Reissner-Nordstrom: Charged Black Hole.

Look for spherically symmetric and static (or just stationary?)

(so $ds^2 = -T(r,t)\,dt^2 + R(r,t)\,dr^2 + r^2 d\Omega_2^2$)

solutions to Einstein's Equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

with matter given by electromagnetic field.

Recall $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

(For contact with conventional non-relativistic notation in Minkowski space the $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$, Φ is A^0 and $\vec{\nabla} = \partial_i$. With low indices,

$$E_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}$$

Similarly $\vec{B} = (\vec{\nabla}\times\vec{A})$ or $B_i = \epsilon_{ijk}\partial_j A_k = \frac{1}{2}\epsilon_{ijk} F_{jk}$

Note that $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$ as it should).

Then, as we saw earlier

$$T_m^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad S = \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

$$\Rightarrow \quad T_m^{\mu\nu} = \int d^4x \left[\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \right]$$

The first term requires $\delta(\det A) = \delta \underbrace{\prod \lambda}_{\substack{\text{eigenvalues} \\ \lambda \text{ of } A}} = \delta e^{\sum \ln\lambda} = e^{\sum\ln\lambda} \sum \frac{1}{\lambda}\delta\lambda$
$= \det A \,\mathrm{Tr}\, A^{-1}\delta A$

or $\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}$ so $\frac{1}{\sqrt{-g}}\delta\sqrt{-g} = \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu}$

For the second term use

$$\mathcal{L} = -\frac{1}{4} F_{\mu\lambda} F_{\nu\rho} g^{\mu\nu} g^{\lambda\rho}$$

so that there is no implicit dependence on $g_{\mu\nu}$ in F, and then

$$\frac{\delta g^{\lambda\rho}}{\delta g_{\mu\nu}} = -g^{\lambda\mu} g^{\rho\nu} - g^{\lambda\nu} g^{\rho\mu}, \text{ so}$$

$$\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = F^{\mu}{}_{\lambda} F^{\nu}{}_{\rho} g^{\lambda\rho}$$

or

$$T^{\mu\nu} = F^{\mu}{}_{\lambda} F^{\nu}{}_{\rho} g^{\lambda\rho} - \frac{1}{4} g^{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}$$

or

$$\boxed{T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}}$$

For spherical symmetry need radial $\vec{E}$ (and possibly $\vec{B}$), so in radial coordinates we have $F_{t\theta} = F_{t\phi} = 0$ and

$$F_{tr} = -F_{rt} = f(t,r)$$

For $\vec{B}$ to be radial we need to generalize $B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$: "$B_\mu$" $= \frac{1}{2}\epsilon_{0\mu\nu\lambda} F^{\nu\lambda}$ and use $\epsilon^{\mu\nu\lambda\rho} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\mu\nu\lambda\rho}$. Or, more directly, but pedestrianly, go back to cartesian $x\,y\,z$. Then

$$F_{\theta\phi} = F_{ij} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi} \quad \text{and} \quad F_{ij} = \epsilon_{ijk} B^k \propto \epsilon_{ijk} x^k \text{ for radial (times some } g\text{)}$$

But then $F_{\theta\phi} = g(r)\, \epsilon_{ijk} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi} x^k = \sin\theta\, g(r)$.

(The factor $\epsilon_{ijk} \frac{\partial x^i}{\partial \theta} \frac{\partial x^j}{\partial \phi} x^k$ is just the determinant $\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} & x \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} & y \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} & z \end{vmatrix}$

which is the measure for the volume integral at $r=1$, $\sin\theta$).

So we take

$$ds^2 = -T(\cancel{t}, r)\,dt^2 + R(r)\,dr^2 + r^2 d\Omega_2^2$$

static

$$F_{tr} = f(r)$$

$$F_{\theta\phi} = g(r)\sin\theta$$

and plug into Einstein's.

Quick computation $\Gamma_{\mu\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\nu\lambda,\mu})$ **NOT FOR CLASS**

$$\Gamma_{rtt} = +\tfrac{1}{2}T' \qquad \Gamma^r_{tt} = \tfrac{1}{2}T'/R$$

$$\Gamma_{trt} = -\tfrac{1}{2}T' \qquad \Gamma^t_{rt} = \tfrac{1}{2}T'/T$$

$$\Gamma_{rrr} = \tfrac{1}{2}R' \qquad \Gamma^r_{rr} = \tfrac{1}{2}R'/R$$

$$\Gamma_{r\theta\theta} = -r \qquad \Gamma^r_{\theta\theta} = -r/R$$

$$\Gamma_{\theta r\theta} = r \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}$$

$$\Gamma_{r\phi\phi} = -r\sin^2\theta \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta/R$$

$$\Gamma_{\phi r\phi} = r\sin^2\theta \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma_{\theta\phi\phi} = -r^2\sin\theta\cos\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma_{\phi\theta\phi} = r^2\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \cos\theta/\sin\theta$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$R_{tt} = \tfrac{1}{2}T''/R - \tfrac{1}{2}\frac{T'R'}{R^2} + \left(\tfrac{1}{2}\frac{T'}{T} + \tfrac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(\tfrac{1}{2}T'/R\right) - 2\left(\tfrac{1}{2}\frac{T'}{R}\right)\left(\tfrac{1}{2}\frac{T'}{T}\right)$$

$$= \frac{1}{2}\frac{T''}{R} - \frac{1}{4}\frac{T'R'}{R^2} - \frac{1}{4}\frac{T'^2}{RT} + \frac{1}{r}\frac{T'}{R}$$

$$R_{rr} = \frac{1}{2}\frac{R''}{R} - \frac{1}{2}\frac{R'^2}{R^2} - \partial_r\left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right) + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\frac{1}{2}\frac{R'}{R}$$
$$- \left(\frac{1}{2}\frac{T'}{T}\right)^2 - \left(\frac{1}{2}\frac{R'}{R}\right)^2 - 2\frac{1}{r^2}$$

$$= -\frac{1}{2}\frac{T''}{T} + \frac{1}{4}\frac{T'^2}{T^2} + \frac{1}{4}\frac{T'R'}{TR} + \frac{R'}{rR}$$

$$R_{rt} = R_{tr} = \cancel{\partial_t\left(\tfrac{1}{2}\tfrac{T'}{T}\right)} + (\cdot) - 0 = 0$$

$$R_{\theta\theta} = \partial_r\left(-\frac{r}{R}\right) + \partial_\theta \cot\theta + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(-\frac{r}{R}\right) - 2\left(-\frac{r}{R}\right)\left(\frac{1}{r}\right) - \left(\frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{1}{2}\frac{rR'}{R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1$$

$$R_{\phi\phi} = \partial_r\left(-\frac{r\sin^2\theta}{R}\right) + \partial_\theta(-\sin\theta\cos\theta) + \left(\frac{1}{2}\frac{T'}{T} + \frac{1}{2}\frac{R'}{R} + \frac{2}{r}\right)\left(-\frac{r\sin^2\theta}{R}\right) + \left(\frac{\cos\theta}{\sin\theta}\right)(-\sin\theta\cos\theta)$$

$$- 2\left(\frac{1}{r}\right)\left(-\frac{r\sin^2\theta}{R}\right) - 2(-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin^2\theta\left[\frac{rR'}{2R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1\right] = \sin^2\theta\, R_{\theta\theta}$$

Ricci scalar

$R = g^{\mu\nu}R_{\mu\nu} = \cdots$ better use trace of T

So write
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$
$$\rightarrow \quad -R = 8\pi G T$$
$$\Rightarrow R_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(-8\pi G T) = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Now, compute $T_{\mu\nu}$:

$$T_{tt} = F_{t\lambda}F_t{}^\lambda - \frac{1}{4}g_{tt}F_{\lambda\rho}F^{\lambda\rho} = \frac{1}{R}f^2 + \frac{1}{2}T\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$
$$= \frac{1}{2R}f^2 + \frac{1}{2}\frac{Tg^2}{r^4}$$

$$T_{rr} = F_{r\lambda}F_r{}^\lambda - \frac{1}{4}g_{rr}F_{\lambda\rho}F^{\lambda\rho} = -\frac{f^2}{T} - \frac{1}{2}R\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$
$$= -\frac{1}{2}\frac{f^2}{T} - \frac{g^2R}{2r^4}$$

$$T_{\theta\theta} = F_{\theta\lambda}F_\theta{}^\lambda - \frac{1}{4}g_{\theta\theta}F_{\lambda\rho}F^{\lambda\rho} = \frac{1}{r^2\sin^2\theta}\sin^2\theta\, g^2 - \frac{1}{2}r^2\left(-\frac{1}{RT}f^2 + \frac{g^2}{r^4}\right)$$
$$= \frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2f^2}{RT}$$

$$T_{\phi\phi} = \sin^2\theta\, T_{\theta\theta}$$

So $T = g^{\mu\nu}T_{\mu\nu} = -\frac{1}{T}\left(\frac{1}{2R}f^2 + \frac{1}{2}\frac{Tg^2}{r^4}\right) + \frac{1}{R}\left(-\frac{1}{2}\frac{f^2}{T} - \frac{g^2 R}{2r^4}\right)$

$$+ \frac{2}{r^2}\left(\frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2 f^2}{RT}\right) = 0$$

And we have

$$\left.\begin{array}{ll} tt: & \frac{1}{2}\frac{T''}{R} - \frac{1}{4}\frac{T'R'}{R^2} - \frac{1}{4}\frac{T'^2}{RT} + \frac{1}{r}\frac{T'}{R} = \left(\frac{f^2}{2R} + \frac{1}{2}\frac{Tg^2}{r^4}\right)8\pi G \\ rr: & -\frac{1}{2}\frac{T''}{T} + \frac{1}{4}\frac{T'^2}{T^2} + \frac{1}{4}\frac{T'}{T}\frac{R'}{R} + \frac{R'}{rR} = \left(-\frac{1}{2}\frac{f^2}{T} - \frac{g^2 R}{2r^4}\right)8\pi G \end{array}\right] \text{Same eqn.}$$

$\theta\theta$: $\frac{1}{2}\frac{rR'}{R^2} - \frac{1}{R} - \frac{1}{2}\frac{T'r}{TR} + 1 = 8\pi G\left(\frac{1}{2}\frac{g^2}{r^2} + \frac{1}{2}\frac{r^2f^2}{RT}\right)$

2 eqs, 4 unknowns; need ~~two~~ more; Maxwell's Equations

$$g^{\mu\nu}\nabla_\mu F_{\nu\lambda} = 0 \qquad \text{and} \qquad \nabla_{[\mu}F_{\nu\lambda]} = 0$$

Recall $\nabla_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - \Gamma^\rho_{\mu\nu}F_{\rho\lambda} - \Gamma^\rho_{\mu\lambda}F_{\nu\rho}$

So, in components

$$g^{\mu\nu}\nabla_\mu F_{\nu t} = -\frac{1}{T}\left[\cancel{-\frac{1}{2}\frac{T'}{R}(-f) + \dots}\right] + \frac{1}{R}\left(-f' - \frac{1}{2}\frac{R'}{R}(-f) - \frac{1}{2}\frac{T'}{T}(-f)\right)$$

$$+ \frac{1}{r^2}\left[-\left(-\frac{r}{R}\right)(-f)\right] + \frac{1}{r^2\sin^2\theta}\left[-\left(-\frac{r\sin^2\theta}{R}\right)(-f)\right]$$

$$= -\frac{f'}{R} + \frac{1}{2}\frac{R'}{R}f + \frac{1}{2}\frac{T'}{T}f - \frac{2f}{rR} = 0$$

$$g^{\mu\nu}\nabla_\mu F_{\nu r} = -\frac{1}{T}[0] + \frac{1}{R}(0) + \frac{1}{r^2}(0) = 0 \qquad \text{automatic}$$

$$g^{\mu\nu}\nabla_\mu F_{\nu\theta} = -\frac{1}{T}(0) + \frac{1}{R}(0) + \frac{1}{r^2}[0] + \frac{1}{r^2\sin^2\theta}[0] = 0$$

$$g^{\mu\nu}\nabla_\mu F_{\nu\phi} = -\frac{1}{T}(0) + \frac{1}{R}(0) + \frac{1}{r^2}\left[\cos\theta\, g - \frac{\cos\theta}{\sin\theta}\sin\theta\, g\right] + \frac{1}{r^2\sin^2\theta}(0) = 0$$

and for the 2nd equation take

$$\nabla_r F_{\theta\phi} + \nabla_\theta F_{\phi r} + \nabla_\phi F_{r\theta}$$

$$= \left(\sin\theta\, g' - \frac{2}{r}\sin\theta\, g\right) + \left(-\frac{1}{r}(-g\sin\theta)\right) + \left(-\frac{1}{r}(-g\sin\theta)\right)$$

$$= \sin\theta\, g' \qquad ?$$

So $\nabla_{[r} F_{\theta\phi]} = 0 \Leftrightarrow \sin\theta\, g' = 0 \quad \Rightarrow \quad g = \text{constant.}$

And Maxwell's equation gives

$$\frac{f'}{f} = \left[\frac{1}{2}\frac{R'}{R} + \frac{1}{2}\frac{T'}{T} - \frac{2}{rR}\right]R$$

Look for a solution with $TR = 1$ $\quad T'/T = -R'/R$

Then $f'/f = -\frac{2}{r} \qquad \frac{df}{f} = -2\frac{dr}{r} \qquad f = \frac{k}{r^2}$

and

$$\frac{rR'}{R^2} - \frac{1}{R} + 1 = 4\pi G\,\frac{(g^2+k^2)}{r^2} \qquad (@)$$

and

$$-\frac{1}{2}\frac{T''}{T} - \frac{T'}{rT} = -\frac{4\pi G}{T}\,\frac{(k^2+g^2)}{r^4}$$

or $$T'' + \frac{2}{r}T' = \frac{8\pi G(k^2+g^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^2}(r^2T')' = \frac{8\pi G(k^2+g^2)}{r^4} \Rightarrow r^2T' = -\frac{8\pi G(k^2+g^2)}{r} + 2GM$$

$$T = 1 - \frac{2GM}{r} + \frac{4\pi G(g^2+k^2)}{r^2}$$

Check (@): $-r\frac{T'}{TR} - T + 1 = -rT' - T + 1 = \left[\frac{8\pi G(k^2+g^2)}{r^2} - \frac{2GM}{r}\right] + \frac{2GM}{r} - \frac{4\pi G(}{r^2}$

$$= \frac{4\pi G(k^2+g^2)}{r^2} \quad \checkmark \quad \text{Yes!}$$

The solution is $ds^2 = -T\,dt^2 + R\,dr^2 + r^2 d\Omega_2^2$

$$T = \frac{1}{R} = 1 - \frac{2GM}{r} + \frac{4\pi G(p^2+q^2)}{r^2}$$

and $F_{tr} = \frac{q}{r^2}$ $\quad F_{\theta\phi} = p\sin\theta$

(Note that $E_r = F_{tr} = \frac{q}{r^2}$ $\quad B_r = \frac{F_{\theta\phi}}{r^2\sin\theta} = \frac{p}{r^2}$

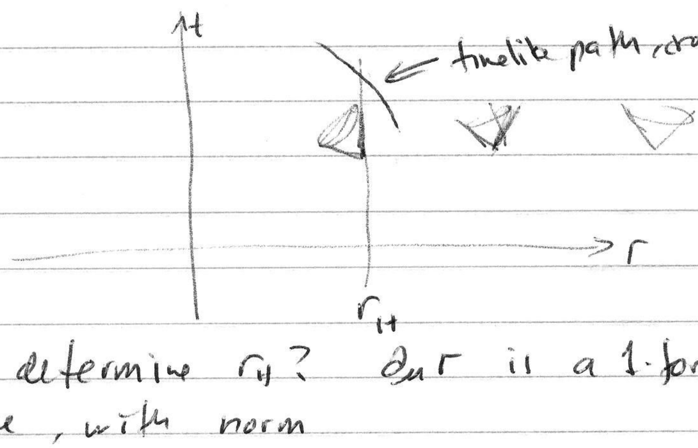
so q & p are electric & magnetic charges, and the notation (q,p) is standard for "dyons").

Singularity at $r=0$. Horizon singularities (see below) are coordinate effects.

<u>Event horizons?</u> In a static space-time (Killing vector ∂_t is asymptotically time-like, $\partial_t g_{\mu\nu} = 0$) choose coordinates (r,θ,ϕ) so metric looks Minkowski as $r\to\infty$.

Hypersurface $r = \text{const}$: timelike "cylinder" (topology $S^2\times\mathbb{R}$) as $r\to\infty$

Now decrease r from infinity to some r_H where the surface becomes null → an event horizon



How to determine r_H? $\partial_\mu r$ is a 1-form normal to $r=\text{const}$ hypersurface, with norm

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}$$

We want this to vanish, so $\boxed{g^{rr}(r_H) = 0}$

This method is very restrictive (to spaces that are static and with coordinates (r,θ,ϕ) as above, found).

Method applies for RN-metric. So

r_H is at $g^{rr} = 1 - \frac{2GM}{r} + \frac{4\pi G(q^2+p^2)}{r^2} = 0$

There are two solutions (at most):

$$r_\pm = \frac{2GM \pm \sqrt{(2GM)^2 - 4^2\pi G(p^2+q^2)}}{2}$$

$$= GM \pm \sqrt{(GM)^2 - 4\pi G(p^2+q^2)}$$

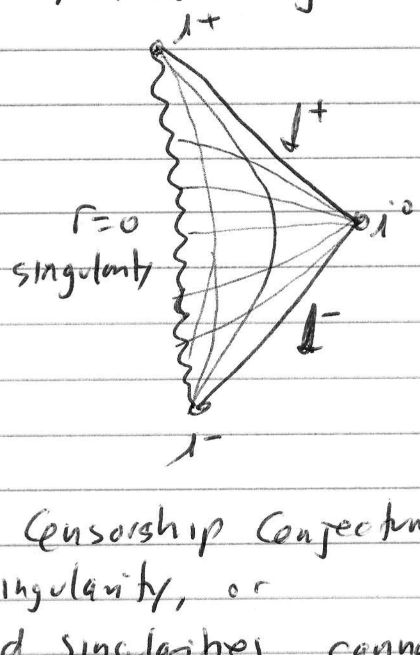
Cases:

(1) $4\pi(p^2+q^2) > GM^2$

No solutions $\Rightarrow$ no event horizon.

"Naked singularity"

Penrose:



Cosmic Censorship Conjecture (Penrose): Nature abhors a naked singularity, or

Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition:

for all timelike vectors t^μ, $T_{\mu\nu} t^\mu t^\nu \geq 0$ (so far, "weak energy condition")

and $T^{\mu\nu} t_\nu$ is a non-spacelike vector (Basically, $\rho > |p|$).

$(T_{\mu\nu} T^\nu{}_\lambda t^\mu t^\lambda \leq 0)$.

(iii) $4\pi(p^2+q^2) > GM^2$

Two distinct solutions, with $r_- < r_+$

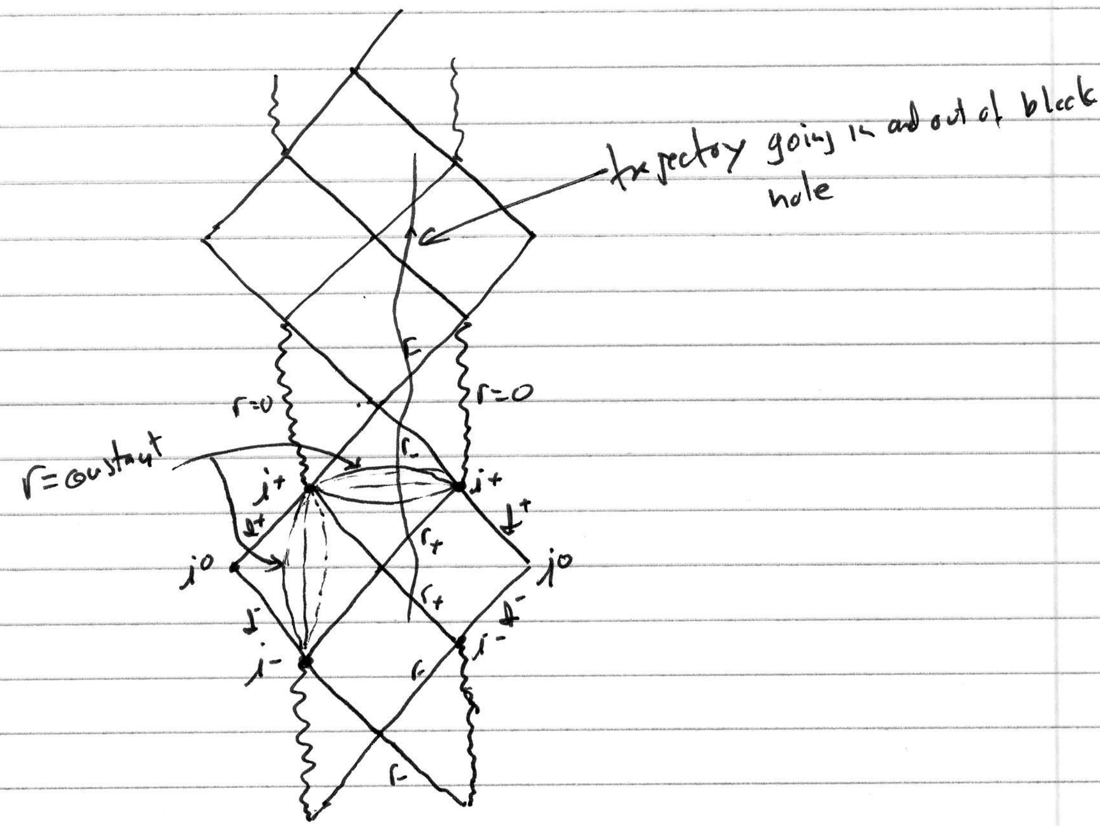
In $r_- < r < r_+$ ∂_r is timelike and ∂_t is spacelike.

But both for $r > r_+$ AND $r < r_-$ ∂_r is spacelike & ∂_t is timelike.

$\Rightarrow$ ① If you fall into this black hole within a spaceship full of gas, once you get to $r = r_+$ you must continue falling towards lesser r, but once you come out to $r < r_-$ you can turn on your thrust engines, turn around before you hit $r=0$, go back to $r = r_-$. Then you must continue, ~~with~~ until you come out to $r = r_+$. You can then decide to continue out to $r=\infty$ or turn around and "re-enter" the black hole?

② $r=0$ singularity is timelike (recall, for Schwarzschild, spacelike)

Conformal diagram



MTW has a step by step on how to derive this.

(iii) $GM^2 = 4\pi(q^2+p^2)$

"Extreme" RN-solution. ~~Here~~

In this case $r_+ = r_- = GM$, and

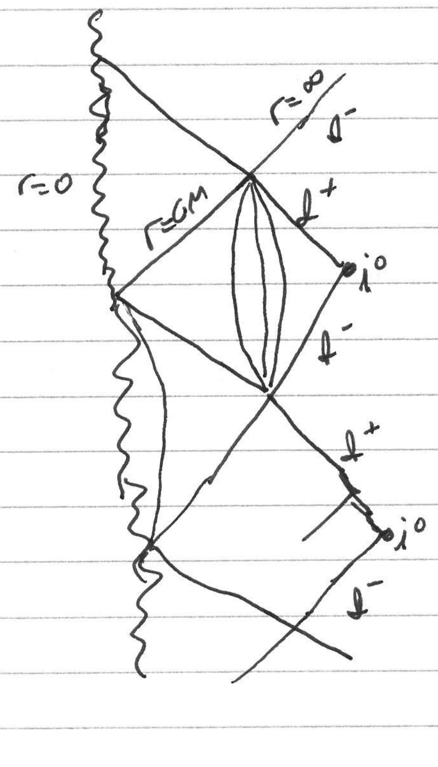
$$g^{rr} = \left(1-\frac{GM}{r}\right)^2$$

In fact

$$ds^2 = -\left(1-\frac{GM}{r}\right)^2 dt^2 + \left(1-\frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

So, there is a horizon at $r=GM$, but r is never timelike.
The singularity is at $r=0$ and it is timelike.

Penrose diagram



Solutions with many extreme RN black holes: remarkably, we can produce metrics which are exact solutions of Einstein's equations in empty space with as many RN black holes as we want.

In $ds^2 = -\left(1-\frac{GM}{r}\right)^2 dt^2 + \left(1-\frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$

let

$$\rho = r - GM$$

so

$$r^2 = (\rho + GM)^2 = \rho^2 H^2(\rho)$$

where $H(\rho) = 1 + \frac{GM}{\rho}$

Also $\quad 1-\frac{GM}{r} = 1 - \frac{GM}{\rho+GM} = \frac{\rho}{\rho+GM} = \frac{1}{1+\frac{GM}{\rho}} = H^{-1}$

so

$$ds^2 = -H^{-2}(\rho)\,dt^2 + H^2(\rho)\left[d\rho^2 + \rho^2 d\Omega_2^2\right]$$

Now, the term in [ ] is just the metric of flat Euclidean 3-space in spherical coordinates, so we can write

$$ds^2 = -H^{-2}(|\vec{x}|)\,dt^2 + H^2(|\vec{x}|)\left[dx^2+dy^2+dz^2\right] \qquad (\ast)$$

where $|\vec{x}|^2 = x^2+y^2+z^2$

If we take the metric $(\ast)$ as an ansatz and plug into Einstein's equations, with $H = H(\vec{x})$, we find it is a solution provided $\nabla^2 H = 0$. To be precise, we need an EM field too. To motivate it, we have, in the single extremal RN solution

$$F_{tr} = \frac{q}{r^2} = -\partial_r A_t + \partial_t A_r \qquad \text{(sign may be wrong)}$$

$$= -\partial_r A_t$$

so $\quad A_t = \frac{q}{r} = \sqrt{\frac{GM^2}{4\pi}}\,\frac{1}{\rho+GM} \quad$ but $\quad \frac{1-H^{-1}}{GM} = \frac{1}{r} = \frac{1}{\rho+GM}$

So $$A_t = \sqrt{\frac{GM^2}{4\pi}}\ \frac{1-H^{-1}}{GM} = \sqrt{\frac{1}{4\pi G}}\left(1-H^{-1}\right) \qquad (**)$$

So one looks for solutions with (*) and (**). Then H must satisfy

$$\vec{\nabla}^2 H = 0 \qquad \text{where} \qquad \vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

The most general solution with $H \to 1$ as $|x| \to \infty$ is

$$H = 1 + \sum_{k=1}^{N} \frac{GM_k}{|\vec{x} - \vec{x}_k|}$$

(Actually, I guess

$$H = 1 + \int d^3x' \ \frac{G\rho(x')}{|\vec{x} - \vec{x}'|}$$

works too, provided ρ has support in a finite region, $|\vec{x}| < R$).
(But $\nabla^2 H = 4\pi G \rho(\vec{x}) \neq 0$ ... so works outside that region).

~~I think, however, these solutions are inconsistent with the electrical forces between the charges! I don't know what the proper resolution of this is ?~~

Note that the electric repulsion between holes cancels the gravitational attraction:

$$F_{12} = -\frac{GM_1M_2}{r^2} + \frac{q_1 q_2}{r^2} = 0 \qquad \text{if} \qquad q_1 q_2 = GM_1M_2$$

or $q_1 = \sqrt{G}M_1 \quad q_2 = \sqrt{G}M_2$

? is there a $\frac{1}{4\pi}$ here?

and we are off by a $\sqrt{4\pi}$.

Note: Verify solution:

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2+dy^2+dz^2)$$

$$\Gamma_{itt} = -\tfrac12 g_{tt,i} = -\tfrac12(-1)(-2H^{-3})\partial_i H = -H^{-3}\partial_i H \qquad \Gamma^i_{tt} = -H^{-5}\partial_i H$$

$$\Gamma_{tit} = H^{-3}\partial_i H \qquad \Gamma^t_{it} = -H^{-1}\partial_i H$$

$$\Gamma_{ijk} = \tfrac12\left((H^2\delta_{ij})_{,k} + (H^2\delta_{ik})_{,j} - (H^2\delta_{jk})_{,i}\right) = H(\delta_{ij}H_k + \delta_{ik}H_j - \delta_{jk}H_i)$$

$$\Gamma^i_{jk} = H^{-1}(\delta_{ij}H_{,k} + \delta_{ik}H_{,j} - \delta_{jk}H_{,i})$$

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

$$R_{tt} = \partial_i(-H^{-5}\partial_i H) + (-H^{-1}\partial_i H + 3H^{-1}\partial_i H)(-H^{-5}\partial_i H) - 2(H^{-1}\partial_i H)(H^{-5}\partial_i H)$$
$$= 5H^{-6}(\partial_i H)^2 - H^{-5}\nabla^2 H - 4H^{-6}(\partial_i H)^2 = H^{-6}(\partial_i H)^2 - H^{-5}\nabla^2 H$$

$$R_{ij} = \partial_k[H^{-1}(\delta_{kj}H_{,i} + \delta_{ik}H_{,j} - \delta_{ji}H_{,k})] - \partial_j[2H^{-1}\partial_i H]$$
$$+ [2H^{-1}\partial_k H][H^{-1}(\delta_{kj}H_{,i} + \delta_{ik}H_{,j} - \delta_{ji}H_{,k})]$$
$$- (H^{-1}\partial_i H)(H^{-1}\partial_j H) - H^{-2}(\delta_{kj}H_{,l} + \delta_{kl}H_{,j} - \delta_{jl}H_{,k})\times$$
$$(\delta_{li}H_{,k} + \delta_{kl}H_{,i} - \delta_{ik}H_{,l})$$
$$= -H^{-2}(\cancel{2H_jH_i} - \delta_{ij}H_{,k}^2) + H^{-1}(\cancel{2H_{,ij}} - \delta_{ij}\nabla^2 H) + 2H^{-2}H_jH_i - \cancel{2H^{-1}H_{,ij}}$$
$$+ 2H^{-2}(2H_jH_i - \delta_{ij}H_{,k}^2) - H^{-2}H_iH_j - H^{-2}(5H_iH_j - 2\delta_{ij}H_{,k}^2)$$
$$= -\delta_{ij}H^{-1}\nabla^2 H + H^{-2}\delta_{ij}H_{,k}^2 - 2H^{-2}H_{,i}H_{,j}$$

The em part: $F_{ti} = -\partial_i A_t = \sqrt{1/4\pi G}\,(-H^{-2}H_{,i})$

$F_{ij} = \partial_i A_j - \partial_j A_i = \cancel{\sqrt{1/4\pi G}\,H^{-2}}\ 0$

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \tfrac14 g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}$$

$$T_{tt} = F_{ti}F_t{}^i - \tfrac14 g_{tt}\,2F_{ti}F^{ti} = H^{-2}\frac{1}{4\pi G}(-H^{-2}H_{,i})^2 - \tfrac12(-H^{-2})(-1)\left(\frac{-H^{-2}H_{,i}}{\sqrt{4\pi G}}\right)^2$$
$$= \frac{1}{8\pi G}H^{-6}H_{,i}^2$$

$$R_{tt} = 8\pi G\,T_{tt} \Rightarrow H^{-6}H_{,i}^2 - H^{-5}\nabla^2 H = 8\pi G\left\{\frac{1}{8\pi G}H^{-6}H_{,i}^2\right\}$$

(using $T_{\lambda\sigma}g^{\lambda\sigma}=0$).
$$\Rightarrow \nabla^2 H = 0$$

Kerr Metric: Rotating Black Holes

No hair "theorem": stationary, asymptotically flat solutions to Einstein's + Maxwell's are fully characterized by M, Q ($+P$) and $J = aM$

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dt\,d\phi + d\phi\,dt)$$

$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where
$$\Delta(r) = r^2 - 2GMr + a^2$$
$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$$

Note: Include charge by changing $2GMr \to 2GMr - 4\pi G(Q^2+P^2)$.
And $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with

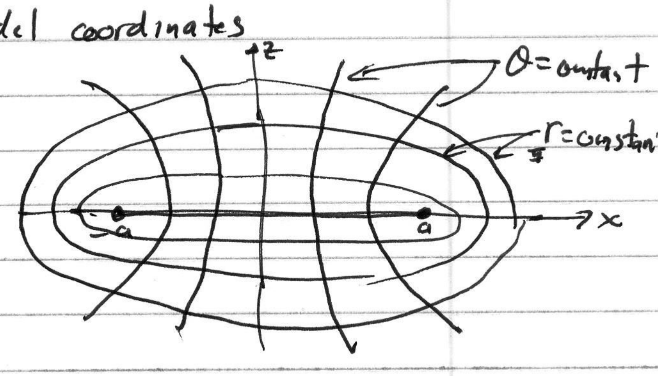
$$A_t = \frac{Qr - Pa\cos\theta}{\rho^2} \qquad A_\phi = \frac{-Qar\sin^2\theta + P(r^2+a^2)\cos\theta}{\rho^2}$$
(Kerr-Newman)

The novel feature is $J = aM \neq 0$, so let's simply set $Q = P = 0$ and study Kerr's solution.

Note that for $M=0$ we have flat space but in weird coordinates (called Boyer-Lindquist coordinates):

$$ds^2 = -dt^2 + \underbrace{\frac{r^2+a^2\cos^2\theta}{r^2+a^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2}_{\text{ellipsoidal coordinates}}$$

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$
$$z = r\cos\theta$$



Killing vectors: ∂_t and ∂_ϕ

Also a killing tensor $K_{\mu\nu} = \cancel{2}\rho^2(l_\mu n_\nu + l_\nu n_\mu) + r^2 g_{\mu\nu}$

with $l^\mu = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a)$ and $n^\mu = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a)$

(they have $l^2 = n^2 = 0$ $\quad l\cdot n = -1$).

So geodesics are easy to find: three constants (use actual energy/ang. mom not just per unit mass)

$$\frac{E}{m} = -(\partial_t)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} \qquad \frac{L}{m} = +(\partial_\phi)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} \qquad S = K_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

included for massive particles only.

plus

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -1 \text{ or } 0 \quad \text{for timelike or null}$$

Note that

$$E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} + \frac{2GMar\sin^2\theta}{\rho^2}\frac{d\phi}{d\lambda} = -g_{tt}\frac{dt}{d\lambda} - g_{t\phi}\frac{d\phi}{d\lambda}$$

$$L = g_{t\phi}\frac{dt}{d\lambda} + g_{\phi\phi}\frac{d\phi}{d\lambda}$$

and

$$\frac{L}{E} = \frac{g_{t\phi} + g_{\phi\phi}\, d\phi/dt}{-g_{tt} - g_{t\phi}\, d\phi/dt}$$

so if $\omega \equiv d\phi/dt$ we have

$$g_{t\phi} + \omega g_{\phi\phi} + \frac{L}{E}(g_{tt} + g_{t\phi}\omega) = 0$$

$$\omega = \frac{-\frac{L}{E}g_{tt} - g_{t\phi}}{g_{\phi\phi} + \frac{L}{E}g_{t\phi}}$$

In particular, even if $L=0$ we can have $\omega \neq 0$ $(\omega = -g_{t\phi}/g_{\phi\phi})$ or with $\omega = 0$ we can have $L \neq 0$.

Horizons: $g^{rr}=0 \Leftrightarrow \frac{\Delta}{\rho^2}=0$. Since $\rho^2\geq 0$ this is $\Delta=0$ or

$$r^2-2GMr+a^2=0$$

or

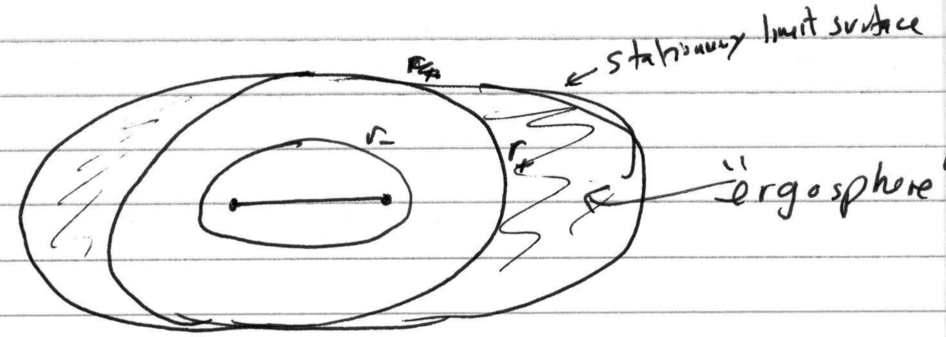
$$r=R_\pm = GM\pm\sqrt{(GM)^2-a^2} \qquad (GM>|a|).$$

Stationary Limit Surface: by definition this is a surface where ∂_t becomes null:

$$g_{\mu\nu}(\partial_t)^\mu(\partial_t)^\nu=0 \Leftrightarrow 1-\frac{2GMr}{\rho^2}=0$$

or

$$r^2+a^2\cos^2\theta-2GMr=0 \qquad (\cancel{\text{vs } \Delta(r)=0})$$



∂_t is spacelike outside the outer horizon! in the "ergosphere".

Moreover, at $r=r_+$ $\quad g_{\mu\nu}(\partial_t)^\mu(\partial_t)^\nu = \frac{r_+^2+a^2\cos^2\theta-2GMr_+}{r_+^2+a^2\cos^2\theta} = \frac{a^2\sin^2\theta}{r_+^2+a^2\cos^2\theta}\geq 0$

and the equality holds only at $\theta=0$ where the stationary limit surface and $r=r_+$ coincide.

The ergosphere is quite peculiar. Consider for simplicity the equator, $\theta=\frac{\pi}{2}$. Null lines have (with $r=$ constant; look at tangential emission).

$$0=g_{tt}dt^2+2g_{t\phi}\,dt\,d\phi+g_{\phi\phi}d\phi^2$$

or

$$\omega=\frac{d\phi}{dt}=-\frac{g_{t\phi}}{g_{\phi\phi}}\pm\sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\left(\frac{g_{tt}}{g_{\phi\phi}}\right)}$$

Ergosphere: $g_{tt}>0$ so $\sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\frac{g_{tt}}{g_{\phi\phi}}} < \left|\frac{g_{t\phi}}{g_{\phi\phi}}\right|$, so both solutions ~~be~~ $\pm$ have same sign!

and at stationary limit surface one solution has $\omega=0$

In fact $-\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2GMar\sin^2\theta}{\sin^2\theta\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]} = \omega$

has the sign determined by by $a = J/M$.

$\Rightarrow$ photons emitted tangentially (with $\dot{r}=0$ and $\dot{\theta}=0$) from the ergosphere move in same direction as rotation of black hole.

Null geodesics in more detail:

We had

$$E = -g_{tt}\frac{dt}{d\lambda} - g_{t\phi}\frac{d\phi}{d\lambda} \qquad L = g_{t\phi}\frac{dt}{d\lambda} + g_{\phi\phi}\frac{d\phi}{d\lambda} \tag{1}$$

Instead of doing most general, using S, we limit ourselves to $\theta = \frac{\pi}{2}$ trajectories. Then

$$g_{\lambda\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \Rightarrow$$

$$g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

Solve (1) above for $\frac{dt}{d\lambda}$ and $\frac{d\phi}{d\lambda}$, write

$$M\begin{pmatrix} dt/d\lambda \\ d\phi/d\lambda \end{pmatrix} = \begin{pmatrix} -E \\ L \end{pmatrix} \quad \text{where } M = \begin{pmatrix} +g_{tt} & +g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

we need M^{-1} which is just the inverse of the metric:

$$M^{-1} = \frac{1}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}$$

and

$$\begin{pmatrix} dt/d\lambda \\ d\phi/d\lambda \end{pmatrix} = M^{-1}\begin{pmatrix} -E \\ L \end{pmatrix} = \frac{1}{\underbrace{g_{tt}g_{\phi\phi} - g_{t\phi}^2}_{D}}\begin{pmatrix} -g_{\phi\phi}E - g_{t\phi}L \\ g_{t\phi}E + g_{tt}L \end{pmatrix}$$

So

$$g_{tt}\frac{(g_{\phi\phi}E + g_{t\phi}L)^2}{D^2} - 2g_{t\phi}\frac{1}{D^2}(g_{\phi\phi}E + g_{t\phi}L)(g_{t\phi}E + g_{tt}L)$$

$$+ \frac{1}{D^2}g_{\phi\phi}(g_{t\phi}E + g_{tt}L)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

This is of the form

$$\left(\frac{dr}{d\lambda}\right)^2 + V_{eff} = 0$$

Compute (at $\theta = \frac{\pi}{2}$) $(\rho^2 = r^2)$ $(\Delta = r^2 + a^2 - 2GMr = \rho^2 - 2GMr)$

$$D = g_{tt} g_{\phi\phi} - g_{t\phi}^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)\left(\frac{(r^2+a^2)^2}{\rho^2} - \frac{a^2\Delta}{\rho^2}\right) - \left(\frac{2GMar}{\rho^2}\right)^2$$

$$= -\frac{\Delta}{\rho^4}\left(\rho^4 - a^2\Delta\right) - \frac{(2GMar)^2}{\rho^4}$$

Better yet, since $\omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$ $\quad D = g_{tt} g_{\phi\phi} - \omega^2 g_{\phi\phi}^2 = g_{\phi\phi}(g_{tt} - \omega^2 g_{\phi\phi})$

$$V_{eff} \cdot D^2 g_{rr} = g_{tt}\left(g_{\phi\phi}^2 E^2 + 2EL g_{\phi\phi} g_{t\phi} + L^2 g_{t\phi}^2\right)$$

$$+ 2 g_{t\phi}\left(g_{\phi\phi} g_{t\phi} E^2 + EL \underbrace{\cancel{g_{t\phi}(g_{tt} + g_{\phi\phi})}}_{g_{\phi\phi} g_{tt} + g_{t\phi}^2} + L^2 g_{tt} g_{t\phi}\right)$$

$$+ g_{\phi\phi}\left(g_{t\phi}^2 E^2 + 2 g_{t\phi} g_{tt} EL + g_{tt}^2 L^2\right)$$

$$= E^2\left(g_{tt} g_{\phi\phi}^2 - 2 g_{\phi\phi} g_{t\phi}^2 + g_{\phi\phi} g_{t\phi}^2\right)$$

$$+ 2EL\left(g_{tt} g_{t\phi} g_{\phi\phi} - g_{t\phi} g_{\phi\phi} g_{tt} - g_{t\phi}^3 + g_{\phi\phi} g_{t\phi} g_{tt}\right)$$

$$+ L^2\left(g_{tt} g_{t\phi}^2 - 2 g_{tt} g_{t\phi}^2 + g_{\phi\phi} g_{tt}^2\right)$$

$$= E^2\left(g_{tt} g_{\phi\phi}^2 - \omega^2 g_{\phi\phi}^3\right) + 2EL\left(\cdot\omega^3 g_{\phi\phi}^3 - \omega g_{tt} g_{\phi\phi}^2\right)$$

$$+ L^2\left(g_{\phi\phi} g_{tt}^2 - \omega^2 g_{tt} g_{\phi\phi}^2\right)$$

$$V_{eff} = \frac{g_{\phi\phi}}{g_{rr} D}\left[E^2 - 2EL\omega + L^2 \frac{g_{tt}}{g_{\phi\phi}}\right]$$

$$\cancel{= \frac{\Delta}{\rho^2} \frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]\left[E^2 - 2EL\omega + L^2 \frac{(1 - 2GMr/\rho^2)}{\frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]}\right]}$$

$$\text{prefactor } \cancel{\frac{r^2+a^2-2GMr}{r^4}\left[\underbrace{(r^2+a^2)^2 - a^2(r^2+a^2-2GMr)}_{r^2(r^2+a^2) + 2GMra^2}\right]}$$

$$V_{eff} = \frac{1}{g_{rr}(g_{tt} - \omega^2 g_{\phi\phi})}\left[E^2 - 2EL\omega + L^2 \frac{g_{tt}}{g_{\phi\phi}}\right]$$

Since $\left(\frac{dr}{d\lambda}\right)^2 \geq 0$ solutions only exist for

$(E - V_+)(E - V_-) > 0$ that is both $V_\pm > E$ or both $V_\pm < E$

So study $V_\pm$. As $r \to \infty$ $V_\pm \sim \pm \frac{L}{r}$ Let's take $L > 0$

so that $V_+ > V_-$ (but we should also investigate $L < 0$, since presumably the relative sign of L and $a = \frac{J}{M}$ matter, and we are assuming $a \geq 0$).

Clearly $V_+ = V_-$ at $\Delta = 0 \Rightarrow$ the event horizon $r = R_+$.

There

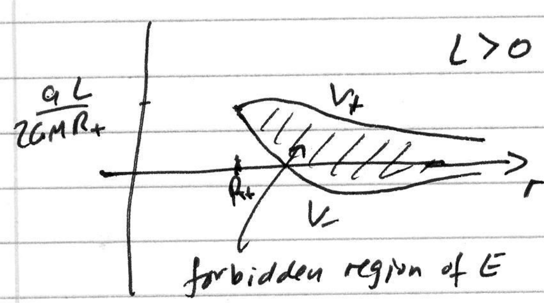
$$V_+ = V_- = \frac{2GMR_+ aL}{(R_+^2 + a^2)^2} = \frac{aL}{2GMR_+}$$

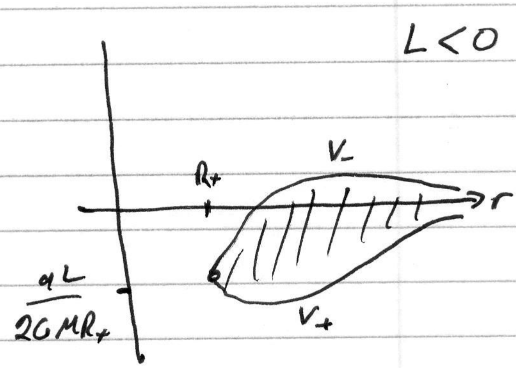
Note that V_+ has no zeroes, while V_- does a zero at

$$2GMr_0 a = r_0^2 \sqrt{\Delta(r_0)}$$

$$\Rightarrow (2GMa)^2 = r_0^2 (r_0^2 + a^2 - 2GMr_0)$$

In principle four zeroes, but note that at $a = 0$ three zeroes are at $r_0 = 0$ while one is at $r_0 \to 2GM$. So we suspect only one zero is in the region $r > R_+$.





Now, writing $-g_{tt} = 1 - \frac{2GMr}{\rho^2} = \frac{\Delta - a^2\sin^2\theta}{\rho^2}$

we have at $\theta = \frac{\pi}{2}$

$$\frac{g_{tt}}{g_{\phi\phi}} = -\frac{\Delta - a^2}{(r^2+a^2)^2 - a^2\Delta} \qquad \omega = \frac{2GMra}{(r^2+a^2)^2 - a^2\Delta}$$

and $$\omega^2 - \frac{g_{tt}}{g_{\phi\phi}} = \frac{(2GMra)^2 + (\Delta - a^2)\left[(r^2+a^2)^2 - a^2\Delta\right]}{\left[(r^2+a^2)^2 - a^2\Delta\right]^2}$$

$$\begin{aligned}
\text{numerator} &= (2GMra)^2 + (r^2 - 2GMr)\left[(r^2+a^2)^2 - a^2(r^2+a^2) + 2GMra^2\right] \\
&= r^2\left[(r^2+a^2)r^2 + 2GMra^2\right] - 2GMr\left[r^2+a^2\right]r^2 \\
&= (r^2+a^2)r^4 - 2GMr^5 \\
&= r^4(r^2+a^2-2GMr) = r^4\Delta
\end{aligned}$$

so

$$\boxed{V_\pm = L\left[\frac{2GMra \pm r^2\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}\right]}$$

and we have

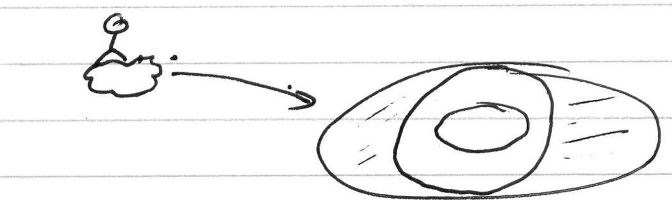
$$\left(\frac{dr}{d\lambda}\right)^2 = -\frac{(E-V_+)(E-V_-)}{g_{rr}(g_{tt} - \omega^2 g_{\phi\phi})}$$

$$\begin{aligned}
\text{den} &= -g_{rr}\,g_{\phi\phi}\left(\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}\right) \\
&= -\frac{\rho^2}{\Delta}\frac{1}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\right]\frac{r^4\Delta}{\left[(r^2+a^2)^2 - a^2\Delta\right]^2} = -\frac{r^4}{\left[(r^2+a^2)^2 - a^2\Delta\right]}
\end{aligned}$$

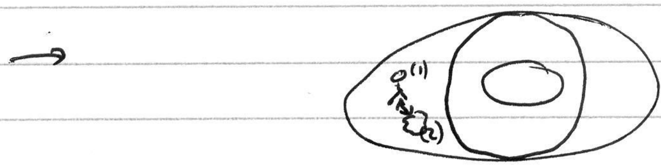
and

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E-V_+)(E-V_-)}$$

Penrose process



jump into ergosphere



push rock away inside ergosphere

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}$$

local conservation of p^μ

or, contracting with $(\partial_t)^\mu$

$$E^{(0)} = E^{(1)} + E^{(2)}$$

Clearly $E^{(0)} > 0$, but if you push $E^{(2)}$ hard enough you can arrange $E^{(2)} < 0$ so

$$E^{(1)} > E^{(0)}$$

$\Rightarrow$ come out of ergosphere with more than the original total energy.

Energy comes from black hole $\Rightarrow$ reduce b.h's angular momentum (rock must be thrown against rotation of b.h.).

To see this lets figure out the condition that the rock (2) crosses the event horizon R_+. We must be slightly careful since $r = R_+$ is a null surface.

Killing Horizons: if a Killing vector χ^μ is null on a null hypersurface Σ, then we say Σ is a Killing Horizon.

For Kerr, $\vec{\partial}_t$ is not null on the event horizon: it is null on the SLS (stationary limit surface) by def'n.

The event horizon is null, and

$$\vec{\chi} = \vec{\partial}_t + \Omega_H \vec{\partial}_\phi$$

is null for some constant Ω_H. Exercise: show $\Omega_H = \frac{a}{a^2+R_+^2}$

[Calculate: $\chi^2 = 0 = \partial_t^2 + 2\Omega_H \vec{\partial}_t \cdot \partial_\phi + \Omega_H^2 \partial_\phi^2$

$$= g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}$$

so $$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

Now on $r = R_+$ $\Delta = r^2 - 2GMr + a^2 = 0$ and

$$g_{tt} = -\left(1 - \frac{2GMr}{\rho^2}\right) = -\frac{1}{\rho^2}\left(\rho^2 - 2GMr\right) = -\frac{1}{\rho^2}\left(r^2 + a^2\cos^2\theta - r^2 - a^2\right)$$

$$= +\frac{1}{\rho^2} a^2 \sin^2\theta$$

$$g_{\phi\phi} = \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] = \frac{\sin^2\theta}{\rho^2}(r^2+a^2)^2$$

$$g_{t\phi} = -\frac{2GMra\sin^2\theta}{\rho^2} = -\frac{a(a^2+r^2)\sin^2\theta}{\rho^2}$$

so $$\Omega_H = \frac{a}{a^2+R_+^2} \pm \sqrt{\frac{a^2}{(a^2+R^2)^2} - \frac{a^2}{(a^2+R^2)^2}} = \frac{a}{a^2+R_+^2}$$]

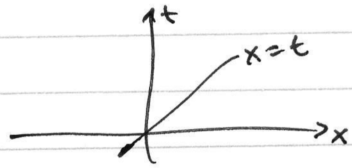
Exercise: show $r = R_+$ is null.

Note: if Σ is defined through $f(x^\mu) = \text{constant}$, then Σ is null $\Leftrightarrow$ $\nabla_\mu f$ is null.

[Calculate: $r = R_+$ is the same as $f(t, r, \theta, \varphi) = r$. Then

$\partial_\mu r$ has $\quad g^{\mu\nu} \partial_\mu r \, \partial_\nu r = 0$ since $g^{rr} = 0$ at $r = R_\pm$.]

To see what condition we want to impose on $p^{(2)}$ (that signals $p^{(2)}$ crosses R_+), look at a null cone in Minkowski space first



The surface is defined by $x - t = \text{constant}$, and $\nabla_\mu (x-t)$ is a null vector both normal and tangent to it.

$$n_\mu = \nabla_\mu (x-t) \Rightarrow n^\mu = (1,1)$$

Now, if we have a particle moving along $x^\mu(\lambda)$, then

$$n \cdot \frac{dx}{d\lambda} = -\frac{dt}{d\lambda} + \frac{dx}{d\lambda}$$

So, if $n \cdot \frac{dx}{d\lambda} < 0 \Rightarrow \frac{dx}{d\lambda} < \frac{dt}{d\lambda}$ or $\frac{dx}{dt} < 0$

while if $n \cdot \frac{dx}{d\lambda} > 0$ then $\frac{dx}{dt} > 0$

Going back to our problem, if χ^μ is a null killing vector tangent to $r = R_+$ then

$$\chi \cdot p^{(2)} < 0 \quad \text{at } r = R_+ \quad \text{signals motion inwards,}$$

but moreover since χ is a killing vector $\chi \cdot p^{(2)} = \text{constant}$.

Now $\vec{\chi} = \vec{\partial}_t + \Omega_H \vec{\partial}_\phi$ with $\Omega_H = \frac{a}{a^2 + R_+^2}$ is a null killing vector on $r = R_+$. Ω_H can be interpreted as the angular velocity of the black hole at the event horizon R_+, ~~as can be~~ as since it corresponds to the minimum ω for a massive particle at $r = R_+$.

Then the condition is $\chi \cdot p^{(2)} < 0$

But $\chi \cdot p^{(2)} = \vec{\partial}_t \cdot p^{(2)} + \Omega_H \vec{\partial}_\phi \cdot p^{(2)} = -E^{(2)} + \Omega_H L^{(2)} < 0$

$$\Rightarrow \quad L^{(2)} < \frac{E^{(2)}}{\Omega_H} < 0$$

So the angular momentum of the black hole decreases by $L^{(2)}$.

$$\delta M = E^{(2)}$$
$$\delta J = L^{(2)}$$

and $$\delta J < \frac{\delta M}{\Omega_H}$$

Conclusion: Energy is extracted from the black hole. As a result the black hole looses ~~energy~~ mass and spin.

However, the process cannot violate the <u>area theorem</u>, that the area of the event horizon is non-decreasing.

The area of the event horizon is

$$A = 4\pi (R_+^2 + a^2)$$

[Calculate from $ds_2^2 = \gamma_{ij} dx^i dx^j = ds^2$ at $dt = dr = 0$, $r = R_+$.

then $$A = \int |\det\gamma|^{1/2} d\theta d\phi$$
$$= \int \sqrt{\rho^2 \cdot \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}\, d\theta d\phi$$

at $r = R_+$, we have $\Delta = 0$ so
$$A = (R_+^2 + a^2)\int \sin\theta\, d\theta d\phi \quad].$$

To show that A is non-decreasing, define the "irreducible mass" through

$$M_{irr}^2 = \frac{A}{16\pi G^2} = \frac{1}{4G^2}(R_+^2 + a^2)$$

$\Delta = 0 \Rightarrow R_+^2 + a^2 = 2GMR_+$
$R_+ = GM + \sqrt{(GM)^2 - a^2}$

$$= \frac{1}{4G^2} 2GM[GM + \sqrt{(GM)^2 - a^2}]$$

or $$M_{irr}^2 = \frac{1}{2}\left[M^2 + \sqrt{M^4 - (J/G)^2}\right]$$ $(J = Ma)$

Now

$$2\cdot 2M_{irr}\,\delta M_{irr} = 2M\delta M + \frac{1}{2}\,\frac{4M^3\delta M - 2J\delta J/G^2}{\sqrt{M^4-(J/G)^2}}$$

$$= \frac{2\left(M\sqrt{M^4-(J/G)^2}+M^3\right)\delta M - J\delta J/G^2}{\sqrt{M^4-(J/G)^2}}$$

we recognize

$$M^3 + M\sqrt{M^4-(J/G)^2} = M\left(M^2+\sqrt{M^4-(J/G)^2}\right)$$

$$= 2M\,M_{irr}^2$$

$$= 2M\,\frac{1}{4G^2}\left(R_+^2+a^2\right)$$

$$= \frac{2Ma}{4G^2}\,\frac{1}{\Omega_H}$$

$$= \frac{J}{2G^2}\,\frac{1}{\Omega_H}$$

so

$$\delta M_{irr} = \frac{J/G^2\left[\,\delta M/\Omega_H - \delta J\,\right]}{4M_{irr}\sqrt{M^4-(J/G)^2}}$$

so our bound that $\delta J < \delta M/\Omega_H$ implies $\delta M_{irr} > 0$

Now $$\delta A = 16\pi G^2\,\delta M_{irr}^2 = \frac{8\pi J\left[\,\delta M/\Omega_H - \delta J\,\right]}{\sqrt{M^4 - J^2/G^2}}$$

or

$$\delta M = \frac{\kappa}{8\pi G}\,\delta A + \Omega_H\,\delta J$$

where $$\kappa = \frac{\sqrt{G^2M^2 - J^2/M^2}\;\Omega_H}{J/M} = \frac{\sqrt{G^2M^2-a^2}}{R_+^2+a^2}$$

or $$\kappa = \frac{\sqrt{G^2M^2-a^2}}{2GM\left(GM+\sqrt{(GM)^2-a^2}\right)}$$

[Note: κ is the surface gravity of the Kerr metric. For a Killing horizon with Killing (null) vector $\vec{\chi}$, the surface gravity is

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)$$

study this later]

Now

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$$

is just like

$$dE = \cancel{k_B} T\, dS - p\, dV$$

for a thermodynamic system, with the association

$$E \Leftrightarrow M$$

$$\frac{A}{8\pi G} \Leftrightarrow S$$

$$T \Leftrightarrow \frac{\kappa}{2\pi}$$

The ambiguity in the association of A & T (where do we put the $8\pi G$) is settled by Hawking's black hole evaporation.

	Thermodynamics	Black Holes
0^{th} Law:	T is constant in thermal equilibrium	Stationary black holes have constant κ.
1^{st} Law:		Energy conservation
2^{nd} Law:	$\delta S > 0$	$\delta A > 0$

Generalized 2nd Law $\delta\left(S + \frac{A}{8\pi G}\right) > 0$.

Note: To make sense of units, S is dimensionless ($k_B = 1$) but $\frac{A}{4\hbar G}$ has units of mass × length, same as $\hbar$. So, it really should be
$S \Leftrightarrow \frac{A}{8\pi\hbar G}$ and $T \Leftrightarrow \frac{\kappa\hbar}{2\pi}$ (or $\hbar \to h$).

Stationary axisymmetric space: general observations.

(i) General case: require $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$ (not of t, ϕ),

and symmetry $t \to -t$, $\phi \to -\phi$ (so $g_{tr} = g_{t\theta} = 0 = g_{\phi r} = g_{\phi\theta}$).

$$\Rightarrow \quad ds^2 = -\tilde{A}\,dt^2 + B\,d\phi^2 - 2B\omega\,dt\,d\phi + C\,dr^2 + D\,d\theta^2$$

$$= -A\,dt^2 + B(d\phi - \omega\,dt)^2 + C\,dr^2 + D\,d\theta^2 \qquad \tilde{A} = A - B\omega^2$$

Note that

$$g^{rr} = \frac{1}{C}, \quad g^{\theta\theta} = \frac{1}{D} \quad \text{if } G = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \Rightarrow G^{-1} = \frac{1}{\det G}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}, \quad \det G = g_{tt}g_{\phi\phi} - g_{t\phi}^2$$

$$= -(A - B\omega^2)B - (B\omega)^2$$

$$= -AB$$

$$\Rightarrow \quad g^{tt} = -\frac{1}{A} \qquad g^{\phi\phi} = \frac{A - B\omega^2}{AB} \qquad g^{t\phi} = -\frac{\omega}{A}$$

For Kerr, plug into $R_{\mu\nu} = 0$. Huge mess!

(ii) Killing vectors $\vec{\partial}_t, \vec{\partial}_\phi \Rightarrow$ conserved quantities

$$L = p_\phi = m\,g_{\phi\mu}\frac{dx^\mu}{d\tau} \quad \text{and} \quad E = p_t = m\,g_{t\mu}\frac{dx^\mu}{d\tau}$$

or, replace $\tau/m \to \lambda$, for massless.

More explicitly

$$L = g_{\phi\phi}\frac{d\phi}{d\lambda} + g_{\phi t}\frac{dt}{d\lambda}$$

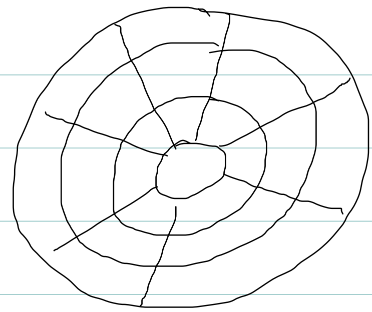
$$E = g_{t\phi}\frac{d\phi}{d\lambda} + g_{tt}\frac{dt}{d\lambda}$$

$\underline{L = 0}$: $\quad \dfrac{d\phi}{dt} = -\dfrac{g_{\phi t}}{g_{\phi\phi}} = \omega(r,\theta)$ ANGULAR VELOCITY WITHOUT ANGULAR MOMENTUM!

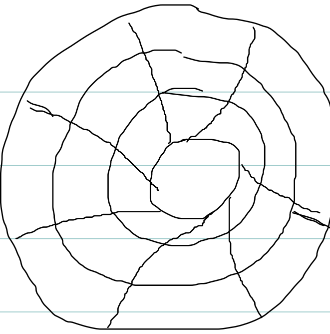
Suppose metric is asymptotically flat (as in Kerr). Then "drop" body from ∞ towards center (from $r=\infty$ towards $r=0$) with $\frac{d\phi}{dt} = 0$ originally (since $\omega(r,\theta)\big|_{r=\infty} = 0$).

Then $\frac{d\phi}{dt}$ will change as body drops.

"Dragging of inertial frames": our test body is free falling, so locally it is moving in straight line $\to$ interpret $\frac{d\phi}{dt} \neq 0$ as moving/rotating inertial frames.



no dragging



dragging

iii) Stationary limit surface

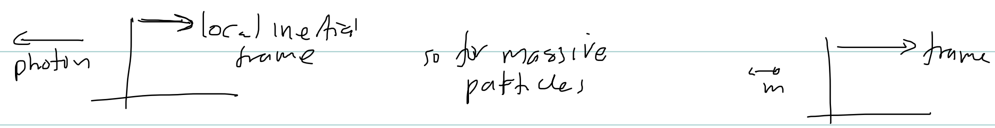
Consider photon emitted in ϕ-direction (from (r,θ,ϕ)).

At emission $d\theta = 0 = dr \Rightarrow ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dt\,d\phi + g_{\phi\phi}d\phi^2$

$$\Rightarrow \quad \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \sqrt{\frac{A}{B}}$$

- While $g_{tt}/g_{\phi\phi} < 0$ get $\begin{cases} d\phi/dt > 0 \rightarrow \gamma \text{ emitted in + dir} \\ d\phi/dt < 0 \rightarrow \gamma \text{ emitted in } - \text{ dir} \end{cases}$

- On a $g_{tt} = 0$ surface $\left(\frac{A}{B} = \omega^2\right)$ $\frac{d\phi}{dt} = \begin{cases} 2\omega \rightarrow \text{twice as fast!} \\ 0 \rightarrow \text{going nowhere!?!} \end{cases}$



Massive particles all dragged in same direction on $g_{tt} = 0$ surface.

"stationary limit surface" = any surface with $g_{tt} = 0$

Q: Schwarszchild? (leave for student to ponder)

Inside stationary limit surface all bodies and radiation are forced to move in same direction, <u>cannot remain fixed</u>.

To see this

Suppose U^μ is 4-vel of body, $U^2=-1$. If we take $U^\mu = (U^t,0,0,0)$

$\Rightarrow$ $g_{tt} = -1/(U^t)^2 < 0$, incompatible with interior of lim. surf.

But nothing wrong with $g_{tt}(U^t)^2 + 2g_{t\phi}U^tU^\phi + g_{\phi\phi}U^{\phi 2} = -1$ since $g_{t\phi} = -\omega g_{\phi\phi}$

and the relative sign of U^t & U^ϕ not fixed. Just off $g_{tt}=0$ we neglect $g_{tt}(U^t)^2$

and get $U^\phi(U^\phi - 2\omega U^t) = -1/g_{\phi\phi}^2 < 0$ which is easily satisfied.

(iv) Redshift: recall for comoving observers (fixed coordinates)

emitting/receiving light,

$$\frac{\lambda_{rec}}{\lambda_{emit}} = \sqrt{\frac{g_{tt}(rec)}{g_{tt}(emit)}}$$

For an observer at stationary limit surface, $g_{tt}(emit) \to 0$ $\Rightarrow$ $\lambda_{rec} \to \infty$.

This is just as with Schwarzschild

(v) Event Horizons. Again we look for null 3-surfaces.

$f(x^\mu) = 0$ defines surface

$\partial_\mu f$ = gradient = normal to surface $= n_\mu$

Tangent $f(x^\mu(\lambda)) = 0 \Rightarrow 0 = \frac{d}{d\lambda} f(x^\mu(\lambda)) = \frac{dx^\mu}{d\lambda}\partial_\mu f \Rightarrow \frac{dx^\mu}{d\lambda} n_\mu = 0 \Rightarrow$ vectors "$\perp$" to n_μ.

In particular, if n_μ is null then $n^\mu = g^{\mu\nu}n_\nu$ is $\perp$ to n_μ ($g^{\mu\nu}n_\mu n_\nu = 0$).

$\Rightarrow$ Look for $g^{\mu\nu}\partial_\mu f \partial_\nu f = 0$

Recall in spherically symmetric case we take $f = f(r)$ $\Rightarrow$

$$g^{\mu\nu}\partial_\mu f \partial_\nu f = 0 \Rightarrow g^{rr}(\partial_r f)^2 = 0 \Rightarrow g^{rr} = 0$$

Now, with axial symmetry, $f = f(r,\theta)$, $g^{rr}(\partial_r f)^2 + 2g^{r\theta}\partial_r f \partial_\theta f + g^{\theta\theta}(\partial_\theta f)^2 = 0$

We can still look for solutions with $f = f(r)$ and $g^{rr} = 0$. Let's look at this (and previous) in Kerr metric

Back to Kerr

(i) Singularities. From $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ one finds $\rho=0$ is a singularity.

Now $\rho^2 = r^2 + a^2\cos^2\theta = 0 \Rightarrow r=0 \quad \theta = \frac{\pi}{2}$

Recall Boyer-Lindquist coordinates in Cartesian:

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$$
$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$$
$$z = r\cos\theta$$

so $r=0$ is $z=0$, $x^2+y^2 \le a^2$ disk:

$$x = (a\sin\theta)\cos\phi$$
$$y = (a\sin\theta)\sin\phi$$

$\Rightarrow \theta = \frac{\pi}{2}$ is $x^2+y^2=a^2$ circle ("equator?").

The singularity is not a point but S^1!

(ii) Event Horizons: look for $g^{rr}=0$. Now $g^{rr} = \frac{\Delta}{\rho^2}$, so need $\Delta=0$.

$\Rightarrow r^2 - 2GMr + a^2 = 0 \Rightarrow r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$

Note, if $|a| < GM \Rightarrow r_- < a < r_+$ and the singularity is behind the horizon (singularity at $r=0$, less than r_-).

For $|a| > GM$, no horizon, naked singularity.

$|a| = GM = r_\pm$ is "extreme Kerr B.H." [It is belived, through calculations, that realistic BH's are near extremal Kerr BH's, since accretion increases $a = J/M$. Limited only by accreting matter radiating away some angular momentum. Calculations give $a \gtrsim 0.998GM$ – see text by Hobson, Efstathiou & Lasenby, p. 324).

Geometry: take $r = \text{const} (=r_\pm)$ $t = \text{const}$ 2-dim surface. Line element is

$$ds^2 = \rho_\pm^2\, d\theta^2 + \left(\frac{2GMr_\pm}{\rho_\pm}\right)^2 \sin^2\theta\, d\phi^2$$

Not the geometry of S^2 embedded in R^3. Rather a pancake, or more techically, an axisymmetric ellipsoid (embedded in R^3).

(iii) Stationary Limit Surface: $g_{tt}=0$

$$1-\frac{2GMr}{\rho^2}=0 \Rightarrow r^2+a^2\cos^2\theta-2GMr=0$$

$$r_\pm(\theta)=GM\pm\sqrt{(GM)^2-a^2\cos^2\theta}$$

(Not the same $r_\pm$ as before, excuse the notation)

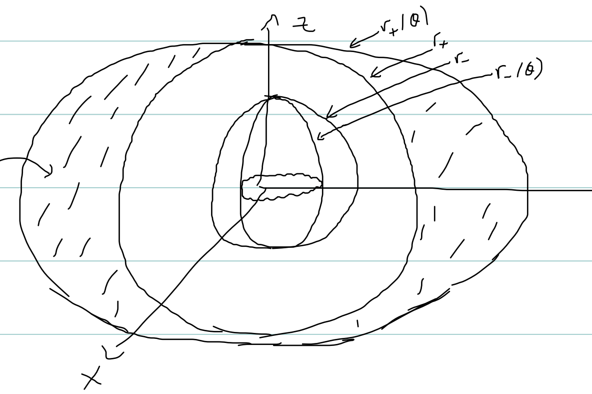
Geometry (as previous)

$$ds^2=\rho_\pm^2\,d\theta^2+\frac{2GMr_\pm(2GMr_\pm+2a^2\sin^2\theta)}{\rho_\pm}\sin^2\theta\,d\phi^2$$ with $r_\pm=r_\pm(\theta)$ and $\rho_\pm=\rho_\pm(\theta)$ & $\rho_\pm^2=2GMr_\pm$.

Which is bigger? $r_\pm=GM\pm\sqrt{(GM)^2-a^2}$ or $r_\pm(\theta)=GM\pm\sqrt{(GM)^2-a^2\cos^2\theta}$

$\Rightarrow r_+(\theta)>r_+>r_->r_-(\theta)$, with $r_+(0,\pi)=r_+$ and $r_-(0,\pi)=r_-$ (equal at poles)

Also $r_-(\frac{\pi}{2})=0\equiv$ singularity.



region with $g_{tt}>0$
not behind event horizon.

In ergosphere everything is forced to move. As before $U^2=-1$ with $U^\mu=(U^t,0,0,U^\phi)$

$\Rightarrow (U^t)^2(g_{tt}+2g_{t\phi}\Omega+g_{\phi\phi}\Omega^2)=-1$ where $\Omega=\frac{U^\phi}{U^t}=\frac{d\phi}{dt}$

U^t real $\Rightarrow$ $g_{tt}+2g_{t\phi}\Omega+g_{\phi\phi}\Omega^2<0 \Rightarrow \Omega\in(\Omega_-,\Omega_+)$ with

$$\Omega_\pm=-\frac{g_{t\phi}}{g_{\phi\phi}}\pm\sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\frac{g_{tt}}{g_{\phi\phi}}}=\omega\pm\sqrt{\frac{A}{B}}$$

Special cases

(i) $A=0$ ($g_{tt}=0$) $\Rightarrow \Omega_-=0$, $\Omega_+=2\omega$. This occurs at $r=r_+(\theta)$ (outer stat. lim. surf)
(already discussed)

(ii) $\omega^2=g_{tt}/g_{\phi\phi}$ $\Rightarrow \Omega_-=\Omega_+=\omega$. This occurs at $r=r_+$, so call this Ω_H.

We have $\Omega_H = \omega(r_+, \theta) = \dfrac{a}{2GMr_+}$ (from Kerr metric).

At the horizon the angular velocity is limited to the one value Ω_H.