

Phonons and Forces: Momentum *versus* Pseudomomentum in Moving Fluids

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Abstract

I provide a pedagogical introduction to the notion of pseudomomentum for waves in a medium, and show how changes in pseudomomentum may sometimes be used to compute real forces. I then explain how these ideas apply to sound waves in a fluid. When the background fluid is in motion, the conservation laws for pseudomomentum and pseudoenergy are most easily obtained by exploiting the acoustic metric and the formalism of general relativity.

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I. INTRODUCTION

This conference is devoted to the physics of waves moving through a medium which affects them as would a background metric. There is therefore a natural analogy with waves propagating in a gravitational field — but we should take care not to push the analogy too far. These systems differ from real general relativity in that the medium constitutes a physical æther. While we may ignore the æther for many purposes, occasionally it is important. For example, if we wish to compute forces exerted by the waves, we must take into account any stress transmitted by the background medium.

The natural tool for computing forces by tracking the flux of energy and momentum is the energy-momentum tensor. This is best defined as the functional derivative of the action with respect to the background metric — but we have two metrics at our disposal: the spacetime metric and the acoustic or other metric which we are exploiting for our GR analogy. Despite the temptation to believe otherwise, we should remember that it is only by differentiating with respect to the “real” metric that we obtain “real” energy and “real” momentum. When we differentiate with respect to the analogy metric, we obtain the density and flux of other quantities. These are usually the pseudoenergy and the pseudomomentum [1].

Failure to distinguish between real energy and momentum and pseudoenergy and pseudomomentum has caused much confusion and controversy over the years. Consider the formula for the momentum of a photon in a dielectric: should the refractive index go in the numerator, where it was placed by Minkowski, or in the denominator, as argued by Abrahams? This dispute was not resolved until Blount [2] identified Minkowski’s expression with the pseudomomentum and Abrahams’ with the true momentum, including the mechanical momentum of the dielectric [3,4].

Although I will not address the problem here, my initial motivation for thinking about these topics was the fear that a similar confusion lies behind some recent controversies [5] involving the Iordanskii force. This force, which has an appealing GR analogue in the

gravitational Aharonov-Bohm effect of a spinning cosmic string [6], is supposed to act on a vortex in a superfluid when it moves relative to the normal component of the fluid.

A related issue, and one that I will address, lies at the heart of the two-fluid model for a superfluid or Bose condensate. It is one of the fundamental assumptions of the two-fluid model that the phonons in a fluid possess momentum $\hbar\mathbf{k}$, and that, unlike that of phonons in a solid, this momentum is true Newtonian momentum, $m\mathbf{v}$. This assumption is essential because we wish to identify the phonon momentum density with the mass current, both being equal to $\rho\mathbf{v}$. The desired identification is supported by the approximate solution of Bogoliubov's weakly interacting Bose gas model, in which the phonon creation operator, $\hat{a}_{\mathbf{k}}^\dagger$, appears to be create bona-fide momentum, so it is quite unnerving to discover that in the literature of fluid mechanics the attribution of real momentum to a sound wave is regarded as a naïve and dangerous fallacy. A particularly forceful statement of this opinion is to be found in the paper [7] *On the Wave Momentum Myth*, by Michael McIntyre.

In the present article we will focus the difference between the true momentum and the pseudomomentum associated with the acoustic metric. The first part is a general pedagogical account of the distinction between momentum and pseudomomentum, and the circumstances under which the latter may be used for computing forces. The second part will discuss the energy and momentum associated with sound waves in a background flow [8].

II. MOMENTUM AND PSEUDOMOMENTUM

The distinction between true momentum and pseudomomentum is especially clear when we consider the problem of transverse vibrations on an elastic string. The action

$$S = \int dx dt \left\{ \frac{\rho}{2} \dot{y}^2 - \frac{T}{2} y'^2 \right\} \quad (2.1)$$

gives rise to the familiar wave equation

$$\rho \ddot{y} - T y'' = 0, \quad (2.2)$$

with $c = \sqrt{T/\rho}$ being the wave speed. By manipulating the wave equation we can establish two local conservation laws. The first,

$$\frac{\partial}{\partial t} \left\{ \frac{\rho}{2} \dot{y}^2 + \frac{T}{2} y'^2 \right\} + \frac{\partial}{\partial x} \{-T\dot{y}y'\} = 0, \quad (2.3)$$

is immediately recognizable as an energy conservation law, with the flux $-T\dot{y}y'$ being the rate of doing work by an element of the string on its neighbor. The second,

$$\frac{\partial}{\partial t} \{-\rho\dot{y}y'\} + \frac{\partial}{\partial x} \left\{ \frac{\rho}{2} \dot{y}^2 + \frac{T}{2} y'^2 \right\} = 0, \quad (2.4)$$

is slightly more obscure in its interpretation.

In a relativistic system the appearance of $-\rho\dot{y}y' = (\text{energy flux})/c^2$ as the local density of a conserved quantity would not be surprising. The symmetry of the energy-momentum tensor requires that $T^{j0} = T^{0j}$, so the energy flux T^{j0} is also (after division by c^2) the density of 3-momentum. Here, however, we are dealing with non-relativistic classical mechanics — and with transverse waves. Whatever the quantity $-\rho\dot{y}y'$ may be, it is *not* the density of the x component of the string's momentum. It is instead the density of *pseudomomentum*.

To understand the origin of pseudomomentum, observe that our elastic string may be subjected to two quite distinct operations either of which might be called “translation in the x direction”:

- An operation where the string, together with any disturbance on it, is translated in the x direction.
- An operation where the string itself is left fixed, but the disturbance is translated in the x direction.

The first operation leaves the action invariant provided space is homogeneous. The associated conserved quantity is true Newtonian momentum. The second operation is a symmetry only when both the background space *and the string* are homogeneous. The conserved quantity here is pseudomomentum. Such a distinction between true and pseudomomentum is necessary whenever a medium (or æther) is involved.

Adding to the confusion is that, although the pseudomomentum is conceptually distinct from the true momentum, there are many circumstances in which changes in pseudomomentum can be used to compute real forces. As an example consider a high speed train picking up its electrical power from an overhead line.

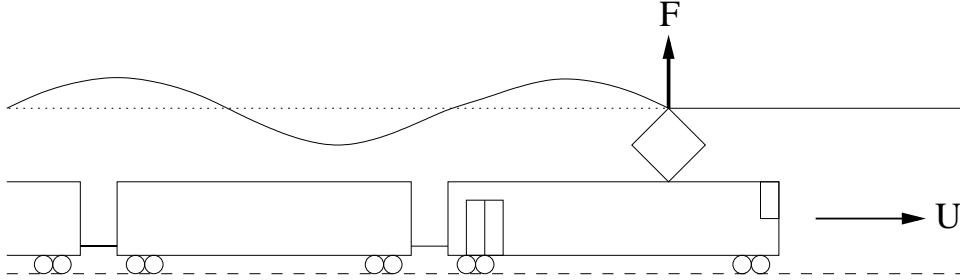


Fig 1. A high-speed train .

The locomotive is travelling at speed U and the pantograph pickup is exerting a constant vertical force F on the power line. We make the usual small amplitude approximations and assume (not unrealistically) that the line is supported in such a way that its vertical displacement obeys an inhomogeneous Klein-Gordon equation

$$\rho\ddot{y} - Ty'' + \rho\Omega^2 y = F\delta(x - Ut), \quad (2.5)$$

with $c = \sqrt{T/\rho}$, the velocity of propagation of short wavelength disturbances.

If $U < c$, the vertical displacement relaxes symmetrically about the point of contact. Once U exceeds c , however, the character of the problem changes from elliptic to hyperbolic, and an oscillatory “wake” forms behind the pantograph. As with all such wakes, the disturbance is stationary when viewed from the frame of the train. With this in mind, we seek a solution to (2.5) of the form $y = y(x - Ut)$. Since the overhead line is undisturbed ahead of the locomotive, we find

$$\begin{aligned} y &= \frac{\gamma F c}{\Omega T} \sin \frac{\Omega \gamma}{c} (Ut - x), & x < Ut \\ y &= 0, & x > Ut. \end{aligned} \quad (2.6)$$

Here $\gamma = (U^2/c^2 - 1)^{-\frac{1}{2}}$ is the Lorentz contraction factor modified for tachyonic motion. The condition that the phase velocity, ω/k , of the wave constituting the wake be equal to the

forward velocity of the object creating it is analogous to the Landau criterion determining the critical velocity of a superfluid. There are no waves satisfying this condition when $U < c$, but they exist for all $U > c$.

In the wake, the time and space averaged energy density $\langle \mathcal{E} \rangle = \langle \frac{1}{2}\rho\dot{y}^2 + \frac{1}{2}Ty'^2 + \frac{1}{2}\Omega^2y^2 \rangle$ is given by

$$\langle \mathcal{E} \rangle = \frac{1}{2}\rho\gamma^4 \left(\frac{F}{T} \right)^2 U^2. \quad (2.7)$$

The expression for the pseudomomentum density for the Klein-Gordon equation is the same as that for the wave equation, and the average pseudomomentum density is

$$\langle -\rho\dot{y}y' \rangle = \frac{1}{2}\rho\gamma^4 \left(\frac{F}{T} \right)^2 U. \quad (2.8)$$

Because energy is being transferred from the locomotive to the overhead line, it is clear that there must be some induced drag, F_d , on the locomotive. This is most easily computed from energy conservation. The rate of working by the locomotive, $F_d U$, must equal the energy density times the rate of change of the length of the wave-train. Thus

$$F_d U = \langle \mathcal{E} \rangle (U - U_g) \quad (2.9)$$

where

$$U_g = \frac{\partial\omega}{\partial k} = \frac{c^2 k}{\omega} = \frac{c^2}{U} \quad (2.10)$$

is the group velocity of the waves, and so the speed of the trailing end of the wake wave-train.

After a short calculation we find that the wave-induced drag force is

$$F_d = \frac{1}{2} \frac{F^2 \gamma^2}{T}. \quad (2.11)$$

Since the average pseudomomentum density turned out to be the average energy density divided by U , we immediately verify that we get exactly the same answer for F_d if we equate the wave drag to the time rate of change of the total pseudomomentum.

For the sceptic we note that we may also obtain the same answer by a more direct evaluation of the force required to deflect the overhead wire.

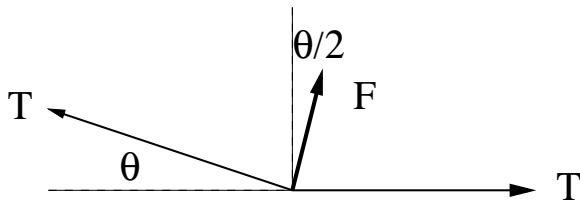


Fig 2. Force on overhead wire.

From the solution (2.6) we see that the force F is related to the angle of upward deflection, θ , by $\theta \approx \tan \theta = -y'(Ut, t) = \gamma^2 F/T$. By balancing the acceleration of the power line against the force and the tension in the line, we see that the force cannot be exactly vertical, but must be symmetrically disposed with respect to the horizontal and deflected parts of the line. The force exerted by the pantograph thus has a small horizontal component $F \sin \theta/2$. The wave drag is therefore $F_d = \frac{1}{2}F^2\gamma^2/T$, as found earlier.

Since there is a real horizontal force acting on the wire, true Newtonian momentum must also be being transferred to the wire. Indeed the wire behind the train is being stretched, while that in front is being compressed. A section of length $2c_{\text{long}}(t - t_0)$, where c_{long} is the velocity of *longitudinal* waves on the wire and $(t - t_0)$ is the elapsed time, is in uniform motion in the x direction. Since usually $c_{\text{long}} \gg c$, this true momentum is accounted for by an almost infinitesimal motion over a large region of the wire. The pseudomomentum and the true momentum are to be found in quite different places — but the divergence of their flux tensors, and hence the associated forces, are equal.

III. RADIATION PRESSURE

In this section we will consider the “radiation pressure” exerted by sound waves incident on an object immersed in the medium. This is a subject with a long history of controversy [9,10]. The confusion began with the great Lord Rayleigh who gave several inequivalent answers to the problem. Our discussion will follow that of Leon Brillouin [11] who greatly clarified the matter. We begin by considering some analogous situations where the force is

exerted by transverse waves on a string.

Consider a standing wave on a semi-infinite elastic string.

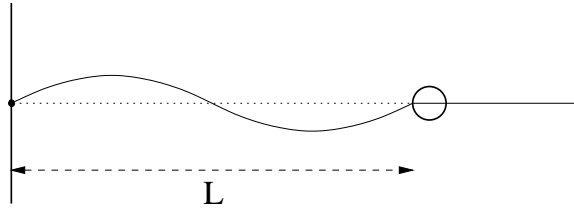


Fig 3. A vibrating string exerts a force on a bead.

We have restricted the vibration to the finite interval $[0, L]$ by means of a frictionless bead which forces the transverse displacement of the string to be zero at $x = L$, but allows free passage to longitudinal motion, and so does not affect the tension.

Suppose the transverse displacement is

$$y = A \sin \omega t \sin \frac{\pi n x}{L} \quad (3.1)$$

with $\omega = c\pi n/L$. The total energy of the motion is

$$E = \int dx \left\{ \frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 \right\} = \frac{1}{4} \rho \omega^2 A^2 L. \quad (3.2)$$

If we alter the size of the vibrating region by slowly moving the bead, we will alter the energy in the oscillations. This change in energy may be found by exploiting the Boltzmann-Ehrenfest principle which says that during an adiabatic variation of the parameters of a harmonic oscillator the quantity E/ω remains constant. Thus $\delta(E/\omega) = 0$ or $\delta E = \left(\frac{E}{\omega}\right) \delta\omega$. To apply this to the string, we note that

$$\delta\omega = \delta\left(\frac{cn\pi}{L}\right) = -\omega \frac{\delta L}{L}. \quad (3.3)$$

The change in energy, and hence the work we must do to move the bead, is then

$$\delta E = -\left(\frac{E}{L}\right) dL. \quad (3.4)$$

The “radiation pressure” is therefore $E/L = \langle \mathcal{E} \rangle$, the mean energy density. This calculation can be confirmed by examining the forces on the bead along the lines of Fig. 2.

The average density of pseudomomentum in each of the two travelling wave components of the standing wave is $\pm \frac{1}{2} \langle \mathcal{E} \rangle / c$. The radiation “pressure” can therefore be accounted for by the $2 \times c \times \frac{1}{2} \langle \mathcal{E} \rangle / c$ rate of change of the pseudomomentum in the travelling waves as they bounce off the bead. Thinking through this example shows why pseudomomentum can be used to compute real forces: On a homogeneous string the act of translating the bead and the wave together, while keeping the string fixed, leaves the action invariant. The associated conserved quantity is the sum of the *pseudomomentum* of the wave and the *true* momentum of the bead.

Keeping track of pseudomomentum cannot account for all forces, however. There is another plausible way of defining the radiation pressure. This time we use a finite string and attach its right-hand end to a movable wall

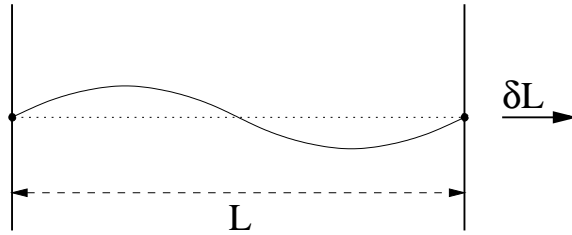


Fig 4. Another way to define radiation pressure.

Now, as we alter the length of the string, we will change its tension, and so alter the value of c . We must take the effect of this into account in the variation of the frequency

$$\delta\omega = \delta\left(\frac{cn\pi}{L}\right) = -\omega\frac{\delta L}{L} + \omega\frac{\delta c}{c}. \quad (3.5)$$

The change in the energy of the vibrating system is therefore

$$\delta E = -\langle \mathcal{E} \rangle \left(1 - \frac{\partial \ln c}{\partial \ln L}\right) \delta L. \quad (3.6)$$

The radiation pressure is thus given by

$$p = \langle \mathcal{E} \rangle \left(1 - \frac{\partial \ln c}{\partial \ln L}\right). \quad (3.7)$$

This force is in addition to the steady pull from the static tension T in the string.

The generality of the Boltzmann-Ehrenfest principle allows us to apply the previous discussion with virtually no changes to compute forces exerted by a sound wave. We need no explicit details of the wave motion beyond it being harmonic. The two ways of defining the radiation pressure for a vibrating string correspond to two different experimental conditions that we might use for measuring the radiation pressure for sound waves. The movable end condition corresponds to what is called the *Rayleigh* sound pressure.

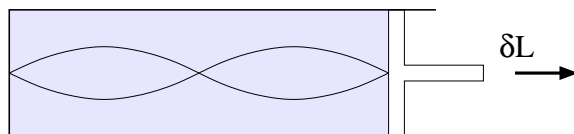


Fig 4. The Rayleigh Radiation Pressure.

We establish a standing wave in a cylinder closed at one end and having a movable piston at the other. Moving the piston confining the sound wave changes both the wavelength of the sound and the speed of propagation, producing a sound radiation pressure

$$p = \langle \mathcal{E} \rangle \left(1 + \left(\frac{\partial \ln c}{\partial \ln \rho} \right)_S \right). \quad (3.8)$$

The subscript S on the derivative indicates that it is being taken at fixed entropy. As with the string, this radiation pressure is in addition to the equilibrium hydrostatic pressure, P_0 , on the piston.

The analogue of the string with the sliding bead leads to the *Langevin* definition of the radiation pressure. Here we insert a bypass so that moving the piston confining the sound wave does not change the density or pressure of the fluid. The radiation pressure on the piston is simply $\langle \mathcal{E} \rangle$.

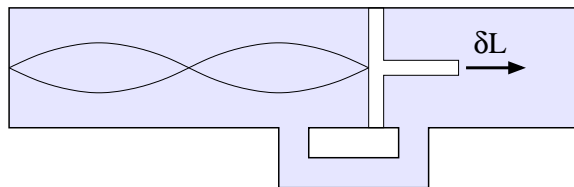


Fig 5. Langevin Radiation Pressure.

The difference in the two definitions of radiation pressure arises because, if we keep the

mean pressure fixed, the presence of a sound wave produces an $O(A^2)$ change in the volume of the fluid. If, instead, the mean density is held fixed, as it is in the Rayleigh definition, then this volume change is resisted by an additional hydrostatic pressure on the walls of the container.

The most common way of measuring sound pressure involves a sound beam in an open tank of fluid. Since the fluid is free to expand, this corresponds to the Langevin pressure.

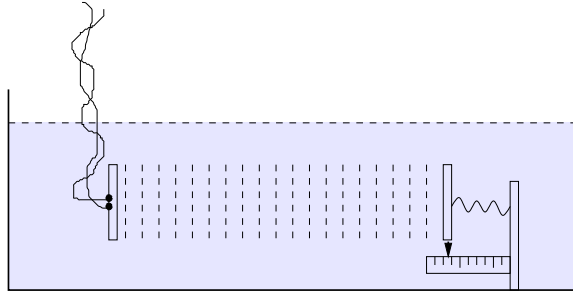


Fig 6. The usual experimental situation.

The radiation pressure is, in reality, a radiation *stress*

$$\Sigma_{ij} = \langle \mathcal{E} \rangle \left(\frac{k_i k_j}{k^2} + \delta_{ij} \left(\frac{\partial \ln c}{\partial \ln \rho} \right)_S \right). \quad (3.9)$$

The anisotropic part depends on the wave-vector \mathbf{k} of the sound beam, and may be accounted for by keeping track of the pseudomomentum changes. The isotropic part cannot be computed from the linearized sound-wave equation since it requires more information about the equation of state of the fluid medium than is used in deriving the wave equation. The extra information is encapsulated in the parameter $\left(\frac{\partial \ln c}{\partial \ln \rho} \right)_S$, a fluid-state analogue of the Grüneisen parameter which characterizes the thermal expansion of a solid. When the experimental situation is such that this isotropic pressure is important, the force associated with the sound field cannot be obtained from the pseudomomentum alone.

IV. MASS FLOW AND THE STOKES DRIFT

Further confusion involving momentum and pseudomomentum in acoustics is generated by the need to distinguish between the Euler (velocity field at a particular point) description

of fluid flow, and the Lagrangian (following the particles) description.

Suppose the velocity field in a sound wave is

$$v_{(1)}(x, t) = A \cos(kx - \omega t). \quad (4.1)$$

Using the continuity equation $\partial_x \rho v + \partial_t \rho = 0$, setting $\rho = \rho_0 + \rho_1$, and approximating $\rho v \approx \rho_0 v_{(1)}$, we find that

$$\rho_1 = \frac{1}{\omega} \rho_0 k A \cos(kx - \omega t) + O(A^2). \quad (4.2)$$

The time average of the momentum density ρv is therefore

$$\langle \rho v \rangle = \langle \rho_1 v_{(1)} \rangle = \frac{k \rho_0}{\omega} \frac{1}{2} A^2, \quad (4.3)$$

to $O(A^2)$ accuracy. This Newtonian momentum density is clearly non-zero, and numerically equal to the pseudomomentum density. Here, unlike the case of the elastic string, there is only one velocity of wave propagation and so the pseudomomentum and true momentum, although logically distinct, are to be found in the same place.

Further $\langle \rho v \rangle$ is both the momentum density and the mass-current. A nonzero average for the former therefore implies a steady drift of particles in the direction of wave propagation, in addition to the back-and-forth motion in the wave. We can confirm this by translating the Eulerian velocity field $v_{(1)} = A \cos(kx - \omega t)$ into Lagrangian language. The trajectory $\xi(t)$ of a particle initially at x_0 is the solution of the equation

$$\frac{d\xi}{dt} = v_{(1)}(\xi(t), t) = A \cos(k\xi - \omega t), \quad \xi(0) = x_0. \quad (4.4)$$

Since the quantity ξ appears both in the derivative and in the cosine, this is a nonlinear equation. We solve it perturbatively by setting

$$\xi(t) = x_0 + A \Xi_1(t) + A^2 \Xi_2(t) + \dots \quad (4.5)$$

We find that

$$\Xi_1(t) = -\frac{1}{\omega} \sin(kx_0 - \omega t). \quad (4.6)$$

Substituting this into (4.4) we find

$$\frac{d\Xi_2}{dt} = \frac{k}{\omega} \sin^2(kx_0 - \omega t). \quad (4.7)$$

Thus $\dot{\Xi}_2$ has a non-vanishing time average, $k/(2\omega)$, leading to a secular drift velocity $\bar{v}_L = \frac{1}{2}kA^2/\omega$ that is consistent with (4.3). This motion is called the *Stokes drift*.

We also see why there is no net Newtonian momentum associated with phonons in a crystal. The atomic displacements in a harmonic crystal are given by

$$\eta_n = A \cos(k(na) - \omega t), \quad (4.8)$$

so the crystal equivalent of (4.4) is

$$\frac{d\eta_n}{dt} = A \cos(k(na) - \omega t). \quad (4.9)$$

Thus η does not appear on the right-hand side of this equation. It is a linear equation and gives rise to no net particle drift.

So, a sound wave does have real momentum? — But wait! The momentum density we have computed is *second order* in the amplitude A . The wave equation we have used to compute it is accurate only to *first order* in A . We may expand the velocity field as

$$\mathbf{v} = \mathbf{v}_{(1)} + \mathbf{v}_{(2)} + \cdots, \quad (4.10)$$

where the second-order correction $\mathbf{v}_{(2)}$ arises because the equations of fluid motion are non-linear. This correction will possess both oscillating and steady components. The oscillatory components arise because a strictly harmonic wave with frequency ω_0 will gradually develop higher frequency components due to the progressive distortion of the wave as it propagates. (A plane wave eventually degenerates into a sequence of shocks.) These distortions are usually not significant in considerations of energy and momentum balance. The steady terms, however, represent $O(A^2)$ alterations to the mean flow caused by the sound waves, and these may possess energy and momentum comparable to that of the sound field.

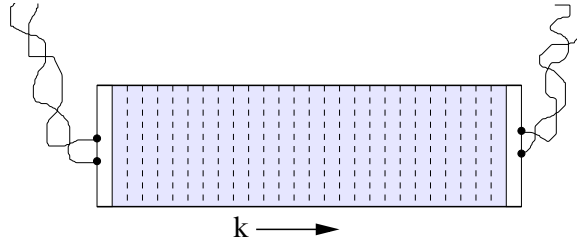


Fig 7. Momentum flux without mass-flow.

For example, we may drive a transducer so as to produce a beam of sound which totally fills a closed cylinder of fluid. At the far end of the cylinder a second transducer with suitably adjusted amplitude and phase absorbs the beam without reflection. Since the container is not going anywhere, it is clear that the average velocity of the center of mass of the fluid must be zero, despite the presence of the sound wave. An exact solution of the non-linear equation of motion for the fluid must provide a steady component in $\mathbf{v}_{(2)}$ and this counterflow completely cancels the $\langle \rho_1 v_{(1)} \rangle$ term. Indeed in Lagrangian coordinates the fluid particles simply oscillate back and forth with no net drift. The wave momentum and its cancelling counterflow are simply artifacts of our Eulerian description. This is one reason why professional fluid mechanics dislike the notion of momentum being associated with sound waves.

The outlook for the two-fluid model is not entirely bleak, however. The $\mathbf{v}_{(2)}$ corrections do not always exactly cancel the momentum. Any non-zero value for $\nabla \cdot \rho_1 \mathbf{v}_{(1)}$ — such as occurs at the transducers at the ends of the cylinder — will act as a source or sink for a $\mathbf{v}_{(2)}$ counterflow, but its exact form depends on the shape of the container and other effects extrinsic to the sound field. For transducers immersed in an infinite volume of fluid, for example, the counterflow will take the form of a source-sink dipole field, and, although the total momentum of the fluid will remain zero, there will be a non-zero momentum density. Because it is not directly associated with the sound field, in the language of the two-fluid model the induced $\mathbf{v}_{(2)}$ counterflow is not counted as belonging to the normal component of the fluid, *i.e.* to the phonons, but is lumped into the background superflow. It is determined

by enforcing mass conservation,

$$\nabla \cdot \rho_s \mathbf{v}_s + \nabla \cdot \rho_n \mathbf{v}_n = 0. \quad (4.11)$$

The difference of opinion between the physics and fluid mechanics communities over whether phonons have real momentum reduces, therefore, to one of different accounting conventions.

V. THE UNRUH WAVE EQUATION

Now we will examine the energy and momentum in a moving fluid. The flow of an irrotational fluid is derivable from the action [12]

$$S = \int d^4x \left\{ \rho \dot{\phi} + \frac{1}{2} \rho (\nabla \phi)^2 + u(\rho) \right\}. \quad (5.1)$$

Here ρ is the mass density, ϕ the velocity potential, and the overdot denotes differentiation with respect to time. The function u may be identified with the internal energy density.

Varying with respect to ϕ yields the continuity equation

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (5.2)$$

where $\mathbf{v} \equiv \nabla \phi$. Varying ρ gives a form of Bernoulli's equation

$$\dot{\phi} + \frac{1}{2} \mathbf{v}^2 + \mu(\rho) = 0, \quad (5.3)$$

where $\mu(\rho) = du/d\rho$. In most applications μ would be identified with the specific enthalpy. For a superfluid condensate the entropy density, s , is identically zero and μ is the local chemical potential.

The gradient of the Bernoulli equation is Euler's equation of motion for the fluid. Combining this with the continuity equation yields a momentum conservation law

$$\partial_t(\rho v_i) + \partial_j(\rho v_j v_i) + \rho \partial_i \mu = 0. \quad (5.4)$$

We simplify (5.4) by introducing the pressure, P , which is related to μ by $P(\rho) = \int \rho d\mu$.

Then we can write

$$\partial_t(\rho v_i) + \partial_j \Pi_{ji} = 0, \quad (5.5)$$

where Π_{ij} is given by

$$\Pi_{ij} = \rho v_i v_j + \delta_{ij} P. \quad (5.6)$$

This is the usual form of the momentum flux tensor in fluid mechanics.

The relations $\mu = du/d\rho$ and $\rho = dP/d\mu$ show that P and u are related by a Legendre transformation: $P = \rho\mu - u(\rho)$. From this and the Bernoulli equation we see that the pressure is equal to minus the action density:

$$-P = \rho\dot{\phi} + \frac{1}{2}\rho(\nabla\phi)^2 + u(\rho). \quad (5.7)$$

Consequently, we can write

$$\Pi_{ij} = \rho\partial_i\phi\partial_j\phi - \delta_{ij}\left\{\rho\dot{\phi} + \frac{1}{2}\rho(\nabla\phi)^2 + u(\rho)\right\}. \quad (5.8)$$

VI. THE ACOUSTIC METRIC

To obtain Unruh's wave equation we set

$$\begin{aligned} \phi &= \phi_0 + \phi_1 \\ \rho &= \rho_0 + \rho_1. \end{aligned} \quad (6.1)$$

Here ϕ_0 and ρ_0 define the mean flow. We assume that they obey the equations of motion. The quantities ϕ_1 and ρ_1 represent small amplitude perturbations. Expanding S to quadratic order in these perturbations gives

$$S = S_0 + \int d^4x \left\{ \rho_1\dot{\phi}_1 + \frac{1}{2}\left(\frac{c^2}{\rho_0}\right)\rho_1^2 + \frac{1}{2}\rho_0(\nabla\phi_1)^2 + \rho_1\mathbf{v}\cdot\nabla\phi_1 \right\}. \quad (6.2)$$

Here $\mathbf{v} \equiv \mathbf{v}_{(0)} = \nabla\phi_0$. The speed of sound, c , is defined by

$$\frac{c^2}{\rho_0} = \left. \frac{d\mu}{d\rho} \right|_{\rho_0}, \quad (6.3)$$

or more familiarly

$$c^2 = \frac{dP}{d\rho}. \quad (6.4)$$

The terms linear in the perturbations vanish because of our assumption that the zeroth-order variables obey the equation of motion.

The equation of motion for ρ_1 derived from (6.2) is

$$\rho_1 = -\frac{\rho_0}{c^2} \{ \dot{\phi}_1 + \mathbf{v} \cdot \nabla \phi_1 \}. \quad (6.5)$$

Since ρ_1 occurs quadratically, we may use (6.5) to eliminate it and obtain an effective action for the potential ϕ_1 only

$$S_2 = \int d^4x \left\{ -\frac{1}{2} \rho_0 (\nabla \phi_1)^2 + \frac{\rho_0}{2c^2} (\dot{\phi}_1 + \mathbf{v} \cdot \nabla \phi_1)^2 \right\}. \quad (6.6)$$

The resultant equation of motion for ϕ_1 is [13,14]

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v} \right) \frac{\rho_0}{c^2} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \phi_1 = \nabla \cdot (\rho_0 \nabla \phi_1). \quad (6.7)$$

This can be written as¹

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi_1 = 0, \quad (6.8)$$

where

$$\sqrt{-g} g^{\mu\nu} = \frac{\rho_0}{c^2} \begin{pmatrix} 1, & \mathbf{v}^T \\ \mathbf{v}, & \mathbf{v}\mathbf{v}^T - c^2 \mathbf{1} \end{pmatrix}. \quad (6.9)$$

We find that $\sqrt{-g} = \sqrt{\det g_{\mu\nu}} = \rho_0^2/c$, and the covariant components of the metric are

$$g_{\mu\nu} = \frac{\rho_0}{c} \begin{pmatrix} c^2 - v^2, & \mathbf{v}^T \\ \mathbf{v}, & -\mathbf{1} \end{pmatrix}. \quad (6.10)$$

The associated space-time interval can be written

¹I use the convention that Greek letters run over four space-time indices 0, 1, 2, 3 with $0 \equiv t$, while Roman indices refer to the three space components.

$$ds^2 = \frac{\rho_0}{c} \left\{ c^2 dt^2 - \delta_{ij} (dx^i - v^i dt)(dx^j - v^j dt) \right\}, \quad (6.11)$$

and metrics of this form, although without the overall conformal factor ρ_0/c , appear in the Arnowitt-Deser-Misner (ADM) formalism of general relativity [15]. There, c and $-v^i$ are referred to as the *lapse function* and *shift vector* respectively. They serve to glue successive three-dimensional time slices together to form a four-dimensional space-time [16]. In our present case, provided ρ_0/c can be regarded as a constant, each 3-space is ordinary flat \mathbf{R}^3 equipped with the rectangular Cartesian metric $g_{ij}^{(space)} = \delta_{ij}$ — but the resultant space-time is in general curved, the curvature depending on the degree of inhomogeneity of the mean flow \mathbf{v} .

In the geometric acoustics limit, sound will travel along the null geodesics defined by $g_{\mu\nu}$. Even in the presence of spatially varying ρ_0 we would expect the ray paths to depend only on the local values of c and \mathbf{v} , so it is perhaps a bit surprising to see the density entering the expression for the Unruh metric. An overall conformal factor, however, does not affect *null* geodesics, and thus variations in ρ_0 do not influence the ray tracing.

VII. SECOND-ORDER QUANTITIES

We are going to derive various energy and momentum conservation laws from our wave equation. Before we do, let us consider what sort of quantities we would want them to contain.

It is reasonable to define the momentum density and the momentum flux tensor associated with the sound field as the second order averages

$$\mathbf{j}^{(\text{phonon})} = \langle \rho_1 \mathbf{v}_{(1)} \rangle + \mathbf{v} \langle \rho_2 \rangle, \quad (7.1)$$

and

$$\Pi_{ij}^{(\text{phonon})} = \rho_0 \langle v_{(1)i} v_{(1)j} \rangle + v_i \langle \rho_1 v_{(1)j} \rangle + v_j \langle \rho_1 v_{(1)i} \rangle + \delta_{ij} \langle P_2 \rangle + v_i v_j \langle \rho_2 \rangle. \quad (7.2)$$

In these expressions I have taken no account of any steady part of $\mathbf{v}_{(2)}$. This is not a quantity intrinsic to the sound field and has to be found by methods outside the purely acoustic. The other second-order quantities P_2 and ρ_2 can be computed in terms of first-order amplitudes.

For P_2 we combine

$$\Delta P = \frac{dP}{d\mu} \Delta\mu + \frac{1}{2} \frac{d^2 P}{d\mu^2} (\Delta\mu)^2 + O((\Delta\mu)^3) \quad (7.3)$$

and Bernoulli's equation in the form

$$\Delta\mu = -\dot{\phi}_1 - \frac{1}{2} (\nabla\phi_1)^2 - \mathbf{v} \cdot \nabla\phi_1, \quad (7.4)$$

with

$$\frac{dP}{d\mu} = \rho, \quad \frac{d^2 P}{d\mu^2} = \frac{d\rho}{d\mu} = \frac{\rho}{c^2}. \quad (7.5)$$

Expanding out and grouping terms of appropriate orders gives

$$P_1 = -\rho_0(\dot{\phi}_1 + \mathbf{v} \cdot \nabla\phi_1) = c^2 \rho_1, \quad (7.6)$$

which we already knew, and

$$P_2 = -\rho_0 \frac{1}{2} (\nabla\phi_1)^2 + \frac{1}{2} \frac{\rho_0}{c^2} (\dot{\phi}_1 + \mathbf{v} \cdot \nabla\phi_1)^2. \quad (7.7)$$

We see that $P_2 = \sqrt{-g}L$ where L is the Lagrangian density for our sound wave equation.

For a plane wave $\langle P_2 \rangle = 0$

The second order change in the density, ρ_2 , may be found similarly. It is

$$\rho_2 = \frac{1}{c^2} P_2 - \frac{1}{\rho_0} \rho_1^2 \left(\frac{d \ln c}{d \ln \rho} \right)_S. \quad (7.8)$$

For a plane wave the time average $\langle P_2 \rangle = 0$, but because ρ_2 contains ρ_1^2 , the time average of this quantity is non-zero. The resulting change of volume of the fluid, or, if the volume is held fixed, the resulting pressure change, is the origin of the isotropic terms in the radiation stress tensor discussed earlier.

VIII. CONSERVATION LAWS

Because the linear wave equation does not have access to information about counterflows or second-order density changes, we will not be able to derive the real energy and momentum fluxes from its solution. We can still derive from the wave equation what look superficially like conservation laws for these quantities, however, and these laws give us insight into the behaviour of the solutions. The conserved quantities are of course the pseudoenergy and pseudomomentum.

We begin by defining a (pseudo)-energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_2}{\delta g_{\mu\nu}}. \quad (8.1)$$

Let us recall how such a tensor comes to be associated with conservation laws. Suppose that we have an action $S(\varphi, g_{\mu\nu})$ which is a functional of some field variables $\varphi(x^\mu)$ and the metric $g_{\mu\nu}$. If we reparameterize spacetime so that the point that had coordinates x^μ is now denoted by $x^\mu + \epsilon^\mu$, then we have $\varphi \rightarrow \varphi + \delta\varphi$ and $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, where

$$\begin{aligned} \delta\varphi &= \epsilon^\mu \partial_\mu \varphi \\ \delta g_{\mu\nu} &= D_\mu \epsilon_\nu + D_\nu \epsilon_\mu. \end{aligned} \quad (8.2)$$

Here D_μ is the covariant derivative containing the Riemann connection compatible with the metric. The variation in the metric comes from the computation

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x) dx^\mu dx^\nu \\ &\rightarrow g_{\mu\nu}(x^\alpha + \epsilon^\alpha) d(x^\mu + \epsilon^\mu) d(x^\nu + \epsilon^\nu) \\ &= (\epsilon^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \epsilon^\alpha + g_{\mu\alpha} \partial_\nu \epsilon^\alpha) dx^\mu dx^\nu \\ &= (D_\mu \epsilon_\nu + D_\nu \epsilon_\mu) dx^\mu dx^\nu. \end{aligned} \quad (8.3)$$

The assembly of the terms into covariant derivatives in the last line is most easily established by using geodesic coordinates and the fact that $\delta g_{\alpha\nu}$ is a tensor. The combination $D_\mu \epsilon_\nu + D_\nu \epsilon_\mu$ is the Lie derivative, $\mathcal{L}_\epsilon g_{\mu\nu}$, of the metric with respect to the vector field ϵ^μ .

Since a mere re-coordinatization does not change the numerical value of the action, we must have

$$0 = \delta S = \int d^4x \sqrt{-g} \left\{ (D_\mu \epsilon_\nu + D_\nu \epsilon_\mu) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} + (\epsilon^\mu \partial_\mu \varphi) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \varphi} \right\}. \quad (8.4)$$

Now the equations of motion state that S is unchanged by any variation in φ , including, *a fortiori* the change $\delta\varphi = \epsilon^\mu \partial_\mu \varphi$. Thus

$$0 = \int d^4x \sqrt{-g} (D_\mu \epsilon_\nu) \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \int d^4x \sqrt{-g} \epsilon_\nu D_\mu \left(\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \right), \quad (8.5)$$

where, in the last equality, we have integrated by parts by using the derivation property of the covariant derivative and the expression

$$D_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu) \quad (8.6)$$

for the divergence of a vector. Since (8.5) is true for arbitrary $\epsilon^\mu(x)$, we deduce that

$$D_\mu \left(\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \right) = D_\mu T^{\mu\nu} = 0. \quad (8.7)$$

Although (8.7) has the appearance of a conservation law, and has useful applications in itself, we have not yet exploited any symmetries of the system — and it is symmetries that lead to conserved quantities. To derive a genuine local conservation law we need to assume that the metric admits a Killing vector, η^μ . This means that the particular reparameterization $\epsilon^\mu = \eta^\mu$ is actually an isometry of the manifold and so leaves the metric invariant

$$\mathcal{L}_\eta g_{\mu\nu} = D_\mu \eta_\nu + D_\nu \eta_\mu = 0. \quad (8.8)$$

Combining (8.8) with (8.7) and using the symmetry of $T^{\mu\nu}$ we find that

$$D_\mu (T^{\mu\nu} \eta_\nu) = 0. \quad (8.9)$$

Using (8.6), this can be written

$$\partial_\mu (\sqrt{-g} T^{\mu\nu} \eta_\nu) = 0. \quad (8.10)$$

Thus the 4-vector density $Q^\mu = \sqrt{-g} T^{\mu\nu} \eta_\nu$ is conventionally conserved, and

$$Q = \int d^3x Q^0 \quad (8.11)$$

is independent of the time slice on which it is evaluated.

Since (6.6) can be written as the usual action for a scalar field

$$S_2 = \int d^4x \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi_1 \partial_\nu \phi_1, \quad (8.12)$$

we have

$$T^{\mu\nu} = \partial^\mu \phi_1 \partial^\nu \phi_1 - g^{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi_1 \partial_\beta \phi_1 \right). \quad (8.13)$$

The derivatives with raised indices in (8.13) are defined by

$$\partial^0 \phi_1 = g^{0\mu} \partial_\mu \phi_1 = \frac{1}{\rho_0 c} (\dot{\phi}_1 + \mathbf{v} \cdot \nabla \phi_1), \quad (8.14)$$

and

$$\partial^i \phi_1 = g^{i\mu} \partial_\mu \phi_1 = \frac{1}{\rho_0 c} (v_i (\dot{\phi}_1 + \mathbf{v} \cdot \nabla \phi_1) - c^2 \partial_i \phi_1). \quad (8.15)$$

Thus

$$\begin{aligned} T^{00} &= \frac{1}{\rho_0^3} \left(\rho_0 \frac{1}{2} (\nabla \phi_1)^2 + \frac{1}{2} \frac{\rho_0}{c^2} (\dot{\phi}_1 + \mathbf{v} \cdot \nabla \phi_1)^2 \right) \\ &= \frac{c^2}{\rho_0^3} \left(\frac{\mathcal{E}_r}{c^2} \right) \\ &= \frac{c^2}{\rho_0^3} \tilde{\rho}_2. \end{aligned} \quad (8.16)$$

The last two equalities serve as a definition of \mathcal{E}_r and $\tilde{\rho}_2$. The quantity \mathcal{E}_r is often described as the acoustic energy density relative to the frame moving with the local fluid velocity [17].

It is, of course, more correctly a pseudo-energy density.

We can express the other components of (8.13) in terms of physical quantities. We find that

$$T^{i0} = T^{0i} = \frac{c^2}{\rho_0^3} \left(\frac{1}{c^2} (P_1 v_{(1)i} + v_i \mathcal{E}_r) \right)$$

$$= \frac{c^2}{\rho_0^3} (\rho_1 v_{(1)i} + v_i \tilde{\rho}_2). \quad (8.17)$$

The first line in this expression shows that, up to an overall factor, T^{i0} is an energy flux — the first term being the rate of working by a fluid element on its neighbour, and the second the advected energy. The second line is written so as to suggest the usual relativistic identification of (energy-flux)/ c^2 with the density of momentum. This interpretation, however, requires that $\tilde{\rho}_2$ be the second-order correction to the density, which, sadly, it is not.

Similarly

$$T^{ij} = \frac{c^2}{\rho_0^3} (\rho_0 v_{(1)i} v_{(1)j} + v_i \rho_1 v_{(1)j} + v_j \rho_1 v_{(1)i} + \delta_{ij} P_2 + v_i v_j \tilde{\rho}_2). \quad (8.18)$$

We again see that if we were only able to identify $\tilde{\rho}_2$ with ρ_2 then T^{ij} has precisely the form we expect for the second-order momentum flux tensor. Although it comes close, the inability of the pseudomomentum flux to exactly capture the true-momentum flux is inevitable as we know that computing the true stresses in the medium requires more information about the equation of state than is available to the linearized wave equation.

We can also write the mixed co- and contra-variant components of the energy momentum tensor $T^\mu_\nu = T^{\mu\lambda} g_{\lambda\nu}$ in terms of physical quantities. This mixed tensor turns out to be more useful than the doubly contravariant tensor. Because we no longer enforce a symmetry between the indices μ and ν , the quantity \mathcal{E}_r is no longer required to perform double duty as both an energy and a density. We find

$$\begin{aligned} \sqrt{-g} T^0_0 &= (\mathcal{E}_r + \rho_1 \mathbf{v}_{(1)} \cdot \mathbf{v}) \\ \sqrt{-g} T^i_0 &= \left(\frac{P_1}{\rho_0} + \mathbf{v} \cdot \mathbf{v}_{(1)} \right) (\rho_0 v_{(1)i} + \rho_1 v_{(0)i}), \end{aligned} \quad (8.19)$$

and

$$\begin{aligned} \sqrt{-g} T^0_i &= -\rho_1 \mathbf{v}_{(1)i} \\ \sqrt{-g} T^i_j &= -(\rho_0 v_{(1)i} v_{(1)j} + v_i \rho_1 v_{(1)j} + \delta_{ij} P_2). \end{aligned} \quad (8.20)$$

We see that $\tilde{\rho}_2$ does not appear here, and all these terms may be identified with physical quantities which are reliably computed from solutions of the linearized wave equation.

Now we turn to the local conservation laws. In what follows I will consider only a steady background flow, and further one for which ρ_0 , c , and hence $\sqrt{-g} = \rho_0^2/c$ can be treated as constant. To increase the readability of some expressions I will also choose units so that ρ_0 and c become unity and no longer appear as overall factors in the metric or the four-dimensional energy-momentum tensors. I will, however, reintroduce them when they are required for dimensional correctness in expressions such as $\rho_0 \mathbf{v}_{(1)}$ or \mathcal{E}_r/c^2 .

From the acoustic metric we find

$$\begin{aligned}
\Gamma_{00}^0 &= \frac{1}{2}(\mathbf{v} \cdot \nabla)|v|^2 \\
\Gamma_{i0}^0 &= -\frac{1}{2}\partial_i |v|^2 + \frac{1}{2}v_j(\partial_i v_j - \partial_j v_i) \\
\Gamma_{00}^i &= \frac{1}{2}v_i(\mathbf{v} \cdot \nabla)|v|^2 - \frac{1}{2}\partial_i |v|^2 \\
\Gamma_{ij}^0 &= \frac{1}{2}(\partial_i v_j + \partial_j v_i) \\
\Gamma_{j0}^i &= -\frac{1}{2}v_i \partial_j |v|^2 + \frac{1}{2}(\partial_j v_k - \partial_k v_j)(v_k v_i - c^2 \delta_{ik}) \\
\Gamma_{jk}^i &= \frac{1}{2}v_i(\partial_j v_k + \partial_k v_j).
\end{aligned} \tag{8.21}$$

We now evaluate

$$\begin{aligned}
D_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma_{\mu\gamma}^\mu T^{\gamma 0} + \Gamma_{\mu\nu}^0 T^{\mu\nu} \\
&= \partial_\mu T^{\mu 0} + \Gamma_{\mu\nu}^0 T^{\mu\nu}.
\end{aligned} \tag{8.22}$$

After a little algebra we find

$$\Gamma_{\mu\nu}^0 T^{\mu\nu} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)(\rho_0 v_{(1)i} v_{(1)j} + \delta_{ij} P_2). \tag{8.23}$$

Note the non-appearance of ρ_1 and $\tilde{\rho}_2$ in the final expression — even though both quantities appear in $T^{\mu\nu}$.

The conservation law therefore becomes

$$\partial_t \mathcal{E}_r + \partial_i (P_1 v_{(1)i} + v_i \mathcal{E}_r) + \frac{1}{2} \Sigma_{ij} (\partial_i v_j + \partial_j v_i) = 0, \tag{8.24}$$

where

$$\Sigma_{ij} = \rho_0 v_{(1)i} v_{(1)j} + \delta_{ij} P_2. \tag{8.25}$$

This is an example of the general form of energy law derived by Longuet-Higgins and Stuart, originally in the context of ocean waves [18]. The relative energy density, $\mathcal{E}_r \equiv T^{00}$, is *not* conserved. Instead, an observer moving with the fluid sees the waves acquiring energy from the mean flow at a rate given by the product of a radiation stress, Σ_{ij} , with the mean-flow rate-of-strain. This equation is sometimes cited [19] as an explanation for the monstrous ship-destroying waves that may be encountered off the eastern coast of South Africa. Here long wavelength swell from distant Antarctic storms runs into the swift southbound Agulhas current and is greatly amplified by the opposing flow.

We now examine the energy conservation law coming from the zeroth component of the mixed energy-momentum tensor. After a little work we find that the connection contribution vanishes identically and the energy conservation law is therefore

$$\partial_t \left(\mathcal{E}_r + \rho_1 \mathbf{v}_{(1)} \cdot \mathbf{v} \right) + \partial_i \left(\left(\frac{P_1}{\rho_0} + \mathbf{v} \cdot \mathbf{v}_{(1)} \right) (\rho_0 v_{(1)i} + \rho_1 v_{(0)i}) \right) = 0. \quad (8.26)$$

We see that the combination $\mathcal{E}_r + \rho_1 \mathbf{v}_{(1)} \cdot \mathbf{v}$ *does* correspond to a conserved energy — as we should have anticipated since a steady flow provides us with a Killing vector $\mathbf{e}_0 = \partial_t$. This conservation law was originally derived by Blokhintsev [20] for slowly varying flows, and more generally by Cantrell and Hart [21] in their study of the acoustic stability of rocket engines.

Finally the covariant conservation equation $D_\mu T^\mu_j = 0$ reads

$$\partial_t \rho_1 v_{(1)j} + \partial_i \left(\rho_0 v_{(1)i} v_{(1)j} + v_i \rho_1 v_{(1)j} + \delta_{ij} P_2 \right) + \rho_1 v_{(1)i} \partial_j v_i = 0. \quad (8.27)$$

Here connection terms have provided a source term for the momentum density. Thus, in an inhomogeneous flow field, momentum is exchanged between the waves and the mean flow.

IX. PHONONS AND CONSERVATION OF WAVE ACTION

The conservation laws we have derived in the previous section may all be interpreted in terms of the semiclassical motion of phonons. As noted by McIntyre [7], the existence

of such an interpretation is a sure sign that the conservation laws in question are those of pseudomomentum and pseudoenergy.

To make contact with the semiclassical picture we observe that when the mean flow varies slowly on the scale of a wavelength, the sound field can locally be approximated by a plane wave

$$\phi(x, t) = A \cos(\mathbf{k} \cdot \mathbf{x} - \omega t). \quad (9.1)$$

The frequency ω and the wave-vector \mathbf{k} are related by the Doppler-shifted dispersion relation $\omega = \omega_r + \mathbf{k} \cdot \mathbf{v}$, where $\omega_r = c|k|$, is the frequency relative to a frame moving with the fluid. A packet of such waves is governed by Hamilton's ray equations

$$\frac{dx^i}{dt} = \frac{\partial \omega}{\partial k^i}, \quad \frac{dk^i}{dt} = -\frac{\partial \omega}{\partial x^i}. \quad (9.2)$$

In other words the packet moves at the group velocity

$$\mathbf{V}_g = \dot{\mathbf{x}} = c \frac{\mathbf{k}}{|k|} + \mathbf{v}. \quad (9.3)$$

and the change in \mathbf{k} is given by

$$\frac{dk_j}{dt} = -k_i \frac{\partial v_i}{\partial x^j}. \quad (9.4)$$

In this equation the time derivative is taken along the ray:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla. \quad (9.5)$$

This evolution can also be derived from the equation for null geodesics of the acoustic metric [8].

The evolution of the amplitude A is linked with that of the relative energy density, \mathcal{E}_r , through

$$\langle \mathcal{E}_r \rangle = \frac{1}{2} A^2 \rho_0 \frac{\omega_r^2}{c^2}. \quad (9.6)$$

For a homogeneous stationary fluid we would expect our macroscopic plane wave to correspond to a quantum coherent state whose energy is, in terms of the (quantum) average phonon density \bar{N} ,

$$E_{tot} = (\text{Volume})\langle \mathcal{E}_r \rangle = (\text{Volume})\bar{N}\hbar\omega_r. \quad (9.7)$$

Since it is a density of “particles”, \bar{N} should remain the same when viewed from any frame. Consequently, the relation

$$\bar{N}\hbar = \frac{\langle \mathcal{E}_r \rangle}{\omega_r} \quad (9.8)$$

should hold true generally. In classical fluid mechanics the quantity $\langle \mathcal{E}_r \rangle / \omega_r$ is called the *wave action* [22,17,23].

The time averages of other components of the energy-momentum tensor may be also expressed in terms of \bar{N} . For the mixed tensor we find

$$\begin{aligned} \langle \sqrt{-g} T_0^0 \rangle &= \langle \mathcal{E}_r + \mathbf{v} \cdot \rho_1 \mathbf{v}_{(1)} \rangle = \bar{N}\hbar\omega \\ \langle \sqrt{-g} T_0^i \rangle &= \left\langle \left(\frac{P_1}{\rho_0} + \mathbf{v} \cdot \mathbf{v}_{(1)} \right) (\rho_0 v_{(1)i} + \rho_1 v_i) \right\rangle = \bar{N}\hbar\omega (V_g)_i \\ \langle -\sqrt{-g} T_i^0 \rangle &= \langle \rho_1 v_{(1)i} \rangle = \bar{N}\hbar k_i \\ \langle -\sqrt{-g} T_j^i \rangle &= \langle \rho_0 v_{(1)i} v_{(1)j} + v_i \rho_1 v_{(1)j} + \delta_{ij} P_2 \rangle = \bar{N}\hbar k_j (V_g)_i. \end{aligned} \quad (9.9)$$

In the last equality we have used that $\langle P_2 \rangle = 0$ for a plane progressive wave.

If we insert these expressions for the time averages into the Blokhintsev energy conservation law (8.26), we find that

$$\frac{\partial \bar{N}\hbar\omega}{\partial t} + \nabla \cdot (\bar{N}\hbar\omega \mathbf{V}_g) = 0. \quad (9.10)$$

We can write this as

$$\bar{N}\hbar \left(\frac{\partial \omega}{\partial t} + \mathbf{V}_g \cdot \nabla \omega \right) + \hbar\omega \left(\frac{\partial \bar{N}}{\partial t} + \nabla \cdot (\bar{N} \mathbf{V}_g) \right) = 0. \quad (9.11)$$

The first term is proportional to $d\omega/dt$ taken along the rays and vanishes for a steady mean flow as a consequence of the Hamiltonian nature of the ray tracing equations. The second term must therefore also vanish. This vanishing represents the conservation of phonons, or, in classical language, the conservation of wave-action. Conservation of wave action is an analogue of the adiabatic invariance of E/ω in the time dependent harmonic oscillator.

In a similar manner, the time average of (8.27) may be written

$$\begin{aligned}
0 &= \frac{\partial \bar{N} k_j}{\partial t} + \nabla \cdot (\bar{N} k_j \mathbf{V}_g) + \bar{N} k_i \frac{\partial v_i}{\partial x^j} \\
&= \bar{N} \left(\frac{\partial k_j}{\partial t} + \mathbf{V}_g \cdot \nabla k_j + k_i \frac{\partial v_i}{\partial x^j} \right) + k_j \left(\frac{\partial \bar{N}}{\partial t} + \nabla \cdot (\bar{N} \mathbf{V}_g) \right).
\end{aligned} \tag{9.12}$$

We see that the momentum law becomes equivalent to phonon-number conservation combined with the ray tracing equation (9.4).

X. SUMMARY

When dealing with waves in a medium it is essential to distinguish between the conceptually distinct quantities of momentum and pseudomomentum. Under suitable circumstances either may be used for computing forces — but there is no general rule for determining those circumstances. In the case of non-dispersive sound waves, momentum and pseudomomentum often travel together, and are therefore easily confused — but the conservation laws derived by manipulating the wave equation are those of pseudomomentum and pseudoenergy. The GR analogy provided by the acoustic metric provides a convenient and structured route to deriving these laws when the background fluid is moving.

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