Physics 225B, General Relativity. Winter 2015
Homework 1

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DUE: Wednesday, January 21, 2015

1. Let $\vec{K}$ be a Killing vector in a metric space with compatible connection and curvature tensor $R^\rho{}_{\sigma\mu\nu}$. Show

$$K^\rho{}_{;\sigma\mu} = \nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$$

and

$$K^\mu R_{;\mu} = K^\mu \nabla_\mu R = 0.$$

2. Let x, y, z be coordinates of flat Euclidean 3-dimensional space, R^3, with metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 + dz^2.$$

Consider the paraboloid $\mathcal{P}$, a sub-manifold of R^3 defined by the condition

$$z = x^2 + y^2.$$

An embedding of $\mathcal{P}$ in R^3 is a map between manifolds given by

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= \rho^2 \end{aligned}$$

where ρ, ϕ are coordinates on the paraboloid, $\mathcal{P}$, defined in $\rho \in [0, \infty)$ and $\phi \in [0, 2\pi]$.
(a) Determine the *induced metric* in $\mathcal{P}$, that is, the pull-back of $g_{\mu\nu}$ to $\mathcal{P}$. Call this $\hat{g}_{ij}$.
(b) Let $\hat{g}^{ij}$ be the inverse of $\hat{g}_{ij}$. Determine the push-forward of $\hat{g}^{ij}$ to R^3. Call this $\tilde{g}^{\mu\nu}$.
(c) Compare $\tilde{g}^{\mu\nu}$ with $g^{\mu\nu}$, the inverse of $g_{\mu\nu}$. Surprised?

3. (a)Consider the n-dimensional manifold R^n. Find the integral curve of the vector field $V^\mu = x^\mu$ from the point x^μ_o in cartesian coordinates. What goes wrong at the origin, that is, if the point $x^\mu_o = 0$?

(b) Construct explicitly a one parameter family of diffeomorphisms ϕ_t taking the point p_o with coordinates x^μ_o to a point p with coordinates y^μ on the integral curve of V^μ a parameter distance t away.

(c) For an arbitrary vector field $\vec{W}$, find the push-forward (by ϕ_{-t}) of $\vec{W}|_p$ and compute the Lie Derivative from its definition (taking the difference of this push-forward and $\vec{W}$ at p_o).

(d) Compute the commutator $[\vec{V}, \vec{W}]$. Compare your answer with part (c).

4. (Exercise B.1 in Carroll). In Euclidean three-space, find and draw the integral curves of the vector fields

$$A = \frac{y - x}{r}\frac{\partial}{\partial x} - \frac{y + x}{r}\frac{\partial}{\partial y}$$

and

$$B = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}.$$

Calculate $C = \mathcal{L}_A B$ and draw the integral curves of C. (*Note that it says "draw," rather than "find and draw," the integral curves of* C.)