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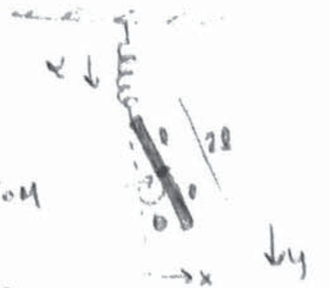
Phys 200A
HW Set 3
Due: Mon, Nov. 14th

1 Uniform Bar mass m, length 2ℓ (Moves in vertical plane)
Spring, constant k (moves up & down)



$$I_{bar} = \frac{1}{12} ML^2$$
$$= \frac{1}{12} m(2\ell)^2$$
$$= \frac{4}{12} m\ell^2$$
$$= \frac{1}{3} m\ell^2$$

The center of mass is in the middle of the bar. We look at how the bar moves using these coordinates.

$x = \ell \sin\theta \qquad y = \ell\cos\theta + \alpha$

$\dot{x} = \ell\cos\theta\,\dot{\theta} \qquad \dot{y} = -\ell\sin\theta\,\dot{\theta} + \dot{\alpha} \qquad \dot{y}^2 = (-\ell\sin\theta\dot{\theta} + \dot{\alpha})(-\ell\sin\theta\dot{\theta} + \dot{\alpha})$

$\dot{x}^2 = \ell^2\cos^2\theta\,\dot{\theta}^2 \qquad\qquad = \ell^2\sin^2\theta\,\dot{\theta}^2 - 2\ell\sin\theta\,\dot{\theta}\dot{\alpha} + \dot{\alpha}^2$

Kinetic Energy:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\dot{\theta}^2$$
$$= \frac{1}{2} m\left(\ell^2\cos^2\theta\,\dot{\theta}^2 + \ell^2\sin^2\dot{\theta}^2 - 2\ell\sin\theta\,\dot{\theta}\dot{\alpha} + \dot{\alpha}^2\right) + \frac{1}{2}\left(\frac{1}{3}m\ell^2\right)\dot{\theta}^2$$
$$= \frac{1}{2} m\left(\ell^2\dot{\theta}^2 + \dot{\alpha}^2 - 2\ell\sin\theta\,\dot{\theta}\dot{\alpha}\right) + \frac{1}{6} m\ell^2\dot{\theta}^2$$
$$\frac{1}{2}\cdot\frac{1}{3} m\ell^2\dot{\theta}^2$$
$$= \frac{1}{2} m\left(\frac{4}{3}\ell^2\dot{\theta}^2 + \dot{\alpha}^2 - 2\ell\sin\theta\,\dot{\theta}\dot{\alpha}\right)$$

Potential Energy:

$$U = -mgy + \frac{1}{2}k\alpha^2$$
$$= mg(\ell\cos\theta + \alpha) + \frac{1}{2}k\alpha^2$$

Lagrangian:

$$L = T - U = \frac{1}{2} m\left[\frac{4}{3}\ell^2\dot{\theta}^2 + \dot{\alpha}^2 - 2\ell\sin\theta\,\dot{\theta}\dot{\alpha}\right] - mg\ell\cos\theta - mg\alpha - \frac{1}{2}k\alpha^2$$

Hamiltonian

$$H = \sum_i p_i\dot{q}_i - L = p_\alpha\dot{\alpha} + p_\theta\dot{\theta} - L$$

$$p_\alpha = \frac{\partial L}{\partial\dot{\alpha}} = -m\ell\sin\theta\,\dot{\theta} + m\dot{\alpha}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{4}{3}m\ell^2\dot{\theta} - m\ell\sin\theta\,\dot{\alpha}$$

(1) cont'd.

$$H = P_\alpha\dot\alpha + P_\theta\dot\theta - \frac{1}{2}m\left[\frac{4}{3}l^2\dot\theta^2 + \dot\alpha^2 - 2l\sin\theta\,\dot\theta\dot\alpha\right] + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

$$\frac{4}{3}l^2\dot\theta^2 + \dot\alpha^2 - l\sin\theta\dot\theta\dot\alpha - l\sin\theta\dot\theta\dot\alpha$$

$$= \dot\theta\left(\frac{4}{3}l^2\dot\theta - l\sin\theta\dot\alpha\right) + \dot\alpha\left(\dot\alpha - l\sin\theta\dot\theta\right)$$

$$= \dot\theta\left(P_\theta/m\right) + \dot\alpha\left(P_\alpha/m\right)$$

$$= P_\alpha\dot\alpha + P_\theta\dot\theta - \frac{1}{2}\cancel{m}\frac{P_\theta}{\cancel{m}}\dot\theta - \frac{1}{2}\cancel{m}\frac{P_\alpha}{\cancel{m}}\dot\alpha + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

$$H = \frac{1}{2}P_\alpha\dot\alpha + \frac{1}{2}P_\theta\dot\theta + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

$$P_\alpha = m\dot\alpha - ml\sin\theta\dot\theta$$

$$m\dot\alpha = P_\alpha + ml\sin\theta\dot\theta$$

$$\dot\alpha = \frac{P_\alpha}{m} + l\sin\theta\dot\theta$$

$$\dot\alpha = \frac{P_\alpha}{m} + \frac{ml\sin\theta}{m}\left[\frac{3P_\theta + 3ml\sin\theta\dot\alpha}{4ml^2}\right]$$

$$= \frac{P_\alpha}{m} + \frac{3\cancel{m}l\sin\theta P_\theta}{4m^{\cancel{2}}l^2} + \frac{3\cancel{m^2}\cancel{l^2}\sin^2\theta\dot\alpha}{4\cancel{m^2}\cancel{l^2}}$$

$$\underbrace{\left(1 - \frac{3\sin^2\theta}{4}\right)}_{\frac{4ml^2 - 3ml^2\sin^2\theta}{4ml^2}}\dot\alpha = \underbrace{\frac{P_\alpha}{m}}_{\frac{4l^2P_\alpha}{4ml^2}} + \frac{3l\sin\theta P_\theta}{4ml^2}$$

$$\dot\alpha = \frac{4l^2P_\alpha + 3l\sin\theta P_\theta}{\cancel{4ml^2}} \cdot \frac{\cancel{4ml^2}}{4ml^2 - 3ml^2\sin^2\theta}$$

$$= \frac{4l^2P_\alpha + 3l\sin\theta P_\theta}{4ml^2 - 3ml^2\sin^2\theta}$$

$$P_\theta = \frac{4}{3}ml^2\dot\theta - ml\sin\theta\dot\alpha$$

$$\frac{4}{3}ml^2\dot\theta = P_\theta + ml\sin\theta\dot\alpha$$

$$\dot\theta = \frac{3P_\theta + 3ml\sin\theta\dot\alpha}{4ml^2}$$

Plug in $\dot\alpha$:

$$\dot\theta = \frac{3P_\theta}{4ml^2} + \frac{3ml\sin\theta}{4ml^2}\left[\frac{P_\alpha}{m} + l\sin\theta\dot\theta\right]$$

$$= \frac{3P_\theta}{4ml^2} + \frac{3\cancel{m}l\sin\theta P_\alpha}{4m^{\cancel{2}}l^2} + \frac{3ml^2\sin^2\theta\dot\theta}{4ml^2}$$

$$\underbrace{\left(1 - \frac{3ml^2\sin^2\theta}{4ml^2}\right)}_{\frac{4ml^2 - 3ml^2\sin^2\theta}{4ml^2}}\dot\theta = \frac{3P_\theta + 3l\sin\theta P_\alpha}{4ml^2}$$

$$\dot\theta = \frac{3P_\theta + 3l\sin\theta P_\alpha}{\cancel{4ml^2}} \cdot \frac{\cancel{4ml^2}}{4ml^2 - 3ml^2\sin^2\theta}$$

$$= \frac{3P_\theta + 3l\sin\theta P_\alpha}{4ml^2 - 3ml^2\sin^2\theta}$$

$$H = \frac{1}{2}P_\alpha\left[\frac{4l^2P_\alpha + 3l\sin\theta P_\theta}{4ml^2 - 3ml^2\sin^2\theta}\right] + \frac{1}{2}P_\theta\left[\frac{3P_\theta + 3l\sin\theta P_\alpha}{4ml^2 - 3ml^2\sin^2\theta}\right] + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

$$= \frac{1}{2}\left[\frac{4l^2P_\alpha^2 + 3l\sin\theta P_\theta P_\alpha + 3P_\theta^2 + 3l\sin\theta P_\alpha P_\theta}{4ml^2 - 3ml^2\sin^2\theta}\right] + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

$$= \frac{1}{2ml^2}\left[\frac{4l^2P_\alpha^2 + 6l\sin\theta P_\theta P_\alpha + 3P_\theta^2}{4 - 3\sin^2\theta}\right] + mgl\cos\theta + mg\alpha + \frac{1}{2}k\alpha^2$$

(1 cont'd)

HEOM given by: $\dot{q}_i = \frac{\partial H}{\partial P_i}$; $\dot{P}_i = -\frac{\partial H}{\partial q_i}$

For α:

$$\dot{\alpha} = \frac{\partial H}{\partial P_\alpha} = \frac{1}{2ml^2}\left[\frac{4l^2(2P_\alpha) + 6l\sin\theta P_\theta}{4-3\sin^2\theta}\right]$$

$$= \frac{4l^2P_\alpha + 3l\sin\theta P_\theta}{4ml^2 - 3ml^2\sin^2\theta}$$ ✓ Same as before.

$$\dot{P}_\alpha = -\frac{\partial H}{\partial \alpha} = -\left[mg + k\alpha\right] = \underline{-mg - k\alpha.}$$

For θ:

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{1}{\cancel{2}ml^2}\left[\frac{3(\cancel{2}P_\theta) + \overset{3}{\cancel{6}}l\sin\theta P_\alpha}{4-3\sin^2\theta}\right]$$

$$= \frac{3P_\theta + 3l\sin\theta P_\alpha}{4ml^2 - 3ml^2\sin^2\theta}$$ ✓ Same as before.

$$\dot{P}_\theta = \frac{-\partial H}{\partial \theta} = -\left[mgl(-\sin\theta) + \frac{1}{2ml^2}\left(\frac{\partial}{\partial\theta}\left\{\frac{4l^2P_\alpha^2}{4-3\sin^2\theta} + \frac{6l\sin\theta P_\theta P_\alpha}{4-3\sin^2\theta} + \frac{3P_\theta^2}{4-3\sin^2\theta}\right\}\right)\right]$$

$$= mgl\sin\theta + \frac{1}{\cancel{2}ml^2}\left[\frac{\overset{2}{\cancel{4}}l^2P_\alpha^2(6\sin\theta\cos\theta)}{(4-3\sin^2\theta)^2} + \frac{3P_\theta^2(\overset{3}{\cancel{6}}\sin\theta\cos\theta)}{(4-3\sin^2\theta)^2} + \frac{\overset{3}{\cancel{6}}lP_\theta P_\alpha\cos\theta(4+3\sin^2\theta)}{(4-3\sin^2\theta)^2}\right]$$

$$= mgl\sin\theta + \frac{1}{ml^2}\left[\frac{12l^2\sin\theta\cos\theta P_\alpha^2 + 9\sin\theta\cos\theta P_\theta^2 + 3l\cos\theta P_\theta P_\alpha(4+3\sin^2\theta)}{(4-3\sin^2\theta)^2}\right]$$

$\nabla^2\psi + \frac{\omega^2}{c_0^2} n(\vec{x})^2 \psi = 0$ let $n(\vec{x})^2 = 1+\delta(\vec{x})$ $\delta << 1$

assume $\hat{z}$ propagation

let $\psi = A(\vec{x}) e^{ik_z z}$

$$\underbrace{\nabla^2\psi}_{①} + \underbrace{\frac{\omega_0^2}{c_0^2}\left[1+\delta(\vec{x})\right]\psi}_{②} = 0$$

& $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

So $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

↓

$\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_\perp^2$

① $\underbrace{\nabla_\perp^2 A(\vec{x}) e^{ik_z z}}_{\psi} + \frac{\partial^2}{\partial z^2}\left[A(\vec{x}) e^{ik_z z}\right]$

$= \nabla_\perp^2 \psi + \frac{\partial}{\partial z}\left[\frac{\partial A(\vec{x})}{\partial z}\cdot e^{ik_z z} + ik_z A(\vec{x}) e^{ik_z z}\right]$

$= \nabla_\perp^2 \psi + \overset{0}{\cancel{\frac{\partial^2}{\partial z^2} A(\vec{x})}} \cdot e^{ik_z z} + \overbrace{ik_z \frac{\partial A(\vec{x})}{\partial z}\cdot e^{ik_z z} + ik_z \frac{\partial A(\vec{x})}{\partial z} e^{ik_z z}}^{\text{add}} - \underbrace{k_z^2 A(\vec{x})\cdot e^{ik_z z}}_{\psi}$

b/c small compared to other terms

$$\frac{\frac{\partial A}{\partial z}}{A} << k_z$$

↓

$$k_z^2 >> \frac{\frac{\partial^2 A}{\partial z^2}}{A^2}$$

$= \nabla_\perp^2 (A(\vec{x}) e^{ik_z z}) + 2ik_z \frac{\partial A(\vec{x})}{\partial z} e^{ik_z z} - k_z^2 A(\vec{x}) e^{ik_z z}$

② $\frac{\omega_0^2}{c_0^2} A(\vec{x}) e^{ik_z z} + \frac{\omega_0^2}{c_0^2}\delta(\vec{x}) A(\vec{x}) e^{ik_z z}$

① + ② ⇒

↙ cancel out ↘

$$\nabla_\perp^2 A(\vec{x})\cancel{e^{ik_z z}} + 2ik_z \frac{\partial A(\vec{x})}{\partial z}\cancel{e^{ik_z z}} - \cancel{k_z^2 A(\vec{x}) e^{ik_z z}} + \cancel{\frac{\omega_0^2}{c_0^2} A(\vec{x}) e^{ik_z z}} + \frac{\omega_0^2}{c_0^2}\delta(\vec{x}) A(\vec{x})\cancel{e^{ik_z z}} = 0$$

let $k_z^2 = \frac{\omega_0^2}{c_0^2}$

$2ik_z \frac{\partial A}{\partial z} + \nabla_\perp^2 A(\vec{x}) + \frac{\omega_0^2}{c_0^2}\delta(\vec{x}) A(\vec{x}) = 0$ where $\psi' = A(\vec{x})$

$$\boxed{2ik_z \frac{\partial \psi'}{\partial z} + \nabla_\perp^2 \psi'(\vec{x}) + \frac{\omega_0^2}{c_0^2}\delta(\vec{x})\psi' = 0}$$ ★

k_r is the wave number $k_r = \frac{\omega}{c_0}$ (a dimensionless value)

i approximation: $|k_z| >> \left| \frac{\frac{\partial A}{\partial z}}{A} \right|$ ∴ $\frac{\partial^2 A}{\partial z^2} = 0$

↗

The amplitude varies slowly since δ is small, and compared to the other terms in *.

(or the change in phase is >> the change in amplitude in the z-direction).

ii

Source term: $2ik_z \frac{\partial A}{\partial z}$ + diffraction: $\nabla_\perp^2 A$ + scattering: $\delta(\vec{x}) \frac{\omega_0^2}{c_0^2} A$

iii Restrictions of $\frac{\partial \psi}{\partial z}'$ etc..

$\frac{\partial A}{\partial z}$ & $\delta(\vec{x}) A(\vec{x})$ should be of the same order

Amplitude should vary more slowly than the frequency (in comparable length scale).

2c. <u>Now</u>: $\psi = A(\vec{x}) e^{i\phi(\vec{x})}$

$$\frac{\partial \psi}{\partial z} = \left[\frac{\partial A(\vec{x})}{\partial z} + i \frac{\partial \phi(\vec{x})}{\partial z} A(\vec{x}) \right] e^{i\phi(\vec{x})}$$

$$\nabla_\perp^2 \psi(\vec{x}) = \nabla_\perp (\nabla_\perp \psi(\vec{x})) = \nabla_\perp \left[(\nabla_\perp A(\vec{x})) + i(\nabla_\perp \phi(\vec{x})) A(\vec{x}) \right] e^{i\phi(\vec{x})}$$

$$= \left[\nabla_\perp^2 A(\vec{x}) + (\nabla_\perp A(\vec{x})) i \nabla_\perp \phi(\vec{x}) + i \nabla_\perp^2 (\phi(\vec{x})) A(\vec{x}) + i \nabla_\perp \phi(\vec{x}) (\nabla_\perp A(\vec{x})) - (\nabla_\perp \phi(\vec{x}))^2 A(\vec{x}) \right] e^{i\phi(\vec{x})}$$

plug into parabolic wave equation. (drop $(\vec{x})$ for simplicity)

$$2ik_z \cdot \left[\frac{\partial A}{\partial z} + i \frac{\partial \phi}{\partial z} A \right] \cancel{e^{i\phi}} + \left[\nabla_\perp^2 A + i(\nabla_\perp A)(\nabla_\perp \phi) + i(\nabla_\perp^2 \phi) A + i(\nabla_\perp \phi)(\nabla_\perp A) - (\nabla_\perp \phi)^2 A \right] \cancel{e^{i\phi}} + \frac{\omega_0^2}{c_0^2} \delta(\vec{x}) \cdot A \cancel{e^{i\phi}} = 0$$

$${}_{z}\frac{\partial A}{\partial z} - 2k_z\frac{\partial \phi}{\partial z} \cdot A + \nabla_\perp^2 A + i(\nabla_\perp A)(\nabla_\perp \phi) + i(\nabla_\perp^2 \phi)A + i(\nabla_\perp \phi)(\nabla_\perp A) - (\nabla_\perp \phi)^2 A$$

$$+ \frac{\omega_0^2}{c_0^2}\delta(\vec{x})A = 0$$

Real part: $-2k_z\frac{\partial \phi}{\partial z} A + \nabla_\perp^2 A - (\nabla_\perp \phi)^2 A + \frac{\omega_0^2}{c_0^2}\delta(\vec{x})A = 0$ (eq 1)

assuming $\frac{\frac{\partial A}{\partial z}}{A} \ll k_z$ then $\frac{\partial^2 A}{\partial z^2}$ is verysmall. This equation would give rise to eikonal theory $\{(\nabla_\perp \phi)^2 = \frac{\omega_0^2}{c_0^2} n^2(\vec{x})\}$? except we separated the z direction to make the approximations leading to the parabolic wave equation.

imaginary part: $2k_z\frac{\partial A}{\partial z} + (\nabla_\perp A)(\nabla_\perp \phi) + (\nabla_\perp^2 \phi)A + (\nabla_\perp \phi)(\nabla_\perp A) = 0$

$$2k_z\frac{\partial A}{\partial z} + 2(\nabla_\perp A)(\nabla_\perp \phi) + (\nabla_\perp^2 \phi)A = 0 \quad \text{(eq 2)}$$

The real equation (eq 1) describes the scattering of the amplitude. The imaginary equation (eq. 2) describes the variation of the wave in the z-direction.

(a)

③ $\frac{\partial \tilde{\rho}}{\partial t} + \vec{v}_0 \cdot \nabla \tilde{\rho} = -\rho_0 \nabla \cdot \vec{\tilde{v}}$ & $\rho_0\left(\frac{\partial \vec{\tilde{v}}}{\partial t} + \vec{v}_0 \nabla \vec{\tilde{v}}\right) = -c^2 \nabla \tilde{\rho}_1$

$\left(\frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla\right)\tilde{\rho} = -\rho_0 \nabla \cdot \vec{\tilde{v}}$ $\qquad$ $\left(\frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla\right)\vec{\tilde{v}} = \frac{-c^2 \nabla \tilde{\rho}}{\rho_0}$

$\tilde{v} = A_1 e^{\phi(\vec{x},t)}$ $\qquad$ $\left(\frac{\partial}{\partial t} + v_0 \cdot \nabla\right)\tilde{v} = -\frac{c^2 \nabla \tilde{\rho}}{\rho_0}$

$\left(\frac{\partial}{\partial t} + v_0 \cdot \nabla\right)\vec{\tilde{v}} = iA_1 e^{i\phi_1} \frac{\partial \phi_1}{\partial t} + v_0 e^{i\phi} \cdot \nabla A_1 + iA_1 e^{i\phi} \nabla \phi$

$\tilde{\rho} = A_2 e^{i\phi(\vec{x},t)}$

$\nabla \tilde{\rho} = \nabla A_2 e^{i\phi} + iA_2 e^{i\phi} \nabla \phi$

Equate imaginary part:

$A_1 \cancel{e^{i\phi}} \frac{\partial \phi}{\partial t} + v_0 (\nabla \phi) A_1 \cancel{e^{i\phi}} = \frac{-c^2}{\rho_0} A_2 \cancel{e^{i\phi}} \nabla \phi$

$\frac{\partial \phi}{\partial t} + v_0(\nabla \phi) = \frac{-c^2}{\rho_0} \frac{A_2}{A_1} \nabla \phi$

$\frac{A_2}{A_1} = \frac{\rho_0}{c} \rightarrow \boxed{\left(\frac{\partial \phi}{\partial t} + v_0(\nabla \phi)\right)^2 = c^2 (\nabla \phi)^2}$

$c = \sqrt{\frac{P}{\rho_0}}$ $\quad$ $\frac{A_2}{A_1} = \sqrt{\frac{\rho_0^2}{P/\rho_0}} = \sqrt{\frac{\rho_0^3}{P}} = \frac{\rho_0^{3/2}}{P^{1/2}}$

$\phi = \omega t + \vec{k} \cdot \vec{x}$ $\qquad$ $\frac{\partial \phi}{\partial t} = \omega$ $\qquad$ $\vec{\nabla} \phi = \vec{k}$

$\vec{k} \cdot \vec{v}_0 = -\vec{v}_0 \cdot \vec{k}$

$\boxed{(\omega - \vec{k} \cdot \vec{v}_0) = c^2 k^2}$

(b) $\frac{\partial \omega}{\partial k} = v$ $\qquad$ $\nabla \omega = -\dot{k}$ $\qquad$ $\omega = \sqrt{c^2 k^2} + \vec{k} \cdot \vec{v}_0 = c\sqrt{k_x^2 + k_y^2 + k_z^2} + k_x v_{0x} + k_y v_{0y} + k_z v_{0z}$

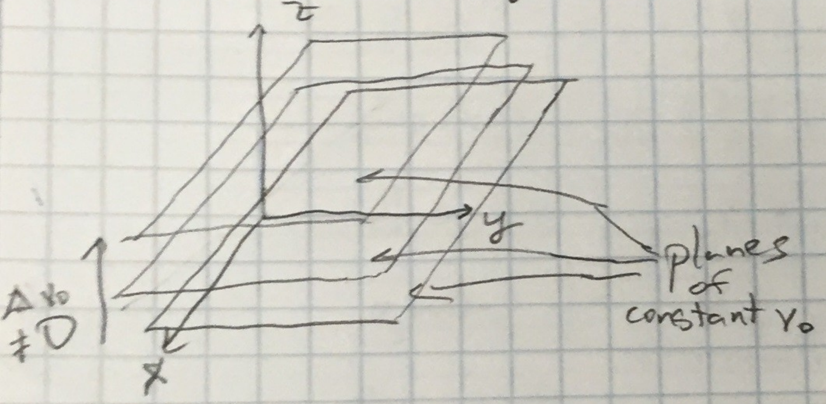
$\frac{\partial \omega}{\partial k_x} = \frac{1}{2} \frac{c\, 2k_x}{\sqrt{k^2}} + v_{0x} = v_{0x} \pm \frac{c k_x}{|k|}$ $\qquad$ $\frac{\partial \omega}{\partial \vec{k}} = \pm \frac{c\vec{k}}{|k|} + \vec{v}$

$\nabla \omega = \pm |k| \nabla c + \vec{k} \cdot \nabla \vec{v} = -\dot{k}$

$$\boxed{\frac{d\vec{x}}{dt} = \vec{v}_0 \pm \frac{\vec{k}c_s}{|\vec{k}|}} \qquad \boxed{\nabla\omega = \pm|k|\nabla c + \vec{k}\cdot\nabla\vec{v} = \frac{-d\vec{k}}{dt}}$$

$$\underbrace{|k|\cdot c}_{\omega_0} \longrightarrow \frac{dk}{dt} = -\frac{\partial}{\partial x}\left(\omega_0 + \vec{k}\cdot\nabla\vec{v}_0\right)$$

(c) Vertically sheared means v_0 is constant in the plane parrallel to the ground but increasing vertically



So $\vec{v}_0 = v_{0x}(y)\hat{x}$

$$\overset{\circ}{k} = \frac{\partial k_x}{\partial t} + \frac{\partial k_y}{\partial t} + \frac{\partial k_z}{\partial t}$$

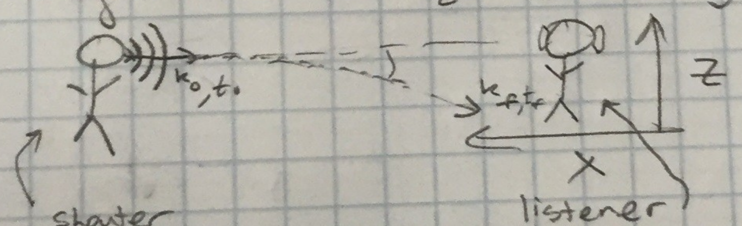
$$\frac{\partial k_x}{\partial t} = -\frac{\partial}{\partial x}\left(\cancelto{0}{\omega_0} + \cancelto{0}{k_x\cdot v_x} + \cancelto{0}{k_y\cdot v_y} + \cancelto{0}{k_z\cdot v_z}\right)$$

↳ same for k_y so k_x & k_y = constant/cyclic

$$\frac{\partial k_z}{\partial t} = -\underset{\substack{\uparrow\\ \text{constant}}}{k_x}\,\frac{\partial}{\partial z}\underset{\substack{\uparrow\\ \text{take this positive}}}{V_x}$$

$\frac{\partial k_z}{\partial t}$ = negative so k_z decreasing in time

so



On a windy day $v_{0,x}$ & $v_{0,y}$ will be largely dependent on z, altitude because of temperature gradients & drag associated w/ the wind & ground so your sound ray will be bent from the direction of the listener

(d) Same argument as above but take $\vec{v}_0 = v_{0x}(y)$ then your k_y will bend as the k_z did above.

$\omega_i = \sqrt{\frac{k_i}{m}}$

4. $$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2 + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2 z^2$$

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_1}, \ldots, q_1, \ldots\right) = 0$$ Hamilton-Jacobi Equation

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S_y}{\partial y}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + \frac{1}{2}m\left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right]$$

Equation does not explicitly involve t so,

$$S(\vec{q}, t) = -Et + S_1(x) + S_2(y) + S_3(z)$$

$$\Rightarrow E = \frac{1}{2m}\left[\left(\frac{\partial S_1}{\partial x}\right)^2 + \left(\frac{\partial S_2}{\partial y}\right)^2 + \left(\frac{\partial S_3}{\partial z}\right)^2\right] + \frac{1}{2}m\left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z z^2\right]$$

Since the variables are seperable the x, y, z parts must be equal to constants

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S_1}{\partial x}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 = \alpha_x$$

$$S_1 = \sqrt{2m\alpha_x}\int dx \sqrt{1 - \frac{m\omega_x^2 x^2}{2\alpha_x}}$$

Analogous for S_2 & S_3

$$S_2 = \sqrt{2m\alpha_y}\int dy \sqrt{1 - \frac{m\omega_y^2 y^2}{2\alpha_y}}$$

$$S_3 = \sqrt{2m\alpha_z}\int dz \sqrt{1 - \frac{m\omega_z^2 z^2}{2\alpha_z}}$$

Remember $E = \alpha_x + \alpha_y + \alpha_z$

Can find x, y, z by transforming the coords

$$Q_i = \beta_i = \partial S(q, \alpha, t)/\partial \alpha_i$$

$$\Rightarrow \beta_x = \frac{\partial S}{\partial \alpha_x} = \sqrt{\frac{m}{2\alpha_x}}\int dx \left(1 - \frac{m\omega_x^2 x^2}{2\alpha_x}\right)^{-1/2} - t$$

Recall, $\arcsin u = \int \frac{du}{\sqrt{1-u^2}}$ $\quad u = \sqrt{m/2\alpha_x}\cdot\omega_x x$

$dx = \sqrt{2\alpha_x/m}\cdot 1/\omega_x \, du$

$$\Rightarrow \beta_x + t = \frac{1}{\omega_x}\arcsin\left[q\sqrt{\frac{m\omega_x^2}{2\alpha_x}}\right]$$

$$\Rightarrow x = \sqrt{\frac{2\alpha_x}{m\omega_x^2}}\sin\left[\omega_x t + \beta_x\right]$$

Analogous for y & z

$$y = \sqrt{\frac{2\alpha_y}{m\omega_y^2}} \sin(\omega_y t + \beta_y)$$

$$z = \sqrt{\frac{2\alpha_z}{m\omega_z^2}} \sin(\omega_z t + \beta_z)$$

Solve for $p_x, p_y, p_z \Rightarrow p_i = \partial S / \partial q_i$

$$p_x = \frac{\partial S}{\partial x} = \frac{\partial W_1}{\partial x} = \sqrt{2m\alpha_x - m^2\omega_x^2 x^2}$$

$$\Rightarrow p_x = \sqrt{2m\alpha_x\left[1 - \sin^2(\omega_x t + \beta_x)\right]} = \sqrt{2m\alpha_x}\cos(\omega_x t + \beta_x)$$

$$p_y = \sqrt{2m\alpha_y}\cos(\omega_y t + \beta_y)$$

$$p_z = \sqrt{2m\alpha_z}\cos(\omega_z t + \beta_z)$$

Now connect α's & β's to initial conditions q_0's, p_0's

$$2m\alpha_x = p_{x_0}^2 + m^2\omega_x^2 x_0^2$$

$$\tan\beta_x = m\omega_x \frac{x_0}{p_{x_0}}$$

Same for α_y, α_z & β_y, β_z

10

5

5. a) $V(r,\phi,z) = a(r) + \frac{b(\phi)}{r^2} + c(z)$

V must have the same form as T

b) $H = \frac{1}{2m}\left[p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right] + V(r,\phi,z)$

$S = S_r(r) + S_\phi(\phi) + S_z(z) - Et$

$\frac{\partial S}{\partial t} + H = 0$

$$\Rightarrow E = \frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 + \left(\frac{\partial S_z}{\partial z}\right)^2\right] + V(r,\phi,z)$$

$$E = \underbrace{\left[\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2 + a(r)\right]}_{E - C_\phi/r^2 - C_z} + \frac{1}{r^2}\underbrace{\left[\frac{1}{2m}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 + b(\phi)\right]}_{C_\phi} + \underbrace{\left[\frac{1}{2m}\left(\frac{\partial S_z}{\partial z}\right)^2 + c(z)\right]}_{C_z}$$

$S_z = \sqrt{2m}\int\left[C_z - c(z)\right]^{1/2} dz$

$S_\phi = \sqrt{2m}\int\left[C_\phi - b(\phi)\right]^{1/2} d\phi$

$S_z = \sqrt{2m}\int\left[E - \frac{C_\phi}{r^2} + C_z + a(r)\right]^{1/2} dr$

$\frac{\partial S}{\partial q_i} = p_i = \frac{m\,dq_i}{dt} \Rightarrow dt = \frac{dq_i}{\partial S/\partial q_i} m$

$\Rightarrow t = \sqrt{\frac{m}{2}}\int\frac{dz}{(C_z - c(z))^{1/2}} + C'$

$t = \sqrt{\frac{m}{2}}\int\frac{d\phi}{(C_\phi - b(\phi))^{1/2}} + C''$

$t = \sqrt{\frac{m}{2}}\int\frac{dr}{(E - C_\phi/r^2 - C_z + a(r))^{1/2}} + C'''$

c) C_z is the energy in the $\hat{z}$ direction

C_ϕ/r^2 is the centrifugal potential acoustic wave lagrangian   Fermats principle:

$$\tau = \int_1^2 \frac{ds}{c_0}\cdot n(x) \qquad \delta\tau = 0$$

$\vec{x}(s)$

$$\equiv \int_1^2 \left(\frac{dx}{ds}\cdot\frac{dx}{ds}\right)^{1/2}\cdot n(x(s))\,ds \qquad |\dot{x}| = \left(\frac{dx}{ds}\cdot\frac{dx}{ds}\right)^{1/2} = 1$$

LEOM:

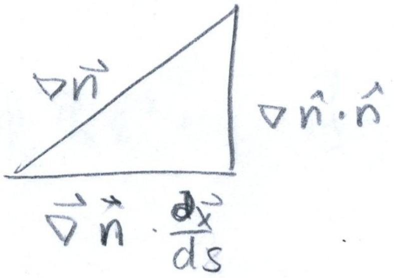
$$\frac{\partial L}{\partial x} - \frac{d}{ds}\frac{\partial L}{\partial \dot{\vec{x}}} = 0$$

$$\frac{\partial n}{\partial \vec{x}} - \frac{d}{ds}\left(n(x)\cdot\frac{\dot{x}}{|\dot{x}|}\right) = \frac{\partial n}{\partial x} - n(\vec{x})\cdot\frac{d^2\vec{x}}{\underbrace{ds^2}_{}} - \frac{\partial n}{\partial \vec{x}}\cdot\frac{d\vec{x}}{ds}\cdot\frac{d\vec{x}}{ds} = 0$$

$$\frac{d^2x}{ds^2} = \frac{1}{n(x)}\left[\frac{\partial n}{\partial \vec{x}}\cdot\frac{d\vec{x}}{ds}\cdot\frac{d\vec{x}}{ds} - \frac{\partial n}{\partial \vec{x}}\right] = \frac{1}{n(\vec{x})}\left[\frac{\partial n}{\partial \vec{x}} - \left(\frac{\partial n(\vec{x})}{\partial \vec{x}}\cdot\frac{d\vec{x}}{ds}\right)\frac{d\vec{x}}{ds}\right]$$

- $\frac{d\vec{x}}{ds}$ is unit tangent to path.

$$\frac{d^2x}{ds^2} = \frac{1}{n(x)}\vec{\nabla}n - \frac{1}{n(\vec{x})}\left(\vec{\nabla}n\cdot\frac{d\vec{x}}{ds}\right)\frac{d\vec{x}}{ds}$$



- $\hat{n}_0$ = unit normal to path

for some vector A

$$\vec{\nabla}\vec{A} - (\vec{\nabla}\vec{A}\cdot\underbrace{\hat{a}_t}_{\text{tangent comp}})\hat{a}_t = (\vec{\nabla}\vec{A}\cdot\underbrace{\hat{a}_n}_{\text{normal comp!}})\hat{a}_n$$

So:

$$\frac{d^2x}{ds^2} = \frac{1}{n(\vec{x})}(\vec{\nabla}n\cdot\hat{n}_0)\hat{n}_0$$

unit normal

$$\frac{\partial^2}{dx^2}\psi + \frac{Q(x)}{\varepsilon^2}\psi = 0 \quad *$$

let $\psi = e^{\underbrace{[\frac{i}{\varepsilon}\sum_{n=0}^{\infty}\varepsilon^n \phi_n(x)]}_{\phi(x) \text{ \& expand later}}} = e^{i/\varepsilon\, \phi(x)}$

$$\frac{d\psi}{dx} = \frac{i}{\varepsilon} e^{\frac{i}{\varepsilon}\phi(x)}\phi'(x)$$

$$\frac{d^2\psi}{dx^2} = \left(\frac{i}{\varepsilon}\right)^2 e^{i/\varepsilon\,\phi(x)}(\phi'(x))^2 + \frac{i}{\varepsilon} e^{i/\varepsilon\,\phi(x)} \cdot \phi''(x)$$

plug into *

$$-\frac{1}{\varepsilon^2}(\phi'_{(x)})^2 \cancel{e^{i/\varepsilon\,\phi(x)}} + \frac{i}{\varepsilon}\cancel{e^{i/\varepsilon\,\phi(x)}}\phi''(x) + \frac{Q(x)}{\varepsilon^2}\cdot\cancel{e^{i/\varepsilon\,\phi(x)}} = 0$$

$$\frac{1}{\varepsilon^2}(\phi'(x))^2 + \frac{i}{\varepsilon}\phi''(x) + \frac{Q(x)}{\varepsilon^2} = 0$$

now expand! $\phi = \sum_{n=0}^{\infty}\varepsilon^n\phi_n$

$$-\frac{1}{\varepsilon^2}\underbrace{\left[\sum_{n=0}^{\infty}\varepsilon^n\phi_n'(x)\right]^2}_{\left[\sum_{n,m}^{\infty}\varepsilon^{n,m}\phi_n'\phi_m'\right]} + \frac{i}{\varepsilon}\sum_{n=0}^{\infty}\varepsilon^n\phi_n''(x) + \frac{Q(x)}{\varepsilon^2} = 0$$

order

only looking at ϕ_0, ϕ_1, & ϕ_2

write out terms

$$-\frac{1}{\varepsilon^2}\Big[\phi_0'^2 + \overbrace{\phi_0'\phi_1'\varepsilon + \phi_1'\phi_0'\varepsilon}^{2\phi_0'\phi_1'\frac{1}{\varepsilon}} + \phi_1'\varepsilon^2 + \phi_0'\phi_2'\varepsilon^2 + \phi_2'\phi_0'\varepsilon^2 + \dots\Big]$$

$$+\frac{i}{\varepsilon}\left[\phi_0'' + \phi_1''\varepsilon + \phi_2''\varepsilon^2 + \dots\right] + \frac{Q(x)}{\varepsilon^2} = 0$$

$\frac{1}{\varepsilon^2}$ order: $-(\phi_0')^2 + Q(x) = 0$

$$\phi_0' = \pm\sqrt{Q(x)} \Rightarrow \phi_0(x) = \pm\int\sqrt{Q(x)}\,dx$$

$\frac{1}{\varepsilon}$ order: $-2\phi_0'\phi_1' + i\phi_0'' = 0 \qquad i\phi_0'' = 2\phi_0'\phi_1'$

ε^0 order: $-\phi_1'^2 + i\phi_1'' - 2\phi_0'\phi_2' = 0$

$$i\phi_1'' = \phi_1'^2 + 2\phi_0'\phi_2'$$

b. ϕ_0 equation is the 1-D eikonal equation!

(Helmholtz 3D eikonal: $(\nabla\phi)^2 = \frac{\omega^2}{c^2}$) our equation is

$$\left(\frac{\partial}{\partial x}\phi_0\right)^2 = Q(x)$$

1D vs. 3D

if you let $Q(x) = \frac{\omega^2}{c(x)^2}$

becomes $\left(\frac{\partial}{\partial x}\phi_0\right)^2 = \frac{\omega^2}{c(x)^2}$

c. solve for S_0 & S_1

from $\frac{1}{\varepsilon^2}$ order: $\phi_0 = \pm\int\sqrt{Q(x)}\,dx$, $\phi_0' = \pm\sqrt{Q(x)}$, $\phi_0'' = \frac{1}{2}\frac{Q'(x)}{\sqrt{Q(x)}}$

from $\frac{1}{\varepsilon}$ order:

$$\phi_1 = \int\frac{i}{2}\frac{\phi_0''}{\phi_0'} = \frac{i}{2}\int\frac{\phi_0''}{\sqrt{Q(x)}}dx = \frac{i}{2}\int\frac{1}{2}\frac{Q'(x)}{Q(x)}$$

$$\phi_1 = \frac{i}{4}\int\frac{Q'(x)}{Q(x)} \Rightarrow \phi_1 = i/4\ln(Q(x)) + c$$

or

$$\underline{\underline{\phi_1 = i\ln(Q(x)^{1/4}) + c}}$$

constructing $\psi(x)$ (first order $1/\varepsilon$)

$$\psi(x) \approx e^{i/\varepsilon[\phi_0 + \varepsilon\phi_1]} \approx e^{i/\varepsilon\int Q(x)dx}\, e^{-\ln(Q(x))+C}$$

$$\downarrow$$

$$\boxed{\psi(x) \approx \frac{A}{Q(x)^{1/4}}e^{\pm i/\varepsilon\int Q(x)dx}}$$

A = some constant

The wave function above shows that the eikonal equation is just the zeroth order WKB. The solution above is the WKB solution, but it is only the 1st order approximation.

d. validity:

- This is only valid for small ϵ & $Q(x)$ so $\Sigma \epsilon^n \phi_n(x)$ converges.
- Q should be slowly varying and terms in the series should be getting smaller.
- This approximation is not valid for $Q(x)=0$ (turning points)
- $Q(x)>0$ Ψ is an oscillating func. $Q(x)<0$ Ψ has amplitudes going as exponentials & can possibly diverg

The eikonal equation is the zeroth order WKB.

d) $Q(x)$ should vary slowly.

8. $n = n(x,y,z)$

$$\nabla^2\psi + \omega^2 n^2(\vec{x})/c_0^2\,\psi = 0 \qquad k = n(\vec{x})/c_0$$

Use eikonal equation $\psi = Ae^{i\phi}$ Assume A constant

$$\Rightarrow (\nabla\phi)^2 - \omega^2 n(\vec{x})^2/c_0^2 = 0$$

$$\Rightarrow \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = \frac{\omega^2 n^2(\vec{x})}{c_0^2}$$

To solve we need Φ & n^2 seperable so

$$\Phi(\vec{x}) = \Phi_x(x) + \Phi_y(y) + \Phi_z(z)$$

$$n^2(\vec{x}) = n^2(x) + n^2(y) + n^2(z)$$

$$\Rightarrow \left(\frac{\partial\Phi_x}{\partial x}\right)^2 + \left(\frac{\partial\Phi_y}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2}\left[n_x(x)^2 + n_y(y)^2 + n_z(z)^2\right]$$

Each term is a constant, so

$$\left(\frac{\partial\Phi_x}{\partial x}\right)^2 - \frac{\omega^2}{c_0^2}n_x(x)^2 = C_x \Rightarrow \frac{\partial\phi_x}{\partial x} = \sqrt{C_x + \frac{\omega^2}{c_0^2}n_x^2(x)}$$

Same for y & z.

So, $C_x + C_y + C_z = 0$

$$\boxed{\Phi(\vec{x}) = \pm\int\sqrt{C_x + \frac{\omega^2}{c_0^2}n_x^2}\,dx \pm \int\sqrt{C_y + \frac{\omega^2}{c_0^2}n_y^2}\,dy \pm \int\sqrt{C_z + \frac{\omega^2}{c_0^2}n_z^2}\,dz}$$

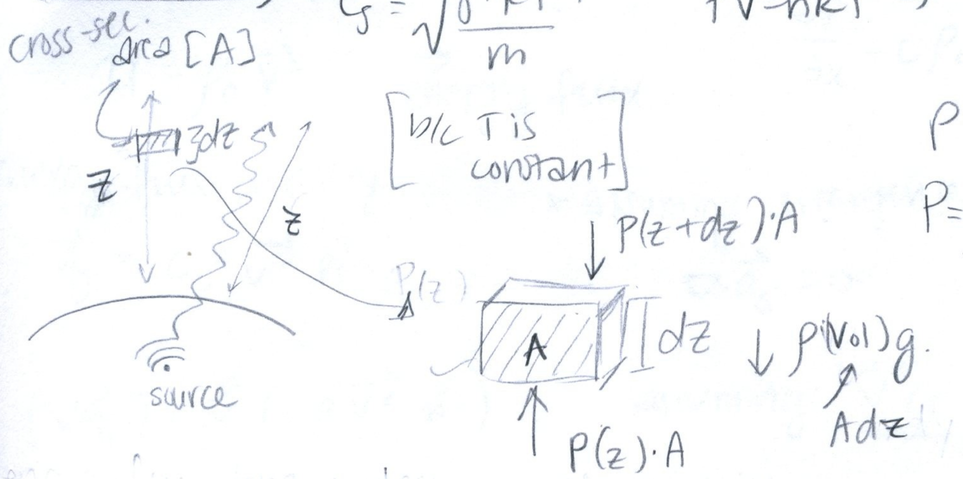
"Pats Quickie"

- isothermal atm (isothermal gas — ideal gas) the velocity of sound is constant $\rightarrow$ $c_s = \sqrt{\frac{\gamma \cdot kT}{m}}$ $\quad PV = nRT \rightarrow P = \frac{n}{V} RT \cdot \left(\frac{M}{M}\right)$

[b/c T is constant]

$P = \rho RT \left(\frac{n}{M}\right) \swarrow \bar{M} \rightarrow$ molecular weight

$P = \frac{\rho RT}{M}$



$$[P(z) - P(z+dz)]\cancel{A} = \rho \cancel{A} dz\, g$$

reduces to

$$\frac{dP}{dz} = -\rho g. \qquad \rho = \frac{MP}{RT}$$

$$\frac{dP}{dz} = -\frac{Mg}{RT} \cdot P$$

$$\int \frac{dP}{P} = -\int \frac{M}{RT} g\, dz \qquad z_0 = 0$$

∴ since T = constant can say $\rho \propto \frac{P}{T}$

$$e^{\ln\left(\frac{P}{P_0}\right)} = e^{\frac{-Mg}{RT} z}$$

$$P(z) = e^{-\frac{Mg}{RT} z}$$

$$\Rightarrow \rho(z) \propto e^{-\frac{Mg}{RT} z} \quad \star$$

ntinued :

- energy flux density along a ray of sound decreases.

mean sound energy density:

$\vec{u} = \rho_0 \vec{v}^2$ $\quad$ $\overrightarrow{\text{energy}}$ flux $\quad$ $\frac{\partial P}{\partial x} = c\rho_0 v_x$

energy flux density $\quad$ * assuming divergence is zero*

$\vec{q}_s = c\rho_0\vec{v}^2\hat{n}$ $\qquad$ $\nabla\cdot\vec{q}_s = 0$

$(\nabla\cdot\vec{q}_s) = \vec{\nabla}\cdot(c\rho\vec{v}^2\hat{n})$ $\quad$ assuming $\underline{\vec{v}^2(r)}$

$= c\vec{\nabla}(\rho\vec{v}^2\hat{n})$ $\quad$ so, $\frac{\partial}{\partial r}(r^2\rho\vec{v}^2)$ must equal zero

$$= \underbrace{c\frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2\rho\vec{v}^2)\right]}_{\nabla\cdot(\vec{v})^2 \text{ in spherical coordinates}} = 0 \Rightarrow \text{so } \rho\vec{v}^2 \propto \frac{1}{r^2} \quad \text{so } \frac{\partial}{\partial r}(\text{const.}) = 0.$$

r - distance from source.

- using equation * $\quad$ $\frac{1}{r} \sim \sqrt{\rho}\cdot v \sim r e^{-\frac{\mu g z}{2RT}}$

So the amplitude of the velocity fluctuations varies along the ray inversely as $r\sqrt{\rho}$ which goes like $r\cdot e^{-\frac{\mu g z}{2RT}}$