad m_{(1)(\mu,\nu)} = 0$$

etc.

Then $P^\mu = T^{\mu\nu} K_\nu$ gives conservation of $E, \vec{P}$, $\vec{J}$ and ? (like $tP + xE$?)
DON'T KNOW

It is clear from the example that ~~manifolds~~ space-times may admit ~~more~~ several (or none) killing vectors.

Since ~~trans for~~ symmetry transformations generally form groups (group multiplication = composition of transformations, ie $\phi_t \circ \phi_s$). and these are continuous transformations generated by $\vec{K}$'s, we expect there to be ~~a~~ Lie groups & the $\vec{K}$'s to form Lie algebras. This is indeed the case, with the Lie bracket being just the commutator, ie.

$$[K_1, K_2] = \mathcal{L}_{K_1} K_2$$