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Physics 200A
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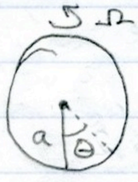
Homework 1 - Physics 200A

3.1



a. $L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\Omega^2\sin^2\theta + mga\cos\theta$ ✓

b. $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$

$ma^2\Omega^2\sin\theta\cos\theta - mga\sin\theta = ma^2\ddot{\theta}$ ✓

An equilibrium circular orbit occurs when $\ddot{\theta} = 0$

$\Rightarrow ma^2\Omega^2\sin\theta_0\cos\theta_0 = mga\sin\theta_0$

$\cos\theta_0 = \frac{g}{a\Omega^2}$ ✓

This is the angle for which the centrifugal force equals the force from gravity:

$F_{c,y} = F_{g,y} \Rightarrow F_c\cos\theta_0 = F_g\sin\theta_0$ ✓

$\frac{m(a\sin\theta_0\Omega)^2}{a\sin\theta_0}\cos\theta_0 = mg\sin\theta_0$ ✓

$\cos\theta_0 = \frac{g}{a\Omega^2}$ ☺ ✓

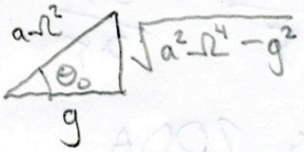
c. $\ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta$ ✓

Let $\theta = \theta_0 + \eta(t)$, where $\eta(t)$ is small

$\ddot{\eta} = \Omega^2\sin(\theta_0+\eta)\cos(\theta_0+\eta) - \frac{g}{a}\sin(\theta_0+\eta)$ ✓

$$\ddot{\eta} = \Omega^2(\sin\theta_0\cos\eta + \cos\theta_0\sin\eta)(\cos\theta_0\cos\eta - \sin\theta_0\sin\eta)$$

$$- \frac{g}{a}(\sin\theta_0\cos\eta + \cos\theta_0\sin\eta)$$



Since η is very small, approximate

$\sin\eta \approx \eta$ and $\cos\eta \approx 1$

So,

$$\ddot{\eta} = -\Omega^2(\sin\theta_0 + \eta\cos\theta_0)(\cos\theta_0 - \eta\sin\theta) - \frac{g}{a}(\sin\theta_0 + \eta\cos\theta_0)$$

$$\approx -\Omega^2(\sin\theta_0\cos\theta_0 + \eta\cos^2\theta_0 - \eta\sin^2\theta_0) - \frac{g}{a}(\sin\theta_0 + \eta\cos\theta_0)$$

$$= -\Omega^2(\sin\theta_0\cos\theta_0 + \eta\cos^2\theta_0 - \eta\sin^2\theta_0) - \Omega^2(\sin\theta_0\cos\theta_0 + \eta^2\cos\theta_0)$$

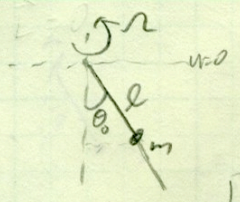
✓

$$\ddot{\eta} = -(\Omega^2\sin^2\theta_0)\eta \Rightarrow \boxed{\omega^2 = \Omega^2\sin^2\theta_0}$$ ✓

d. If $a\Omega^2 < g$, $\cos\theta_0 > 1$

This means there is no equilibrium circular orbit. ✓

3.2



$$T=\frac{1}{2}m\left(\dot r^2+r^2\overset{0}{\cancel{\dot\theta^2}}+r^2\sin^2\theta\,\overset{\Omega^2}{\cancel{\dot\phi^2}}\right)$$

$$=\frac{1}{2}m\left(\dot\ell^2+\ell^2\sin^2\theta_0\,\Omega^2\right)$$ ✓

$$L=T-U$$

$$\boxed{L=\frac{1}{2}m\left(\dot\ell^2+\ell^2\sin^2\theta_0\,\Omega^2\right)+mg\ell\cos\theta_0}$$ ✓

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\ell}\right)-\frac{\partial L}{\partial\ell}=0$$

$$\frac{d}{dt}\left(m\dot\ell\right)-\left(+m\ell\sin^2\theta_0\Omega^2+mg\cos\theta_0\right)=0$$

$$m\overset{0}{\cancel{\ddot\ell}}-m\ell\sin^2\theta_0\Omega^2\mp mg\cos\theta_0=0$$

eq: $\ddot\ell=0$

$$-\cancel{m}\ell_0\sin^2\theta_0\Omega^2=\cancel{m}g\cos\theta_0$$

$$\boxed{\ell_0=-\frac{g\cos\theta_0}{\Omega^2\sin^2\theta_0}}$$ ✓

stability with small displacement: $\ell=\ell_0+\eta$

$$\ddot\ell=\ell\sin^2\theta_0\Omega^2+g\cos\theta_0$$

$$\ddot\eta=(\ell_0+\eta)\sin^2\theta_0\Omega^2+g\cos\theta_0$$

$$-g\cos\theta=\Omega^2\ell_0\sin^2\theta_0$$

$$\ddot\eta=\cancel{\ell_0\Omega^2\sin^2\theta_0}+\eta\sin^2\theta_0\Omega^2-\cancel{\Omega^2\ell_0\sin^2\theta_0}$$

$$\ddot\eta=\eta\sin^2\theta_0\Omega^2$$

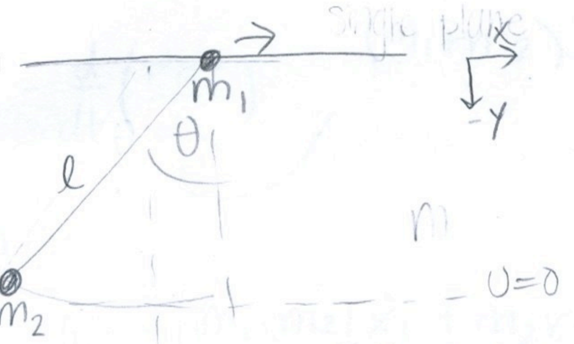
$$\omega^2=\sin^2\theta_0\Omega^2$$

$$\boxed{\omega=\Omega\sin\theta_0}$$

frequency of oscillation about equilibrium orbit, orbit must be above point where U is defined to be 0

✓ - no stable orbit restoring force not opposite to direction of displacement (unlike 3.1 part c)

3.3



we have two particles:

$m_1: \; T_1 = \frac{1}{2} m_1 \dot{x}_1^2$

$m_2: \; T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

* need to include θ, $r = l = \text{const.}$

$x_2 = x_1 + r\sin\theta \rightarrow \dot{x}_2 = \dot{x}_1$

$y_2 = -r\cos\theta$

$x_1 = x_1$

$\dot{x}_2 = \dot{x}_1 + \cancelto{0}{\dot{r}\cos\theta} + r\cos\theta\dot{\theta} \; \dot{\theta} \leftarrow$

$\dot{x}_2 = \dot{x}_1 + r\cos\theta\dot{\theta}$

$\dot{y}_2 = \cancelto{0}{-\dot{r}\cos\theta} + r\sin\theta\dot{\theta}$

$\dot{y}_2 = r\dot{\theta}\sin\theta$

$U = m_2 g y = m_2 g(1-\cos\theta) r$

constructing the lagrangian:

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \Big[\underbrace{(\dot{x}_1 + r\cos\theta\dot{\theta})^2}_{\dot{x}_1^2 + 2r\cos\theta\dot{\theta}\dot{x}_1 + r^2\cos^2\theta\dot{\theta}^2} + (r\dot{\theta}\sin\theta)^2 \Big] + m_2 g r(1-\cos\theta)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \Big[\dot{x}_1^2 + 2r\cos\theta\dot{\theta}\dot{x}_1 + \underbrace{r^2\cos^2\theta\dot{\theta}^2 + r^2\sin^2\theta\dot{\theta}^2}_{r^2\dot{\theta}^2} \Big] + m_2 g r(1-\cos\theta)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 [\dot{x}_1^2 + 2r\dot{\theta}\dot{x}_1\cos\theta + r^2\dot{\theta}^2] + m_2 g r(1-\cos\theta)$$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + m_2 r\dot{\theta}\dot{x}_1\cos\theta + \frac{1}{2} m_2 r^2\dot{\theta}^2 + m_2 g r(1-\cos\theta)}$$

$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)$ $\quad (m_1+m_2)\dot{x}_1 + m_2 r\dot{\theta}\cos\theta +$

$\uparrow$
0

$$0 = \frac{d}{dt}\left((m_1+m_2)\dot{x}_1 + m_2 r\dot{\theta}\cos\theta\right)$$

$$0 = (m_1+m_2)\ddot{x}_1 + m_2 r\ddot{\theta}\cos\theta - m_2 r\dot{\theta}\sin\theta\dot{\theta} = 0$$

$r = l$

$$\frac{\ddot{x}_1(m_1+m_2)}{(m_1+m_2)} = \frac{m_2 r\dot{\theta}\sin\theta\dot{\theta} - m_2 r\ddot{\theta}\cos\theta}{(m_1+m_2)}$$

$$\ddot{x}_1 = \frac{m_2 l(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{M_1+m_2}$$

assuming $\theta \approx$ small

$\cos\theta \approx 1$

$\sin\theta \approx \theta$

$$\boxed{\text{Eq 1: } \ddot{x}_1 = \frac{m_2 l(\dot{\theta}^2\theta - \ddot{\theta})}{(m_1+m_2)}}$$

$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \rightarrow \frac{d}{dt}\left[m_2 l\dot{x}_1\cos\theta + m_2 l^2\dot{\theta}\right]$

$\searrow$ $m_2 l\ddot{x}_1\cos\theta - m_2 l\dot{x}_1\sin\theta\dot{\theta} + m_2 l^2\ddot{\theta}$

$\uparrow$
$(-m_2 l\dot{\theta}\dot{x}_1\sin\theta + m_2 g l\sin\theta)$

$$m_2 g l\sin\theta - m_2 l\dot{\theta}\cancel{\dot{x}_1\sin\theta} = m_2 l\ddot{x}_1\cos\theta - m_2 l\cancel{\ddot{x}_1\sin\theta\dot{\theta}} + m_2 l^2\ddot{\theta}$$

$$m_2 g l\sin\theta = m_2 l\ddot{x}_1\cos\theta + m_2 l^2\ddot{\theta} \Rightarrow \ddot{\theta}(\cancel{m_2} l^{\cancel{2}}) = -\cancel{m_2} g\cancel{l}\sin\theta - \cancel{m_2 l}\ddot{x}_1\cos\theta$$

$$\ddot{\theta} l = -g\underset{\searrow\theta}{\sin\theta} - \ddot{x}_1\underset{\searrow 1}{\cos\theta}$$

assume θ small

$$\text{Eq 2 } \boxed{\ddot{\theta} = -\frac{g\theta}{l} - \frac{\ddot{x}_1}{l}}$$

$$\ddot{\theta} = \frac{-g\theta}{\ell} - \frac{1}{\ell}\left[\frac{m_2\ell(\theta\dot{\theta}^2-\theta)}{(m_1+m_2)}\right]$$

$$\ddot{\theta} + \frac{g}{\ell}\theta + \frac{m_2}{(m_1+m_2)}(\dot{\theta}^2\theta-\ddot{\theta}) = 0$$

$$\ddot{\theta} + \frac{g}{\ell}\theta + \frac{m_2}{(m_1+m_2)}\dot{\theta}^2\theta - \frac{m_2}{(m_1+m_2)}\ddot{\theta} = 0$$

asume θ small

$\dot{\theta}^2 \approx 0$

$$\ddot{\theta}\left(1-\frac{m_2}{m_1+m_2}\right) + g/\ell\,\theta + \cancelto{0}{\frac{m_2}{(m_1+m_2)}}\dot{\theta}^2\theta = 0$$

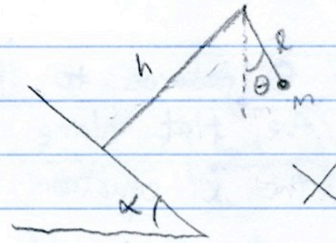
$$\ddot{\theta}\left(1-\frac{m_2}{m_1+m_2}\right) = -g/\ell\,\theta$$

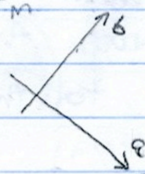
$$\ddot{\theta} = -\frac{g}{\ell}\underbrace{\frac{1}{\left(1-\frac{m_2}{m_1+m_2}\right)}}_{\text{const!}}\theta$$

$$\boxed{\cdot\,\omega^2 = \frac{g}{\ell}\frac{1}{\left(1-\frac{m_2}{m_1+m_2}\right)}}$$

Simple harmonic motion

3.4.





$$A = a + l\sin(\theta + \alpha)$$
$$B = h - l\cos(\theta + \alpha)$$
$$\dot{A} = \dot{a} + l\dot{\theta}\cos(\theta + \alpha)$$
$$\dot{B} = l\dot{\theta}\sin(\theta + \alpha)$$

$$L = \frac{1}{2}m\left(\dot{a}^2 + 2l\dot{\theta}\dot{a}\cos(\theta+\alpha) + l^2\dot{\theta}^2\cos^2(\theta+\alpha) + l^2\dot{\theta}^2\sin^2(\theta+\alpha)\right)$$
$$- mg\left[-(a + l\sin(\theta+\alpha))\sin\alpha + (h - l\cos(\theta+\alpha))\cos\alpha\right]$$

$$\boxed{L = \frac{1}{2}m\left(\dot{a}^2 + 2l\dot{\theta}\dot{a}\cos(\theta+\alpha) + l^2\dot{\theta}^2\right) + mg\left[(a + l\sin(\theta+\alpha))\sin\alpha - (h - l\cos(\theta+\alpha))\cos\alpha\right]}$$

LEOM

$$\frac{\partial L}{\partial a} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{a}} = 0$$

$$mg\sin\alpha - m\ddot{a} - \frac{d}{dt}\left(ml\dot{\theta}\cos(\theta+\alpha)\right) = 0$$

$$g\sin\alpha - \ddot{a} - l\ddot{\theta}\cos(\theta+\alpha) + l\dot{\theta}^2\sin(\theta+\alpha) = 0$$

$$\boxed{\ddot{a} = g\sin\alpha - l\ddot{\theta}\cos(\theta+\alpha) + l\dot{\theta}^2\sin(\theta+\alpha) = 0}$$

$$\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} = 0$$

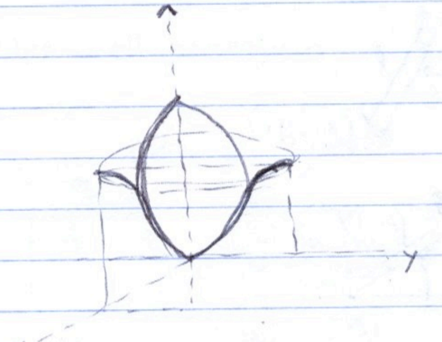
$$-l\dot{\theta}\dot{a}\sin(\theta+\alpha) + gl\cos(\theta+\alpha)\sin\alpha + gl\sin(\theta+\alpha)\cos\alpha$$
$$- \frac{d}{dt}\left[l\dot{a}\cos(\theta+\alpha) + l^2\dot{\theta}\right] = 0$$

$$-l\dot{\theta}\dot{a}\sin(\theta+\alpha) + gl\left[\cos(\theta+\alpha)\sin\alpha + \sin(\theta+\alpha)\cos\alpha\right] - l\ddot{a}\cos(\theta+\alpha)$$
$$+ l\dot{a}\dot{\theta}\sin(\theta+\alpha) - l^2\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{g}{l}\left[\cos(\theta+\alpha)\sin\alpha + \sin(\theta+\alpha)\cos\alpha\right] - \frac{\ddot{a}}{l}\cos(\theta+\alpha)$$

As $\alpha \to 0$, the expression for $\ddot{\theta}$ reduces to that of the pendulum moving across the flat plane (problem 3.3). The $\ddot{\alpha}$ equation reduces to the $\ddot{x}$ equation for 3.3 (with $m_1=0$).

3.8)



$$z = \alpha \sin\left(\frac{r}{R}\right)$$

$$\dot{z} = \frac{\alpha \dot{r}}{R} \cos\left(\frac{r}{R}\right)$$

There is rotational symmetry about z axis. So we use cylindrical coordinates (r,ϕ,z). Also, as a consequence of the symmetry, the angular momentum along z-axis is conserved.

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$$

We are given the constraint $z = \alpha \sin\left(\frac{r}{R}\right)$

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \frac{\alpha^2 \dot{r}^2}{R^2} \cos^2\left(\frac{r}{R}\right)\right)$$

$$V = mg\alpha \sin\left(\frac{r}{R}\right)$$

$$\mathcal{L} = T - V = \frac{1}{2} m(\dot{r}^2)\left(1 + \frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right) + \frac{1}{2} m r^2 \dot{\phi}^2 - mg\alpha \sin\left(\frac{r}{R}\right)$$

The lagrangian EOM are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\Rightarrow \frac{d}{dt}\left(m r^2 \dot{\phi}\right) = 0 \quad \Rightarrow m r^2 \dot{\phi} = \text{constant} = L_z \longrightarrow ①$$

This is expected due to the symmetry about z axis.

The other equation of motion is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$

$$\Rightarrow \frac{d}{dt}\left(m\dot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right)\right) = \frac{-1}{2} m\dot{r}^2 \frac{\alpha^2}{R^2}\, 2\cos\left(\frac{r}{R}\right)\sin\left(\frac{r}{R}\right)\frac{1}{R}$$

$$+ m r \dot{\phi}^2 - \frac{mg\alpha}{R}\cos\left(\frac{r}{R}\right) \longrightarrow ②$$

One could replace $\dot{\phi} = \frac{L_z}{mr^2}$ in ② to reduce this to a one dimensional problem in the r coordinate.

To find the stationary circular orbits, we set $r = r_0$, $\dot{r} = \ddot{r} = 0$. Then ② reduces to

$$0 = m r_0 \dot{\phi}^2 - \frac{mg\alpha}{R}\cos\left(\frac{r}{R}\right)$$

$$\dot{\phi} = \frac{L_z}{m r_0^2}$$

$$\Rightarrow \frac{\cancel{m r_0} L_z^2}{r_0^3} = \frac{m^2 g\alpha}{R}\cos\left(\frac{r_0}{R}\right)$$

The ~~stable~~ stationary points are obtained by solving the equation for r_0.

$$\sec\left(\frac{r_0}{R}\right) = \frac{m^2 g\alpha}{R L_z^2} r_0^3 \longrightarrow ③$$

The first order solution is obtained if $\frac{r_0}{R} \ll 1$ as $r_0 = \left(\frac{R L_z^2}{mg\alpha}\right)^{1/3}$.

Alternatively, solutions can be obtained graphically.
let $\frac{r_0}{R} = x$ & $\frac{m^2 g\alpha R^2}{L_z^2} - k$.

Then ③ becomes $\sec x = k x^3$. We only look for solutions $x > 0$

A little work with wolframalpha reveals that the number of solutions depends on k. At the critical value of k, the ~~[illegible]~~ plots $y = \sec x$ & $y = kx^3$ ~~inter sect~~ meet tangentially. Below that value, $\sec x$ grows too fast for kx^3 to catch up. Above that value, two solutions exist in the interval $(0, \pi/2)$

let us find the critical value k^*. At $k=k^*$ $\exists$ x_0 st.

$$\sec(x_0) = k^* x_0^3$$

& $$\frac{d}{dx}(\sec x)\Big|_{x=x_0} = 3k^* x_0^2$$

$$\Rightarrow \quad \sec x_0 \tan x_0 = 3k^* x_0^2$$

$$\cancel{k^*} x_0^3 \sqrt{k^{*2} x_0^6 - 1} = 3\cancel{k^*} x_0^2$$

$$k^{*2} x_0^6 - 1 = \frac{9}{x_0^2}$$

$$\Rightarrow \quad k^{*2} x_0^8 - x_0^2 - 9 = 0.$$

This equation must have only one solution x_0 in the interval $(0, \pi/2)$ for the parameter value k^*. Again some computations reveals that $k^* \approx 1.5$.

(c) Stability of orbits:

Consider a small perturbation $\eta(t)$ in radial coordinate so that $r = r_0 + \eta(t)$ and linearise the equation of motion to get

$$\left[1 + \frac{\alpha^2}{2R^2} + \frac{\alpha^2}{2R^2}\cos\left(\frac{2r_0}{R}\right)\right]\ddot{\eta} = \left(\frac{g\alpha}{R}\sin\frac{r_0}{R} - \frac{3L_z^2}{m^2 r_0^3}\right)\eta$$

$$\Rightarrow \quad \ddot{\eta} = \frac{g\alpha\sqrt{10}}{R\left(1 + \frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)\right)}\sin\left(\frac{r_0}{R} - \phi\right)\eta \quad , \tan\phi = 3$$

$\Rightarrow$ Oscillations are stable if ~~$\frac{r_0}{R} - \phi > 0$~~ $\sin\left(\frac{r_0}{R} - \phi\right) < 0$.

It can be shown that when 2 solutions exist, the ~~larger~~ smaller value of r_0 corresponds to a stable orbit & the larger value of r_0 to the unstable orbit.