Physics 4A
Feb. 24, 2015

Sunil Sinha

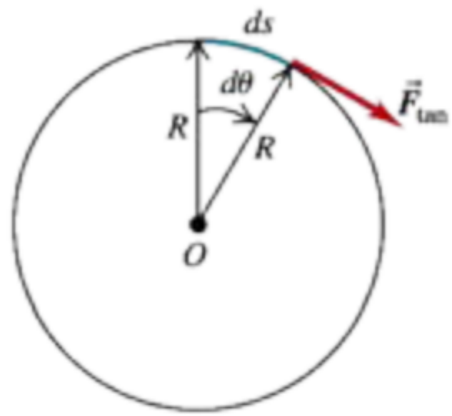
UCSD Physics

Work Done By Torque In Rotational Motion

Tangential force $\vec{F}$ over time dt applied at rim of a disk causes torque $\vec{\tau}$, leads to ang. displacement dè

Work done $dW = F_{tan} ds = F_{tan} R d\theta$

$$\Rightarrow dW = \tau_z d\theta \Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta$$



Overhead view of merry-go-round

If applied torque is constant

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta = \tau_z(\theta_2 - \theta_1)$$

Work & Power In Rotational Motion

As result of work done by $\vec{\tau}$, kinetic energy changes

Since $\vec{\tau} = I\vec{\alpha}_z \;\; \Rightarrow \tau_z d\theta = I\alpha_z d\theta$

$$\tau_z d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega = I\omega_z d\omega_z$$

$$\Rightarrow W = \int_{\theta_1}^{\theta_2}\tau_z d\theta = \int_{\omega_1}^{\omega_2} I\omega_z d\omega_z = \frac{1}{2}I(\omega_2^2 - \omega_1^2) = \Delta K$$

work-energy theorem
for rotating rigid bodies

Power associated with applied external torque:

$$\text{P} = \frac{\text{dW}}{\text{dt}} = \tau_z\frac{d\theta}{dt} = \tau_z\omega_z$$

Cable Unwinding Off A Cylinder

Cable wrapped around cylinder (mass M) is attached to object mass m. As cable unwinds, U_{grav} converted to kinetic energy.

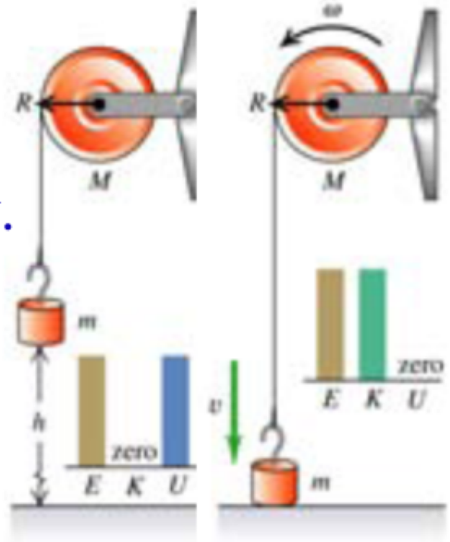
Find speed of object as it hits floor

$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow v = \sqrt{\frac{2gh}{1 + M/2m}}$$



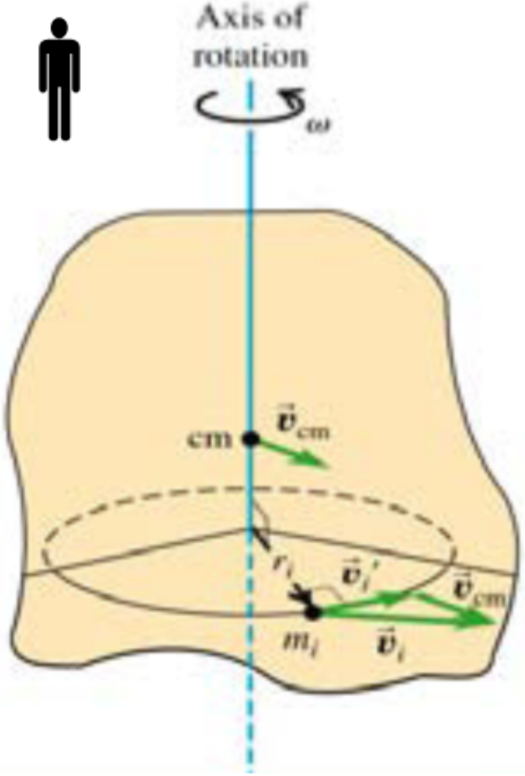
Cylinder is Solid

Rigid Body Rotation About a Moving Axis

A rigid body's motion = sum of translation motion $\vec{v}_{CM}$ of CM

& rotation about an axis through the CM

Component particle m_i at $\vec{r}_i$ has $\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$ $\Leftarrow$ vel. rel. to CM



$$K_i = \frac{1}{2} m_i (\vec{v}_{cm} + \vec{v}_i') . (\vec{v}_{cm} + \vec{v}_i')$$

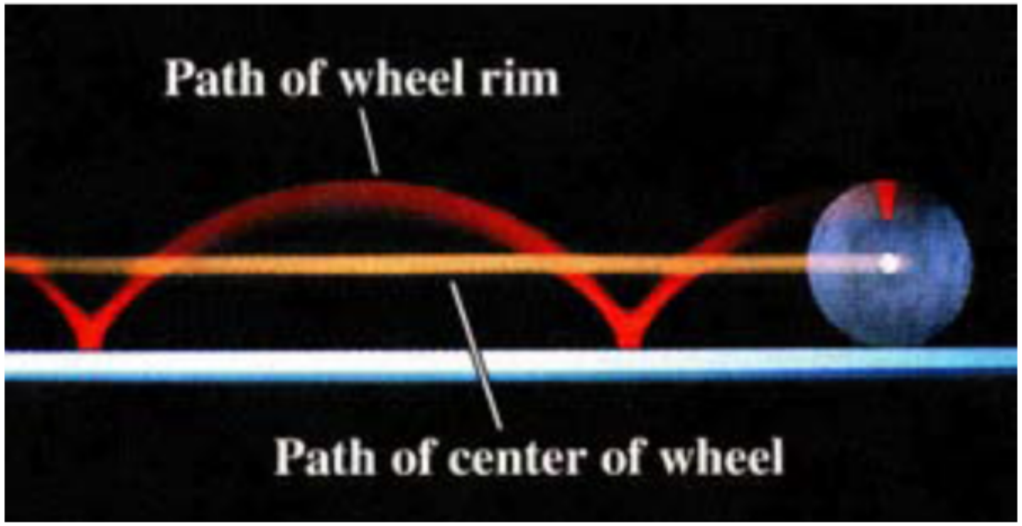
$$\Rightarrow K_i = \frac{1}{2} m_i (v_{cm}^2 + 2\vec{v}_{cm} . \vec{v}_i' + v_i'^2)$$

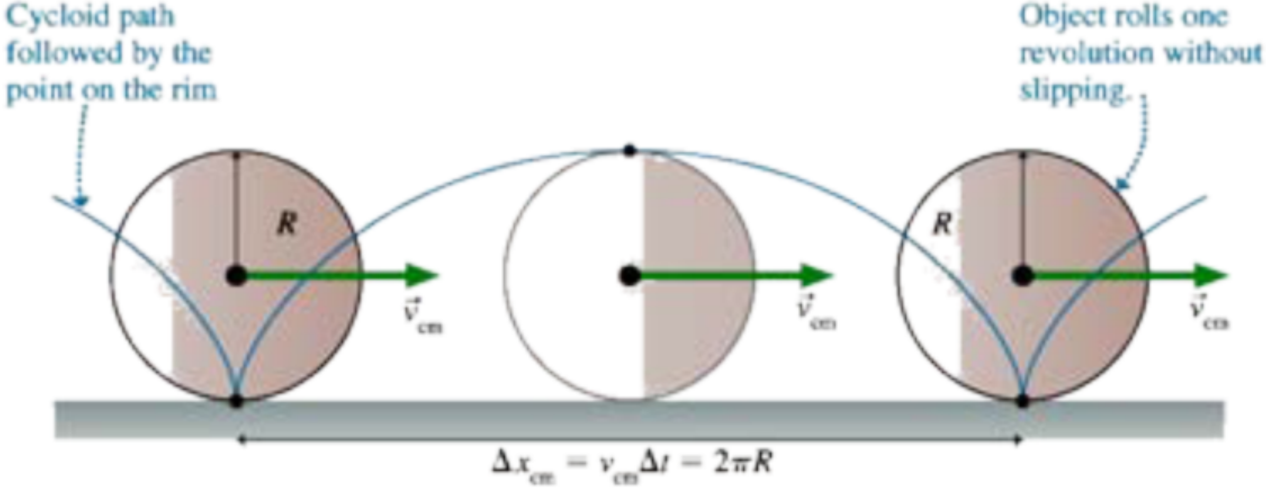
$$\Rightarrow K = \frac{1}{2} (\sum_i m_i) v_{cm}^2 + \vec{v}_{cm} . (\sum_i m_i \vec{v}_i') + \sum_i \left(\frac{1}{2} m_i v_i'^2 \right)$$

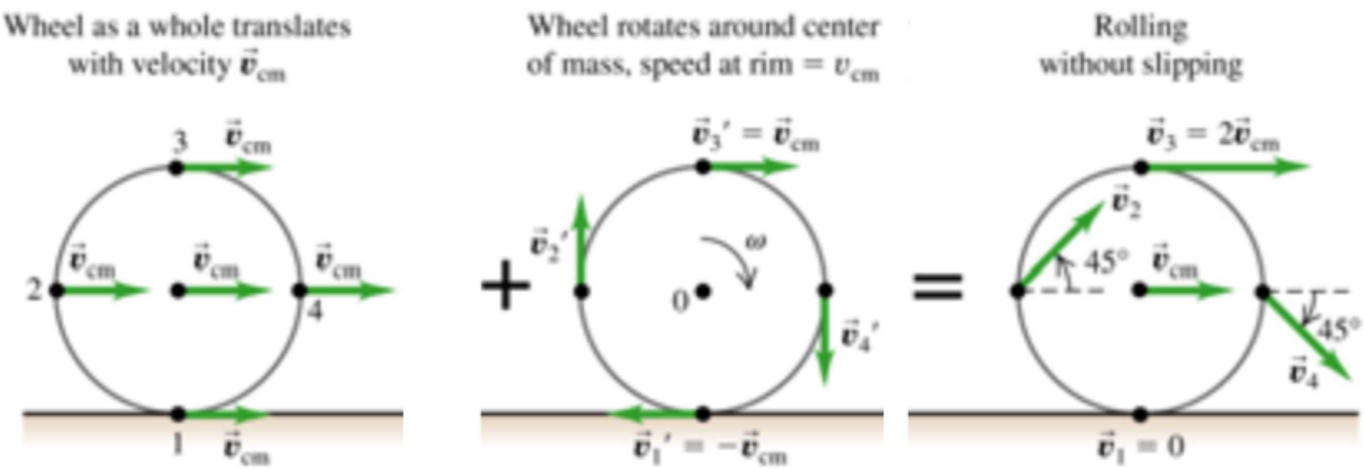
Since $\sum_i m_i \vec{v}_i' = \mathrm{M} \times$ Vel. of CM relative to CM = 0

$$\Rightarrow \boxed{K = \frac{1}{2} M v_{cm}^2 + 0 + \frac{1}{2} I_{cm} \omega^2}$$

Ball Rolling Thru One Revolution







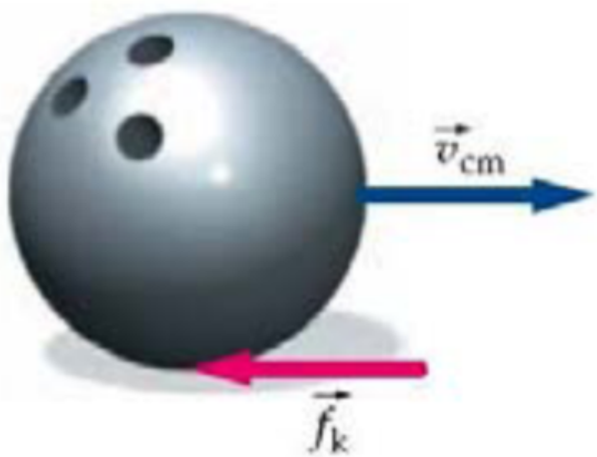
Think of wheel as rotating about an "instantaneous" axis of rotation that passes thru point of contact (labeled 1) with surface. ω is same for this axis (since rigid body) as thru CM. $\Rightarrow$ $\boxed{K = I_1\omega^2}$

Parallel Axis Theorem $\Rightarrow I_1 = I_{CM} + MR^2$

$$\Rightarrow K = \frac{1}{2} I_1 \omega^2 = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MR^2 \omega^2$$

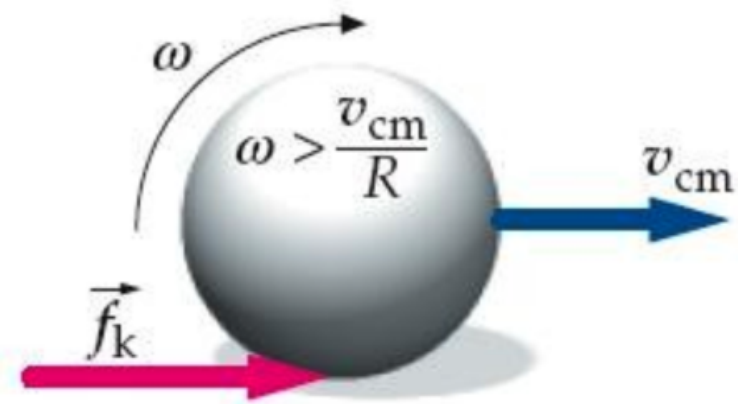
$$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$$

Rolling **With** Slipping ($v_{CM} \neq \omega R$)



A bowling ball moving with no initial angular speed.

The frictional force ***f_k*** exerted by the floor reduces the linear speed and increases the angular speed ω until $\mathbf{v_{cm}=R\omega}$ ***(how)***



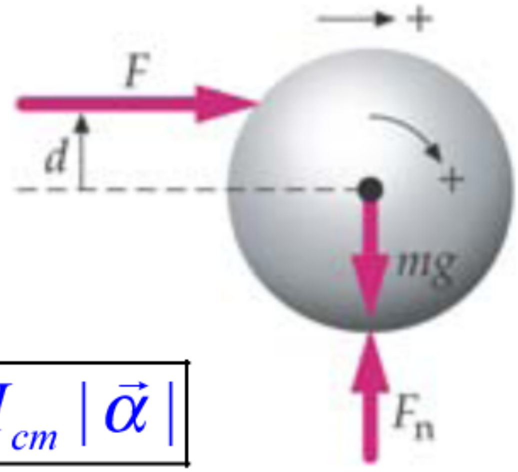
Ball with "excess" topspin initially.

The frictional force accelerates the ball in the direction of motion.

Pool Shark Physics

For rotation about an axis thru CM, magnitude of the only torque is:

$$|\vec{\tau}| = \vec{F}.\vec{d} = Fd$$



Second Law $\Rightarrow$ $\boxed{F = ma_{cm}}$ & $\boxed{|\vec{\tau}| = I_{cm}\,|\vec{\alpha}|}$

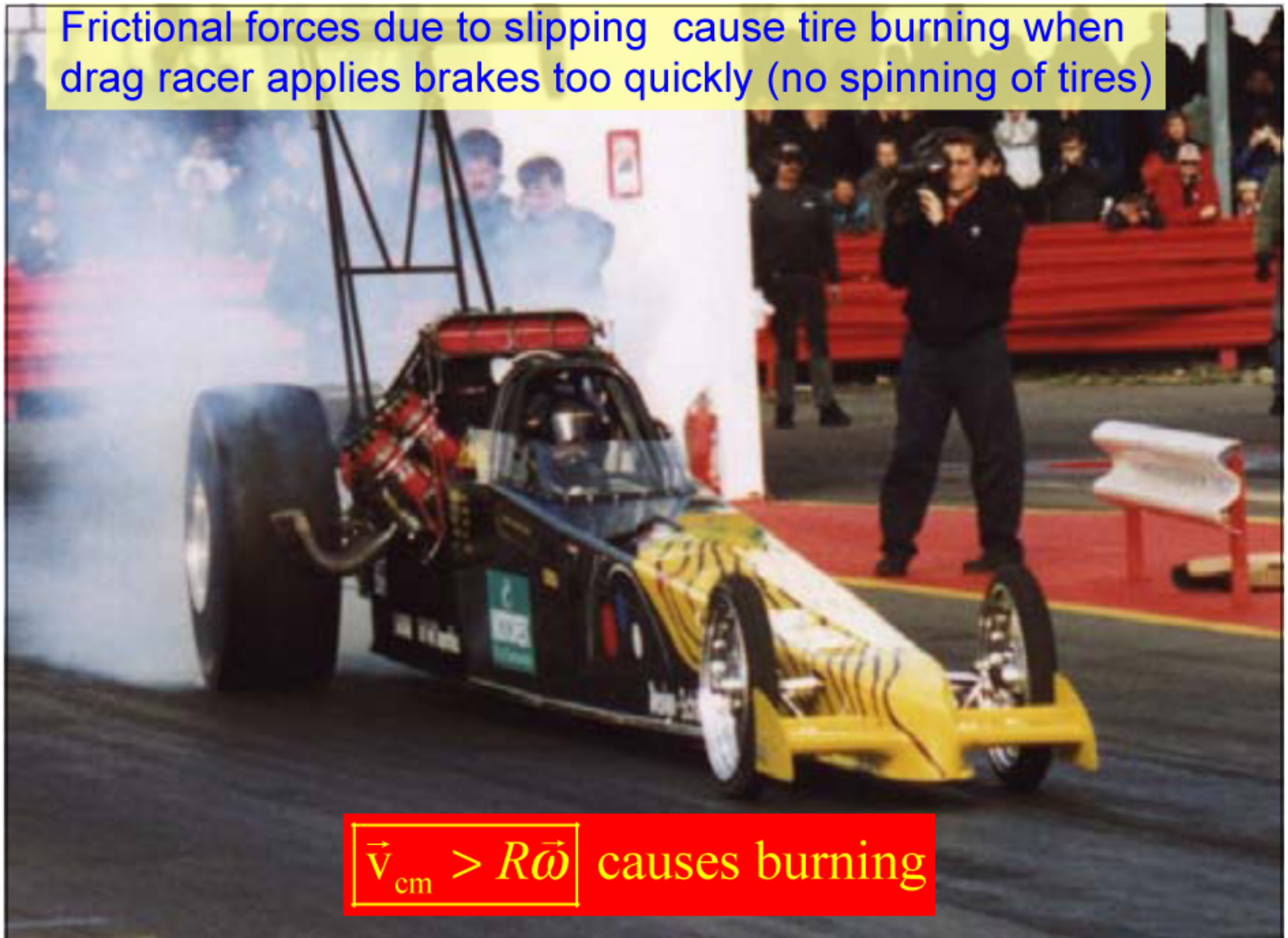
Rolling without slipping means: $a_{cm} = R\alpha$

$$\Rightarrow \frac{F}{M} = R\frac{Fd}{I_{cm}}, \text{ note } \mathrm{I}_{cm} = \frac{2}{5}MR^2$$

$$\Rightarrow \boxed{d = \frac{\mathrm{I}_{cm}}{mR} = \frac{2}{5}R}$$

Always NEED to understand ***which*** forces cause non-zero torques along the axis of rotation !

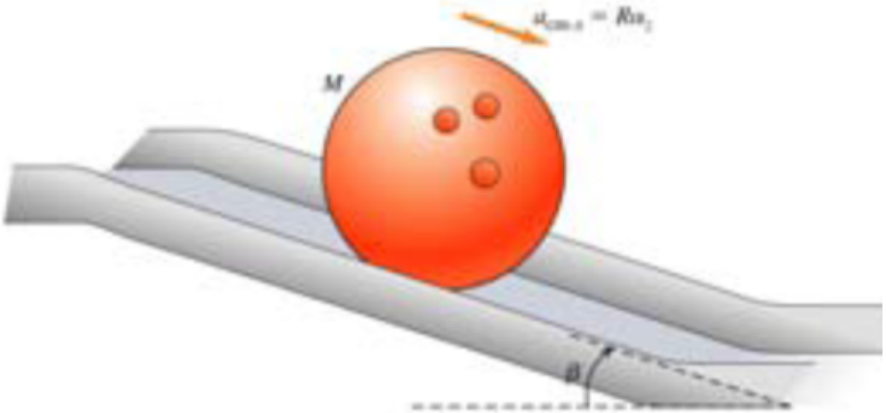
Frictional forces due to slipping cause tire burning when drag racer applies brakes too quickly (no spinning of tires)

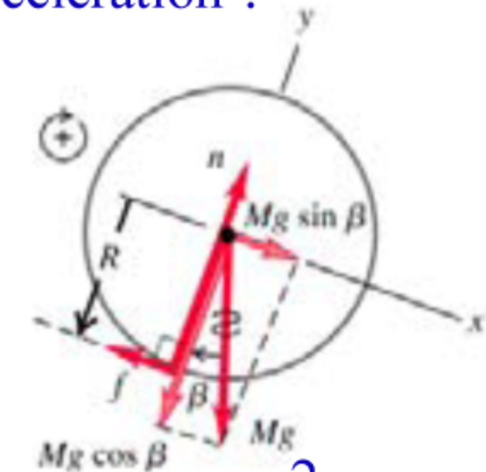


$\vec{v}_{cm} > R\vec{\omega}$ causes burning

Acceleration Of A Rolling Sphere

A giant solid sphere rolls, without slipping, down an incline (with friction) at angle β. What is ball's acceleration ?





$$\sum F_x = Mg\sin\beta + (-f) = Ma_{cm-x} \;\;\&\;\; \sum \tau_z = fR = I_{cm}\alpha_z = \frac{2}{5}MR^2\alpha_z$$

No slipping $\Rightarrow \boxed{a_{cm\text{-}x} = R\alpha_z} \Rightarrow \boxed{f = \frac{2}{5}Ma_{cm\text{-}x}}$ (*static* friction,why?)

$$\therefore f_s = \frac{2}{5}Ma_{cm\text{-}x} = Ma_{cm\text{-}x} - Mg\sin\beta \Rightarrow \boxed{a_{cm\text{-}x} = \frac{5}{7}g\sin\beta}$$

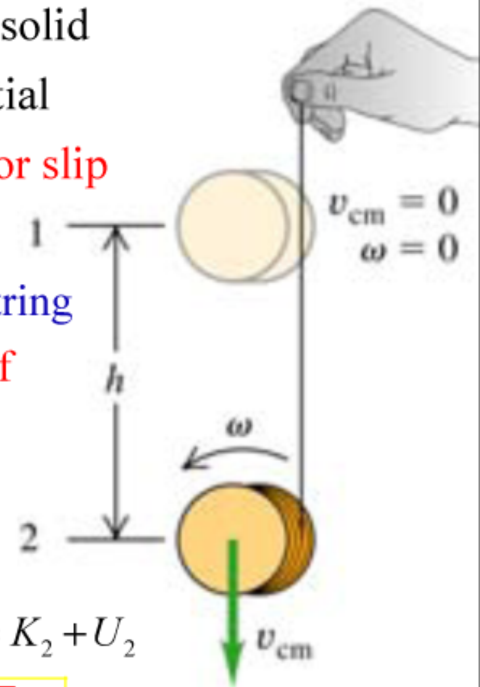
Yo-Yo's Speed: From Energy Perspective

Yo-Yo made of string wrapped several times around a solid cylinder of mass M and radius R. Released with no initial motion. String unwinds & rotates but does not stretch or slip

What is v_{cm} of yo-yo after it drops height =h?

No upward motion ⇒ Hand does no work on cylinder+string

Friction cause rotation but since no slipping on surface of cylinder ⇒ no mech. energy lost ⇒ Use E conservation:



$$K_2 = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{cm}^2}{R}\right) = \frac{3}{4}Mv_{cm}^2$$

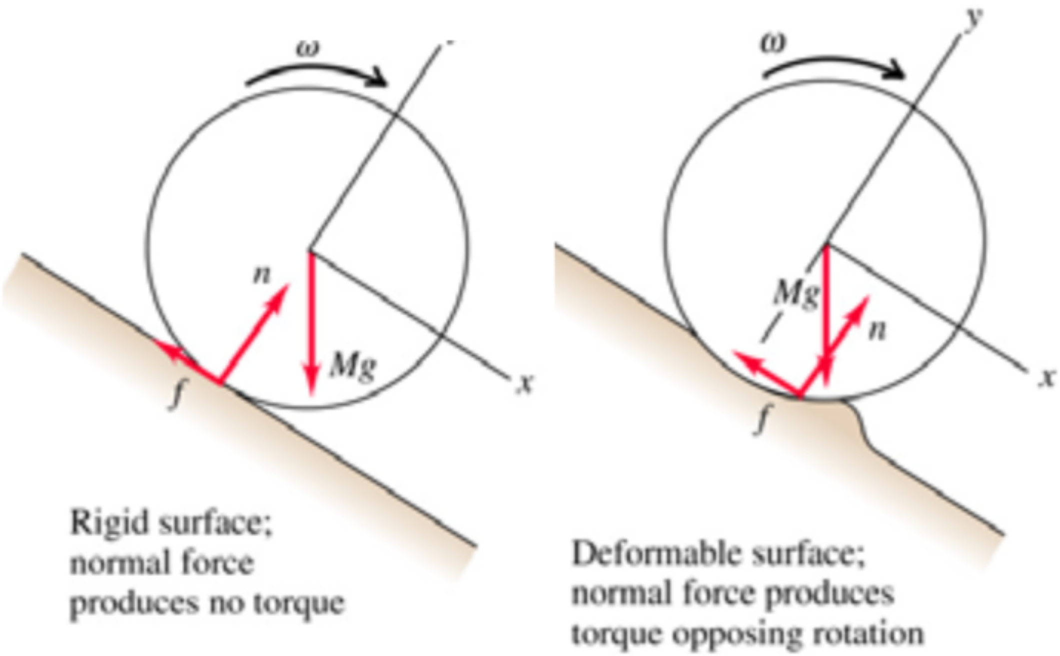
Conservation of Energy $\Rightarrow K_1 + U_1 = K_2 + U_2$

$$\Rightarrow 0 + Mgh = \frac{3}{4}Mv_{cm}^2 + 0 \Rightarrow \boxed{v_{cm} = \sqrt{\frac{4}{3}gh}}$$

v_{cm} is less than if yo-yo was just dropped since some energy went into rotational Kinetic Energy K_{rot}

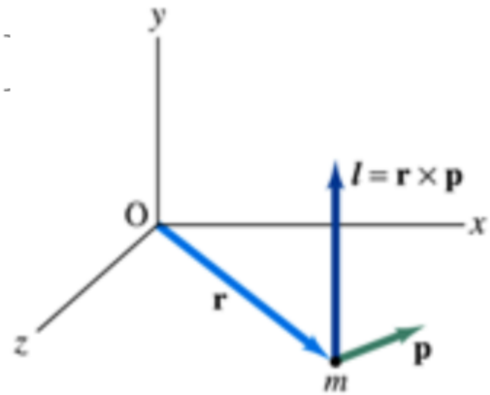
Rolling Friction

No rolling friction if body and surface totally rigid.
Not often the case. If sphere or surface are deformable then normal force (no longer act along a single point) produces a counterclockwise torque that opposes clockwise rotation



Torque Causes Change in Angular Momentum

Angular Momentum



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

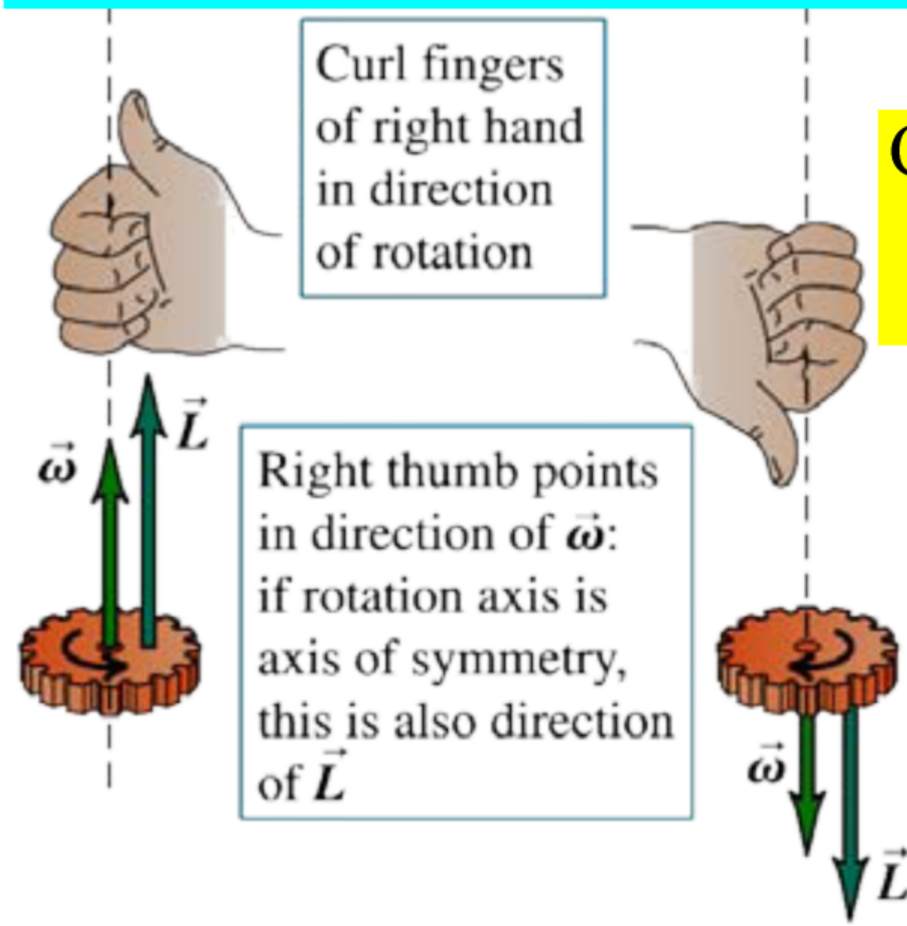
$$|\vec{L}| = r(mv)\sin\phi$$

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

$$\boxed{\frac{d\vec{L}}{dt} = (\vec{r} \times \vec{F}) = \vec{\tau}}$$

rate of change of angular Momentum= torque of net force acting on it

Angular momentum $\vec{L}$ always directed along $\vec{\omega}$



Can show that :

$$\vec{L} = I\vec{\omega}$$

& Rate of change of ang. momentum with time

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation Of Angular Momentum

Rate of change of ang. momentum with time $\sum \vec{\tau} = \frac{d\vec{L}}{dt}$

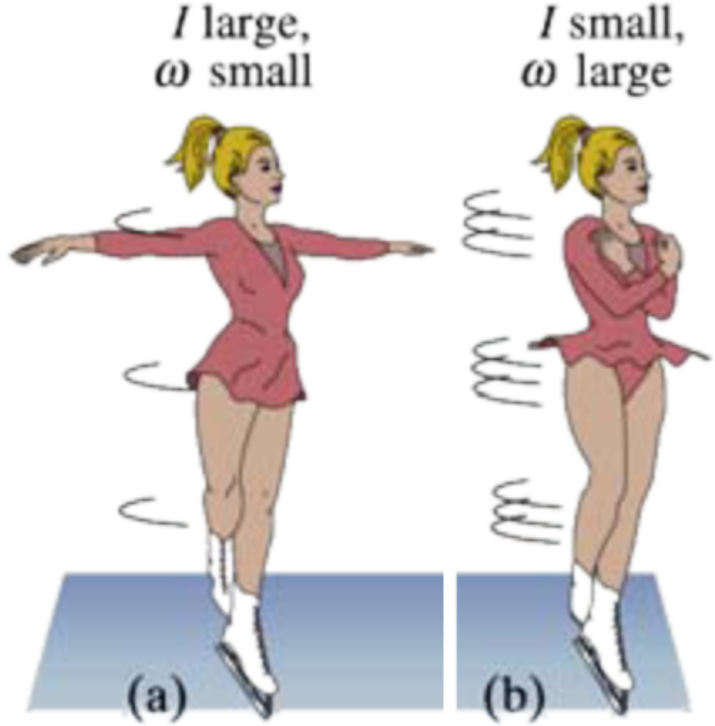
When $\sum \vec{\tau} = 0$, $\vec{L}$=constant

$$\Rightarrow \boxed{I_1 \omega_{1z}}_{before} = \boxed{I_2 \omega_{2z}}_{after}$$

If Inertia I changes because of rearrangement of object's mass $\Rightarrow$ corresponding change in ω

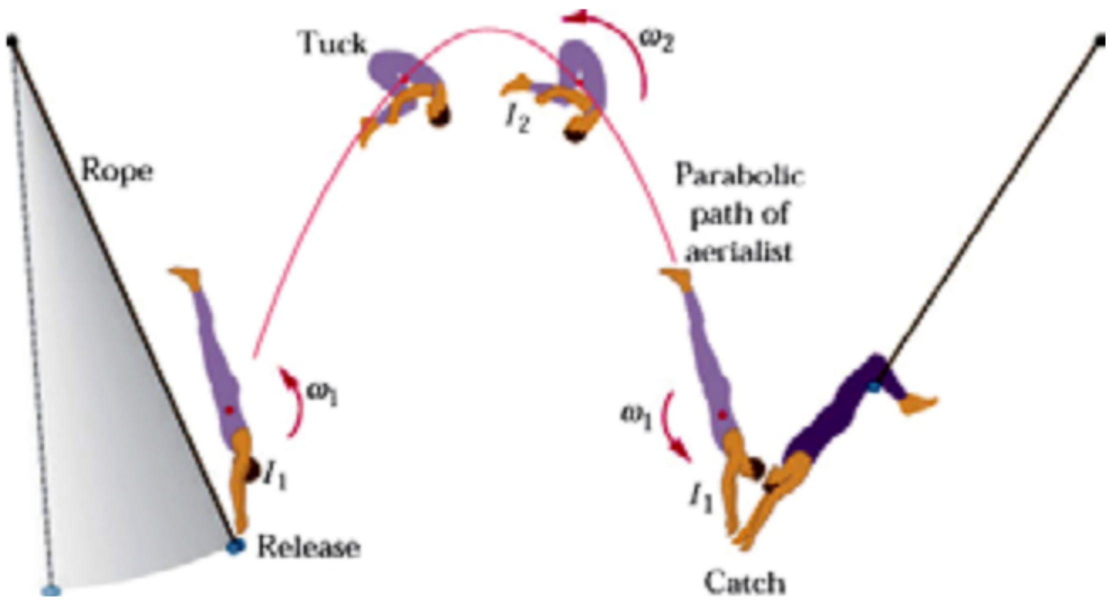
$\Rightarrow$ Many spectacular examples of angular momentum conservation !

Angular Momentum Conservation



Cirque du Soleil !

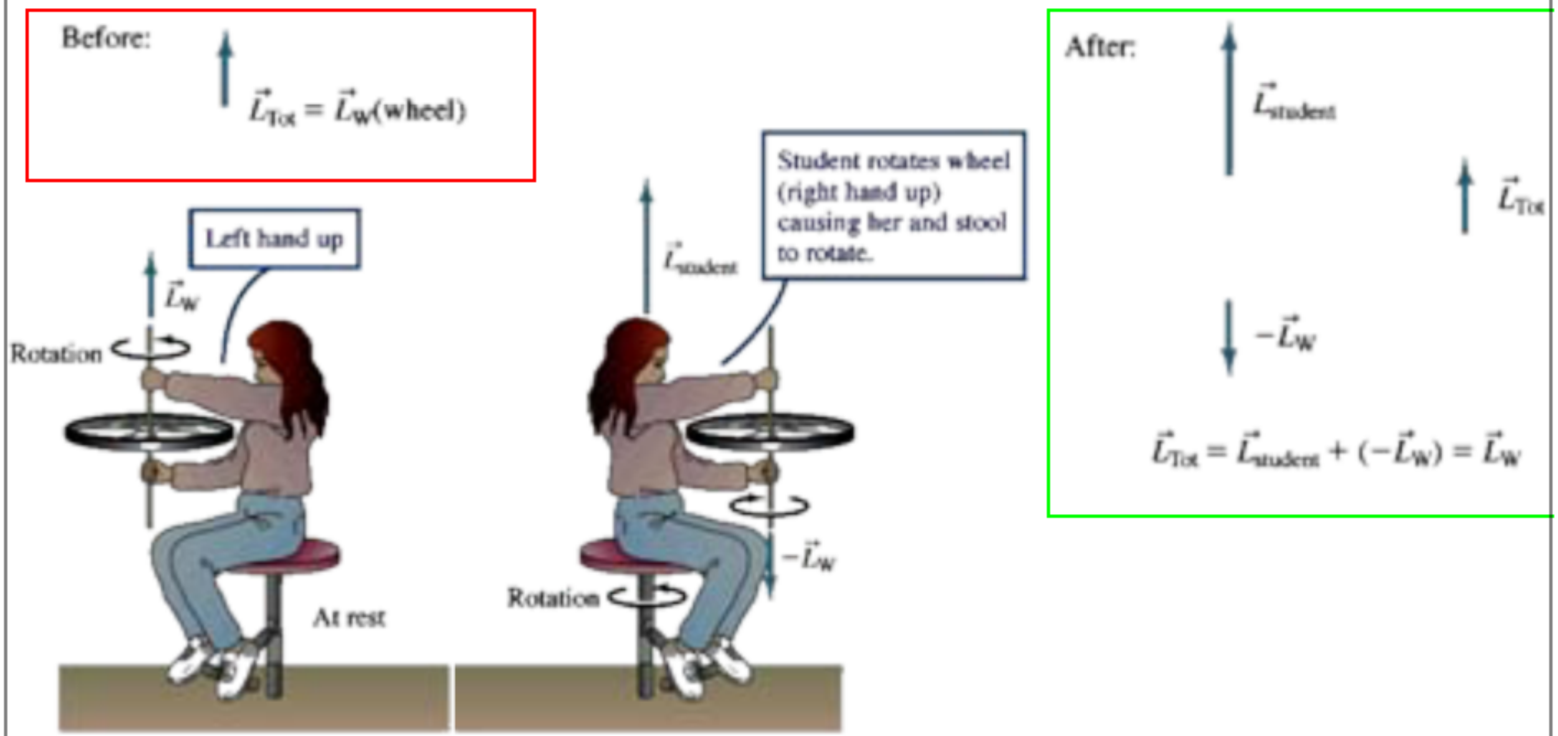
Quadrapule somersault !



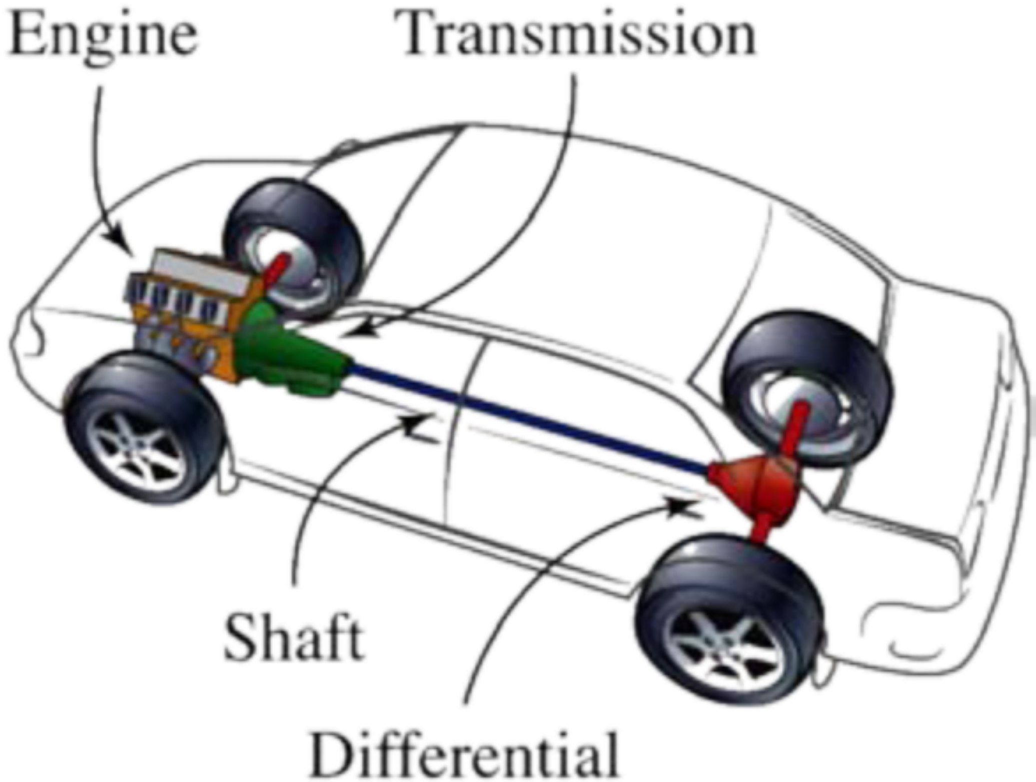
Same Mass But Differently Distributed

$$\sum \vec{\tau} = 0 \Rightarrow \vec{\mathrm{L}} = \text{constant} \Rightarrow \boxed{I_1 \omega_{1z}}_{before} = \boxed{I_2 \omega_{2z}}_{after}$$

Vector nature of angular momentum $\vec{L}$ and Implications of conservation of $\vec{L}$



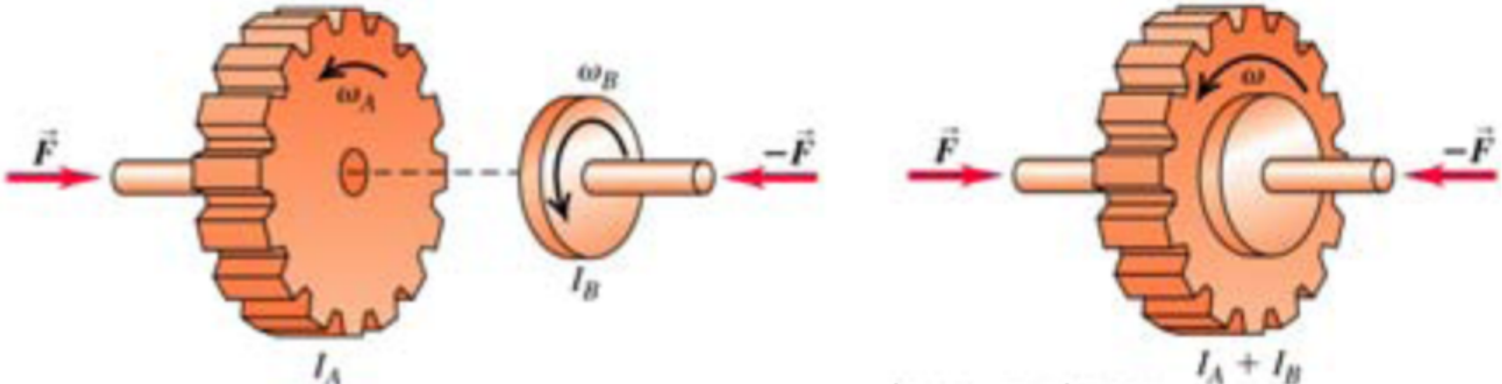
Can Angular Momentum Be Transferred ?



Angular momentum contained in a turning engine is transferred to the wheels when the “clutch” is engaged This engagement is a kind of inelastic “rotational” collision. L is conserved by K is reduced…See next

Engine Flywheel & Transmission Shaft

a flywheel & a clutch plate attached to a transmission shaft each rotating independently, then joined together by forces acting along the axis of rotation (no additional torque). Disks rub against each other and finally reach a common ω



$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega_{final}$$

$$\Rightarrow \omega_{final} = \frac{I_A\omega_A + I_B\omega_B}{(I_A + I_B)}$$

angular momentum conservation

Example of inelastic angular collision of 2 rigid bodies