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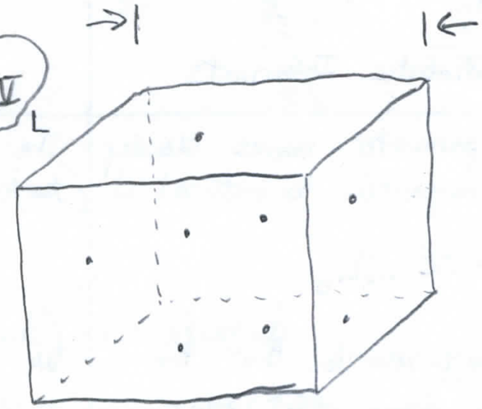
Phys 200A - Problem Set IV

① $V = L^2 L(t)$ $\quad \frac{20.5}{25}$



$\frac{\dot{L}}{L} \ll \frac{\langle v_x \rangle}{L}$ adiabatic regime

For ideal gas

$PV(t) = NKT \quad \text{\&} \quad \frac{1}{2}mv^2 = \frac{3}{2}KT$

$PL^2 L(t) = \frac{N}{3}mv^2$

$I = \oint p\, dq = \int_0^L mv\, dx + \int_L^0 m(-v)\, dx = 2mvL(t) = I$

$\frac{N}{3}mv^2 \cdot A = 2mvL \Rightarrow A = \frac{6mvL}{Nmv^2} = \frac{6L}{Nv}$

$PV \propto \frac{I^2}{L^2}$

$P(VL^2) = \text{const}$

$PV^{5/3} = \text{const}$

In thermodynamics, $PV^\gamma = \text{const}$ for an ideal gas

where γ is the adiabatic index. $\gamma = 5/3$ for monoatomic gas.

↳ i.e. point particles

Method: $\boxed{\text{Adiabatic Invariants}}$

The Approximation Made: $-\frac{\dot{a}}{a} << \Omega_0$ Ω_0 natural frequency of problem
$\dot{a}$ slowly varying parameter

- Or put another way, a changes very little in one period, $1/\Omega$.

Why it works: Because the change in a is small in one cycle of Ω so you can solve the two timescales independently by averaging over the faster timescale.

Key feature:
- $I = \frac{1}{2\pi}\oint p \cdot dq$ = Adiabatic Invariant
  @ fixed λ, E → w/ respect to specific time scale
- $\partial E / \partial I = \Omega$ → Energy changes with I proportional to Ω

The Canonical Example:
- Simple pendulum varying ℓ
- Mass on spring varying k, or m
- Mechanical or Magnetic Mirror

Simple Pendulum "one line summary":

$\dot{\ell}/\ell << \sqrt{\frac{g}{\ell}}$ $I = E/\omega$ → WKB $\tau = \varepsilon t$ (keep Φ_0 & Φ_1) expand in orders ε & average over Ω you get I = constant

$$x(t) = \frac{a_0}{\sqrt{\omega}} e^{i\int \omega(\varepsilon t) dt} \qquad I = \frac{1}{2\pi}\oint p \cdot dq = \frac{1}{2\pi}\oint m\dot{x}\, dx$$

$$I = \frac{1}{2\pi}\oint_0^{\tau = 1/\omega} m\dot{x}^2 dt = \frac{1}{2\pi}\int_0^{2\pi} a_0^2 \frac{\cancel{\omega^2}}{\cancel{\omega}} \sin^2\theta \frac{d\theta}{\cancel{\omega}} = \frac{a_0^2}{2} \rightarrow \text{constant}$$

cont.

Method: $\boxed{\text{Ponderomotive Force}}$

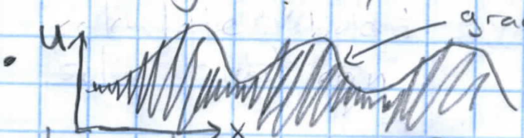
The approximation: $\frac{\dot{a}}{a} >> \Omega_0$ & amplitude of quiver is small

Why it works: Like adiabatic key is ability to separate time scales.

Key Features: • Average over forces expanded to 1st order in ε & all fast pieces average to 0. Ponderomotive force is the beat between $\varepsilon \frac{df}{dy}$ (quiver; f ← force; dy ← slow vary piece)

• $m\ddot{y} = -\frac{dU}{dy} + \underbrace{\left\langle \varepsilon \frac{\partial f}{\partial y} \right\rangle}_{\text{ponderomotive force}}$

• Finding stability condition → minimum force required

• 

gradient of this envelope = ponderomotive force

Canonical Example: Inverted Pendulum

Summary: • Set up L & L.E.O.M & Expand $\ddot{\phi}$ in terms of $(\ddot{\phi}_0 + \ddot{\varepsilon})$

• Solve fast eqn: $m\ddot{\varepsilon} = f(y)$ for ε

• Calculate beat with ε

• Use beat to calculate slow eqn

• Solve $\frac{dU}{d\phi} = 0$ to find extrema & $\frac{d^2U}{d\phi^2}$ for stability

(2) cont.

Method: $\boxed{\text{Parametric Instability}}$

Approximation: • $\ddot{x} + \omega_0^2(1+\lambda\cos(\gamma t))x = 0$

$\gamma \sim 2\omega_0 \rightarrow$ parametric resonance

$\lambda \longrightarrow$ dictates how close to $2\omega_0$ you must be.

Why it works: Your oscillating parameter beats with the natural frequency & when $\gamma \sim 2\omega_0$ this adds energy @ ω_0 creating exponential growth in oscillation amplitude of ω_0 as t increases

Key Features: • $\gamma = 2\omega_0 + \varepsilon$ instability region $|\varepsilon| < \frac{\omega_0 h}{2}$

• For $\gamma = \frac{2\omega_0}{n} + \varepsilon$ instability region $|\varepsilon| < \frac{\omega_0 h^n}{2}$

• with friction α then: $\frac{2\sqrt{\varepsilon^2 + 4\alpha^2}}{\omega_0} < h$

• Requirements on h for instability otherwise stable oscillation

Canonical Example: Pumping on a swing

Summary: Modulating moment of inertia

$I(t) = I_0 + \varepsilon I_1(t)$ $\quad \ddot{\theta} + \frac{g}{\ell}(1+\lambda\cos(\gamma t))\theta + \alpha\dot{\theta} =$

Take solution of the form:

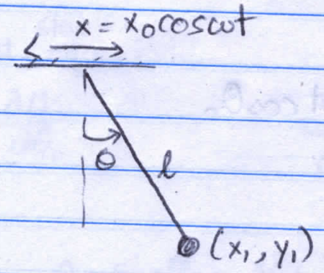
$$\theta = a(t)\sin((\omega_0 + \tfrac{\varepsilon}{2})t) + b(t)\cos((\omega_0 + \tfrac{\varepsilon}{2})t)$$

Throw away terms h.o. in ε & λ & calculate the beat terms.

Collect terms $f_1(\dot{a}, \dot{b}, a, b)\cos[\,] + f_2(\dot{a}, \dot{b}, a, b)\sin[\,] = 0$

find solution for a & b from f_1 & f_2 in the form e^{st}, & e^{-st} & condition for instability is positive eigenvalues.

3(a)



$$x_1 = x_0\cos\omega t + l\sin\theta \qquad \dot{x}_1 = -x_0\omega\sin\omega t + l\cos\theta\dot{\theta}$$

$$y_1 = -l\cos\theta \qquad \dot{y}_1 = l\sin\theta\dot{\theta}$$

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) - mgy_1$$

$$= \frac{1}{2}m\left(x_0^2\omega^2\sin^2\omega t - 2lx_0\omega\cos\theta\sin\omega t\,\dot{\theta} + l^2\dot{\theta}^2\right) + mgl\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = -mlx_0\omega\cos\theta\sin\omega t \;\cancel{mgl\sin\theta} + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = -ml\omega^2x_0\cos\theta\cos\omega t + mlx_0\omega\sin\theta\sin\omega t\,\dot{\theta} + ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = +mlx_0\omega\sin\theta\sin\omega t\,\dot{\theta} - mgl\sin\theta$$

The EOM is

$$-mlx_0\omega^2\cos\theta\cos\omega t + ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{x_0\omega^2}{l}\cos\theta\cos\omega t$$

$$\theta(t) = \underbrace{\theta_0(t)}_{\text{slow}} + \underbrace{\epsilon(t)}_{\text{fast}}, \quad \epsilon \ll 1$$

$$\ddot{\theta}_0 + \ddot{\epsilon} = -\frac{g}{l}(\sin\theta_0 + \epsilon\cos\theta_0) + \frac{x_0\omega^2}{l}\cos\omega t(\cos\theta_0 - \epsilon\sin\theta_0)$$

Taking time average over time $\approx \frac{2\pi}{\omega}$,

$$\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 - \frac{x_0\omega^2}{l}\sin\theta_0\langle\cos\omega t\;\epsilon\rangle \longrightarrow (1)$$

This leaves the fast equation:

$$\ddot{\epsilon} = -\frac{g}{l}\epsilon\cos\theta_0 + \frac{x_0\omega^2}{l}\cos\omega t\cos\theta_0$$

$$\omega \gg \sqrt{g/l}\,,$$

$$\Rightarrow \ddot{\epsilon} \approx \frac{x_0\omega^2}{l}\cos\omega t\cos\theta_0$$

$$\Rightarrow \epsilon(t) = -\frac{x_0\omega^2}{l\omega^2}\cos\omega t\cos\theta_0$$

$$\boxed{\epsilon(t) = -\frac{x_0}{l}\cos\omega t\cos\theta_0}$$

Plug this into ①,

$$\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 - \frac{x_0\omega^2}{l}\sin\theta_0\left\langle\cos\omega t\left(\frac{-x_0}{l}\right)\cos\omega t\cos\theta_0\right\rangle$$

$$\Rightarrow \ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 + \frac{x_0^2\omega^2}{2l^2}\sin\theta_0\cos\theta_0$$

$$\Rightarrow -\frac{\partial U_{eff}}{\partial\theta_0} = -\frac{g}{l}\sin\theta_0 + \frac{x_0^2\omega^2}{4l^2}\sin 2\theta_0$$

$$\frac{\partial U_{eff}}{\partial\theta_0} = \frac{g}{l}\sin\theta_0 - \frac{x_0^2\omega^2}{4l^2}\sin 2\theta_0$$

$$\Rightarrow \boxed{U_{eff}(\theta_0) = U_0 - \frac{g}{l}\cos\theta_0 + \frac{x_0^2\omega^2}{8l^2}\cos 2\theta_0}$$

At stable equilibrium,

$$\frac{\partial U}{\partial \theta_0} = 0 \quad , \quad \frac{\partial^2 U}{\partial \theta_0^2} > 0$$

$$\frac{g}{l} \sin\theta_0 - \frac{x_0^2 \omega^2}{4l^2} \sin 2\theta_0 = 0$$

$$\sin\theta_0 \left(\frac{g}{l} - \frac{x_0^2\omega^2}{2l^2} \cos\theta_0 \right) = 0$$

$$\Rightarrow \theta_0 = 0, \pi, \cos^{-1}\left(\frac{2gl}{x_0^2\omega^2}\right) \text{ are fixed points}$$

$$\left.\frac{\partial^2 U}{\partial \theta_0^2}\right| = \frac{g}{l}\cos\theta_0 - \frac{x_0^2\omega^2}{2l^2}\cos 2\theta_0$$

$$\left.\frac{\partial^2 U}{\partial \theta_0^2}\right|_{\theta_0 = 0} = \frac{g}{l} - \frac{x_0^2\omega^2}{2l^2} > 0 \text{ if } 2gl > x_0^2\omega^2$$

$$\left.\frac{\partial^2 U}{\partial \theta_0^2}\right|_{\theta_0 = \pi} = -\frac{g}{l} - \frac{x_0^2\omega^2}{2l^2} \not> 0$$

$$\left.\frac{\partial^2 U}{\partial \theta_0^2}\right|_{\theta_0 = \cos^{-1}\left(\frac{2gl}{x_0^2\omega^2}\right)} = \frac{g}{l}\left(\frac{2gl}{x_0^2\omega^2}\right) - \frac{x_0^2\omega^2}{2l^2}\left(\frac{2\cdot 4g^2l^2}{x_0^4\omega^4} - 1\right)$$

$$= \frac{2g^2}{x_0^2\omega^2} - \frac{4g^2}{x_0^2\omega^2} + \frac{x_0^2\omega^2}{2l^2}$$

$$= \frac{-2g^2}{x_0^2\omega^2} + \frac{x_0^2\omega^2}{2l^2} > 0 \text{ if } x_0^2\omega^2 > 2gl$$

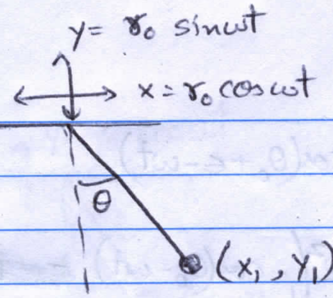
So,

$\theta_0 = \pi$ is never stable.

$\theta_0 = 0$ is stable if $x_0^2\omega^2 < 2gl$

$\theta_0 = \cos^{-1}\left(\frac{2gl}{x_0^2\omega^2}\right)$ is stable if $x_0^2\omega^2 > 2gl$.

3(b)



$$x_1 = r_0 \cos\omega t + l\sin\theta \qquad \dot{x}_1 = -r_0\omega\sin\omega t + l\cos\theta\,\dot{\theta}$$

$$y_1 = r_0 \sin\omega t - l\cos\theta \qquad \dot{y}_1 = r_0\omega\cos\omega t + l\sin\theta\,\dot{\theta}$$

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) - mgy_1$$

$$= \frac{1}{2}m\begin{pmatrix} r_0^2\omega^2\sin^2\omega t - 2lr_0\omega\dot{\theta}\sin\omega t\cos\theta + l^2\cos^2\theta\,\dot{\theta}^2 \\ + r_0^2\omega^2\cos^2\omega t + 2lr_0\omega\dot{\theta}\cos\omega t\sin\theta + l^2\sin^2\theta\,\dot{\theta}^2 \end{pmatrix} \begin{matrix} - mgr_0\sin\omega t \\ + mgl\cos\theta \end{matrix}$$

$$L = \frac{1}{2}m\left(r_0^2\omega^2 + 2lr_0\omega\dot{\theta}\sin(\theta-\omega t) + l^2\dot{\theta}^2\right) - mgr_0\sin\omega t + mgl\cos\theta$$

$$L_{eff} = mlr_0\omega\dot{\theta}\sin(\theta-\omega t) + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

~~The~~ $$\frac{\partial L_{eff}}{\partial\dot{\theta}} = mlr_0\omega\sin(\theta-\omega t) + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial\dot{\theta}}\right) = mlr_0\omega\cos(\theta-\omega t)\left[\dot{\theta}-\omega\right] + ml^2\ddot{\theta}$$

$$\frac{\partial L_{eff}}{\partial\theta} = mlr_0\omega\dot{\theta}\cos(\theta-\omega t) - mgl\sin\theta$$

The equation of motion is:

$$-mlr_0\omega^2\cos(\theta-\omega t) + ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{r_0}{l}\omega^2\cos(\theta-\omega t)$$

$$\theta = \underbrace{\theta_0}_{\text{slow}} + \underbrace{\epsilon}_{\text{fast}}, \quad \epsilon << 1$$

$$\ddot{\theta}_0 + \ddot{\epsilon} = -\frac{g}{l}\sin(\theta_0+\epsilon) + \frac{r_0}{l}\omega^2\cos(\theta_0+\epsilon-\omega t)$$

$$\ddot{\theta}_0 + \ddot{\epsilon} = -\frac{g}{l}\left(\sin\theta_0 + \epsilon\cos\theta_0\right) + \frac{r_0\omega^2}{l}\left(\cos(\theta_0-\omega t) - \epsilon\sin(\theta_0-\omega t)\right)$$

$$\ddot{\theta}_0 + \ddot{\epsilon} = -\frac{g}{l}\left(\sin\theta_0 + \epsilon\cos\theta_0\right) + \frac{r_0\omega^2}{l}\left[\begin{array}{l}\cos\theta_0\cos\omega t + \sin\theta_0\sin\omega t \\ -\epsilon\sin\theta_0\cos\omega t + \epsilon\cos\theta_0\sin\omega t\end{array}\right]$$

Taking average over time $\sim \frac{2\pi}{\omega}$

$$\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 - \frac{r_0\omega^2}{l}\sin\theta_0\langle\epsilon\cos\omega t\rangle + \frac{r_0\omega^2\cos\theta_0}{l}\langle\epsilon\sin\omega t\rangle \quad \to (1)$$

$$\ddot{\epsilon} = -\frac{g}{l}\epsilon\cos\theta_0 + \frac{r_0\omega^2}{l}\cos(\theta_0-\omega t)$$

$$\omega \gg \sqrt{\frac{g}{l}}$$

$$\Rightarrow \quad \ddot{\epsilon} = \frac{r_0\omega^2}{l}\cos(\theta_0-\omega t)$$

$$\Rightarrow \quad \boxed{\epsilon = -\frac{r_0}{l}\cos(\theta_0-\omega t)}$$

Plug this into (1):

$$\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 - \frac{r_0\omega^2}{l}\langle\epsilon\sin(\theta_0-\omega t)\rangle$$

$$\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0 + \frac{r_0^2\omega^2}{2l^2}\langle\sin(2\theta_0-2\omega t)\rangle$$

$$\boxed{\ddot{\theta}_0 = -\frac{g}{l}\sin\theta_0}$$

So, the slow piece is the same as the original problem. The only stable equilibrium is $\theta_0 = 0$.

(4) $y = -l\cos\theta + y_0 \cos\omega t \qquad x = l\sin\theta$

$\dot{y} = l\dot{\theta}\sin\theta - y_0\omega\sin\omega t \qquad \dot{x} = l\dot{\theta}\cos\theta$

$L = \frac{m}{2}\left(l^2\dot{\theta}^2 + y_0^2\omega^2\sin^2\omega t - 2y_0 l\omega\dot{\theta}\sin\theta\sin\omega t\right) - mg\left(y_0\cos\omega t - l\cos\theta\right)$

E.O.M. $\frac{d}{dt}\left(ml^2\dot{\theta} - my_0 l\omega\sin\theta\sin\omega t\right) = -my_0 l\omega\dot{\theta}\cos\theta\sin\omega t - mgl\sin\theta$

$\cancel{m}l^{\cancel{2}}\ddot{\theta} - \cancel{my_0 l\omega\dot{\theta}\cos\theta\sin\omega t} - \cancel{m}y_0\cancel{l}\omega^2\sin\theta\cos\omega t = -\cancel{my_0 l\omega\dot{\theta}\cos\theta\sin\omega t} - \cancel{m}g\cancel{l}\sin\theta$

$\ddot{\theta} + \frac{g}{l}\sin\theta\left(1 - y_0\omega^2\cos\omega t\right) = 0$

take $\sin\theta \sim \theta$ & $\omega_0^2 = \frac{g}{l}$

$\ddot{\theta} + \omega_0^2\left(1 - \underbrace{y_0\omega^2}_{h}\cos\omega t\right)\theta = 0 \qquad y_0\omega^2 = h \ll 1$

take soln $\theta = a(t)\sin\left(\left(\omega_0 + \frac{\varepsilon}{2}\right)t\right) + b(t)\cos\left(\left(\omega_0 + \frac{\varepsilon}{2}\right)t\right)$

$a(t)$ & $b(t)$ are slow bc $h \ll 1$

$\dot{\theta} = \dot{a}\sin[\ ] + \dot{b}\cos[\ ] + a\left(\omega_0 + \frac{\varepsilon}{2}\right)\cos[\ ] - b\left(\omega_0 + \frac{\varepsilon}{2}\right)\sin[\ ]$

$\ddot{\theta} = \underbrace{\cancel{\ddot{a}}}_{0\ (\text{slow})}\sin[\ ] + \underbrace{\cancel{\ddot{b}}}_{0\ (\text{slow})}\cos[\ ] + 2\dot{a}\left(\omega_0 + \cancel{\tfrac{\varepsilon}{2}}^{\,\varepsilon \ll \omega_0 = 0}\right)\cos[\ ] - 2\dot{b}\left(\omega_0 + \cancel{\tfrac{\varepsilon}{2}}^{\,\varepsilon \ll \omega_0 = 0}\right)\sin[\ ]$

$- \underbrace{a\left(\omega_0 + \frac{\varepsilon}{2}\right)^2}_{O(\varepsilon^2) \to 0}\sin[\ ] - \underbrace{b\left(\omega_0 + \frac{\varepsilon}{2}\right)^2}_{O(\varepsilon^2) \to 0}\cos[\ ]$

$\left(2\dot{a}\omega_0 - b\left(\cancel{\omega_0^2} + \omega_0\varepsilon\right)\right)\cos[\ ] + \left(-2\dot{b}\omega_0 - a\left(\cancel{\omega_0^2} + \omega_0\varepsilon\right)\right)\sin[\ ]$

$+ \left(\cancel{\omega_0^2} - h\cos\omega t\right)\left(a\sin[\ ] + b\cos[\ ]\right) = 0$

$\cos\left((2\omega_0 + \varepsilon)t\right)\sin\left(\left(\omega_0 + \frac{\varepsilon}{2}\right)t\right) = \frac{1}{2}\left[\underbrace{\sin\left(\left(3\omega_0 + \frac{3\varepsilon}{2}\right)t\right)}_{0 \text{ bc h.o. in } h} - \sin\left(\omega_0 + \frac{\varepsilon}{2}\right)\right]$

$\cos\left((2\omega_0 + \varepsilon)t\right)\cos\left(\left(\omega_0 + \frac{\varepsilon}{2}\right)t\right) = \frac{1}{2}\left[\cos\left(3\omega_0 + \frac{3\varepsilon}{2}t\right) + \cos\left(\left(\omega_0 + \frac{\varepsilon}{2}\right)t\right)\right]$

$2\dot{a}\omega_0 - b\left(\omega_0\varepsilon + \frac{1}{2}h\right) = 0 \qquad -2\dot{b}\omega_0 + a\left(-\omega_0\varepsilon + \frac{1}{2}h\right) = 0$

$\dot{a} = \left(\frac{\varepsilon}{2} + \frac{h}{4\omega_0}\right)b \qquad \dot{b} = \left(-\frac{\varepsilon}{2} + \frac{h}{4\omega_0}\right)a$

$$\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = \begin{pmatrix}0 & \frac{\varepsilon}{2} + \frac{h}{4\omega_0}\\ -\frac{\varepsilon}{2} + \frac{h}{4\omega_0} & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}$$

$$\det \begin{vmatrix} -\lambda & \frac{\varepsilon}{2} + \frac{h}{4\omega_0} \\ \frac{\varepsilon}{2} + \frac{h}{4\omega_0} & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 - \left(-\frac{\varepsilon^2}{4} + \frac{h^2}{4\omega_0^2} \right) = 0$$

$$\lambda = \pm \sqrt{\frac{h^2}{4\omega_0^2} - \frac{\varepsilon^2}{4}}$$

$$\theta(t) = e^{\lambda t} \sin[\,] + e^{-\lambda t} \cos[\,]$$

If λ real then you have instability

$$\boxed{\frac{h^2}{4\omega_0^2} > \frac{\varepsilon^2}{4}} \rightarrow \frac{y_0^2 (2\omega_0 + \overset{0 \;\; \varepsilon \ll 2\omega_0}{\varepsilon})^2}{4\omega_0^2} = \frac{y_0^2 \cancel{4\omega_0^2}}{\cancel{4\omega_0^2}} > \frac{\varepsilon^2}{4}$$

$$\boxed{y_0^2 > \frac{\varepsilon^2}{4}} \qquad \boxed{-\frac{\varepsilon}{2} < y_0 < \frac{\varepsilon}{2}}$$

(5) $\ddot{\theta} + \alpha\dot{\theta} + \omega_0^2(1 + h\cos(\omega t))\theta = 0$ 10

$$\omega = 2\omega_0 + \varepsilon \qquad \theta = a(t)\sin((\omega_0 + \tfrac{\varepsilon}{2})t) + b(t)\cos((\omega_0 + \tfrac{\varepsilon}{2})t)$$

$$2\dot{a}\omega_0\cos[\;] - 2\dot{b}\omega_0\sin[\;] - a(\omega_0^2 + \varepsilon\omega_0)\sin[\;]$$
$$- b(\omega_0^2 + \varepsilon\omega_0)\cos[\;] + \alpha\dot{a}\sin[\;] + \alpha\dot{b}\cos[\;] + \alpha a\omega_0\cos[\;]$$
$$- \alpha b\omega_0\sin[\;] + \omega_0^2(1 + h\cos(\omega t))(a\underbrace{\sin[\;]}_{-\frac{1}{2}\sin[\;]} + b\underbrace{\cos[\;]}_{\frac{1}{2}\cos[\;]})$$

$$(2\dot{a}\omega_0 - b(\cancel{\omega_0^2} + \varepsilon\omega_0) + \alpha\dot{b} + \alpha a\omega_0 + b\omega_0^2(\cancel{1} + \tfrac{h}{2}))\cos[\;] +$$
$$(-2\dot{b}\omega_0 - a(\cancel{\omega_0^2} + \varepsilon\omega_0) + \alpha\dot{a} - \alpha b\omega_0 + a\omega_0^2(\cancel{1} - \tfrac{h}{2}))\sin[\;] = 0$$

$$\underbrace{\begin{pmatrix} 2\omega_0 & \alpha \\ \alpha & -2\omega_0 \end{pmatrix}}_{A}\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \underbrace{\begin{pmatrix} -(\alpha\omega_0) & -\omega_0(\frac{h\omega_0}{2} - \varepsilon) \\ \omega_0(\frac{h\omega_0}{2} + \varepsilon) & \alpha\omega_0 \end{pmatrix}}_{B}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$A^{-1}B = \frac{-1}{\alpha^2 - (2\omega_0)^2}\begin{pmatrix} 2\alpha\omega_0^2 - \alpha\omega_0(\frac{h\omega_0}{2} + \varepsilon) & -\alpha^2\omega_0 + 2\omega_0^2(\frac{h\omega_0}{2} - \varepsilon) \\ \alpha^2\omega_0^2 + 2\omega_0^2(\frac{h\omega_0}{2} + \varepsilon) & 2\alpha\omega_0^2 + \alpha\omega_0(\frac{h\omega_0}{2} - \varepsilon) \end{pmatrix}$$

$$\lambda_\pm = \frac{2\alpha\omega_0(\varepsilon - \omega_0) \pm \omega_0\sqrt{-4\alpha^4 + \alpha^2(h\omega_0)^2 - 16\alpha^2\varepsilon\omega_0 - 16\varepsilon^2\omega_0^2 + 4h^2\omega_0^4}}{2(\alpha^2 + (2\omega_0)^2)}$$

$$2\alpha\cancel{\omega_0}(\varepsilon - \omega_0) < \cancel{\omega_0} \pm \sqrt{\alpha^2((h\omega_0)^2 - 4\alpha^2 - 16\varepsilon\omega_0) + 4\omega_0^2((h\omega_0)^2 - 4\varepsilon^2)}$$

$$\frac{1}{\omega_0^2}\left[(2\alpha)^2(\omega_0^2 - 2\varepsilon\omega_0) < \alpha^2((h\omega_0)^2 - (2\alpha)^2 - 16\varepsilon\omega_0) + 4\omega_0^2((h\omega_0)^2 + 4\varepsilon^2)\right]$$

$$(2\alpha)^2(1 - \cancelto{0}{\tfrac{2\varepsilon}{\omega_0}}) < \alpha^2(h^2 - \cancelto{0}{\tfrac{4\alpha^2}{\omega_0^2}} - \cancelto{0}{\tfrac{16\varepsilon}{\omega_0}}) + 4((h\omega_0)^2 - \varepsilon^2)$$

$$(2 - h^2)\alpha^2 - 4(h\omega_0)^2 < -4\varepsilon^2 \qquad (\cancelto{0}{\tfrac{h^2}{4}} - \tfrac{1}{2})\alpha^2 + (h\omega_0)^2 > \varepsilon^2$$

$$\pm\left[(h\omega_0)^2 - \frac{\alpha^2}{2}\right]^{1/2} > \varepsilon^2 \qquad \boxed{-\left[(h\omega_0)^2 - \frac{\alpha}{2}\right]^{1/2} < \varepsilon < \left[(h\omega_0)^2 - \frac{\alpha}{2}\right]^{1/2}}$$

$$h\omega_0^2 > \frac{\alpha}{2} \qquad \boxed{h > \frac{\alpha}{2\omega_0^2}}$$

$(q, p, t) = H_0(p, q) + V(q)\frac{d^2A(t)}{dt^2}$ where $A(t)$ is periodic w/ period $\tau \ll T$ $\to A$ is slow $\therefore \frac{d^2A}{dt^2}$ is slow

H_0 has period T.

and $H_0 = \frac{p^2}{2m} + V_0(q)$

a) We start with HEOMs:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The only term that depends on p is the H_0 term, so

$$\dot{q} = \frac{\partial H_0}{\partial p} = \frac{p}{m}$$

Both terms in H depend on q, so

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H_0}{\partial q} - \frac{\partial V}{\partial q}\cdot\frac{d^2A}{dt^2} = \frac{-\partial V_0}{\partial q} - \frac{\partial V}{\partial q}\cdot\frac{d^2A}{dt^2}$$

Now, $$\ddot{q} = \frac{\dot{p}}{m} = \frac{1}{m}\left[-\frac{\partial V_0}{\partial q} - \frac{\partial V}{\partial q}\cdot\frac{d^2A}{dt^2}\right] \quad (*)$$

we can define

$q = y + \varepsilon$ (similar to what was done in class).

where y is slow centroid motion and ε is fast quiver

$\sim$ just means it's fast.

(*) becomes

$$\ddot{y} + \ddot{\varepsilon} = \frac{1}{m}\left[-\frac{\partial V_0}{\partial y} - \tilde{\varepsilon}\frac{\partial^2 V_0}{\partial y^2} - \frac{\partial V}{\partial y}\frac{d^2A}{dt^2} - \tilde{\varepsilon}\frac{\partial^2 V}{\partial y^2}\frac{d^2A}{dt^2}\right]$$

Now we time-average this

$$\langle\ddot{y}\rangle + \cancel{\langle\ddot{\varepsilon}\rangle} = \frac{1}{m}\left[-\langle\partial V_0/\partial y\rangle - \cancel{\langle\tilde{\varepsilon}\,\partial^2 V_0/\partial y^2\rangle} - \cancel{\langle\partial V/\partial y\cdot d^2A/dt^2\rangle} - \langle\tilde{\varepsilon}\,\partial^2 V/\partial y^2\cdot d^2A/dt^2\rangle\right]$$

fast, 0 b/c periodic; slow; fast · slow $\to 0$; periodic $\to 0$; fast, fast?, fast; slow

This gives us

$$\ddot{y} = -\frac{1}{m}\left[\frac{\partial V_0}{\partial y} + \frac{\partial^2 V}{\partial y^2}\left\langle\tilde{\varepsilon}\frac{d^2A}{dt^2}\right\rangle\right]$$

which leaves us with

$$\ddot{\varepsilon} = -\frac{1}{m}\left[\cancel{\varepsilon\frac{\partial^2 V_0}{\partial y^2}} + \frac{\partial V}{\partial y}\left\langle\frac{d^2A}{dt^2}\right\rangle\right]$$

$\to 0$ b/c it oscillates slower

let's say

$\varepsilon = \sin(kx - \omega t)$

$\ddot{\varepsilon} = -\omega^2\sin(kx - \omega t)$

$= -\omega^2\varepsilon$

$$+\omega^2\varepsilon = +\frac{1}{m}\frac{\partial V}{\partial y}\left\langle\frac{d^2A}{dt^2}\right\rangle$$

$A = A_0\cos(\omega t)$

$\ddot{A} = -\omega^2 A$

$$\varepsilon = \frac{1}{m\omega^2}\frac{\partial V}{\partial y}\left\langle\frac{d^2A}{dt^2}\right\rangle$$

continued

Now, we plug our ε into the expression for $\ddot{y}$

$$\ddot{y} = \frac{-1}{m}\left[\frac{\partial V_0}{\partial y} + \frac{\partial^2 V}{\partial y^2}\left\langle \frac{1}{m\omega^2}\frac{\partial V}{\partial y}\cdot \overset{\to \omega^2}{\frac{d^2A}{dt^2}}\cdot\frac{d^2A}{dt^2}\right\rangle\right]$$

$$= \frac{-1}{m}\left[\frac{\partial V_0}{\partial y} + \frac{1}{m}\frac{\partial V}{\partial y}\frac{\partial^2 V}{\partial y^2}\left\langle \frac{d^2A}{dt^2}\right\rangle\right].$$

$$\ddot{y} = \frac{-1}{m}\frac{\partial V_0}{\partial y} - \frac{1}{m^2}\frac{\partial V}{\partial y}\frac{\partial^2 V}{\partial y^2}\left\langle \frac{d^2A}{dt^2}\right\rangle$$

b) $K(p,q) = H_0(p,q) + \frac{1}{4m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\left(\frac{\partial V(q)}{\partial q}\right)^2$

$$\dot{q} = \frac{\partial K}{\partial p} = \frac{\partial H_0}{\partial p} = p/m$$

$$\dot{p} = -\frac{\partial K}{\partial q} = -\frac{\partial H_0}{\partial q} - \frac{1}{4m}\left(\frac{\partial V}{\partial q}\right)^2\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle 2\frac{\partial V}{\partial q}.$$

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{1}{m}\frac{\partial H_0}{\partial q} - \frac{1}{2m^2}\frac{\partial V}{\partial q}\left(\frac{\partial V}{\partial q}\right)^2\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle \quad (*)$$

We know that

$$H_0 = \frac{p^2}{2m} + V_0(q)$$

(*) becomes

$$\ddot{q} = -\frac{1}{m}\frac{\partial V_0}{\partial q} - \frac{1}{2m^2}\frac{\partial V}{\partial q}\left(\frac{\partial V}{\partial q}\right)^2\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle$$

→ This is almost exactly like the (*) equation above, except for the $\frac{1}{2}$ factor.

#7 $\vec{L} = \sum_i \vec{x}_i \times \vec{P}_i$

$$= \begin{vmatrix} i & j & k \\ x_i & x_j & x_k \\ P_i & P_j & P_k \end{vmatrix} = i(x_j P_k - x_k P_j) + j(x_k P_i - x_i P_k) + k(x_i P_j - x_j P_i)$$

$$[F, G]_{PB} = \sum_\sigma \left[\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial P_\sigma} - \frac{\partial F}{\partial P_\sigma} \frac{\partial G}{\partial q_\sigma} \right] \quad ; \quad L_i = q_j P_k - q_k P_j$$

$$[L_i, L_j]_{PB} = \sum_{\substack{\sigma \\ = i,j,k}} \left[\frac{\partial L_i}{\partial q_\sigma} \frac{\partial L_j}{\partial P_\sigma} - \frac{\partial L_i}{\partial P_\sigma} \frac{\partial L_j}{\partial q_\sigma} \right]$$

$$= 0 + 0 + [-P_j \cdot -q_i - q_j \cdot P_i] = q_i P_j - q_j P_i = \underline{L_k} \checkmark$$

$[L^2, L_i]_{PB}$ (assume all $[\,]$ are $[\,]_{PB}$).

$$L^2 = L_x^2 + L_y^2 + L_z^2 \longrightarrow [L^2, L_i] = [L_x^2, L_i] + [L_y^2, L_i] + [L_z^2, L_i]$$

$$= [L_x, L_i]L_x + L_x[L_x, L_i] + [L_y, L_i]L_y + L_y[L_y, L_i] + [L_z, L_i]L_z + L_z[L_z, L_i]$$

if $i \equiv x \longrightarrow [L^2, L_x] = 0 + 0 + -L_z L_y - L_y L_z + L_y L_z + L_z L_y = 0$

if $i \equiv y \longrightarrow [L^2, L_y] = L_z L_x + L_x L_z + 0 + 0 - L_x L_z - L_z L_x = 0$

if $i \equiv z \longrightarrow [L^2, L_z] = -L_y L_x - L_x L_y + L_x L_y + L_y L_x + 0 + 0 = 0$

$\therefore [L^2, L_i]_{PB} = 0 \checkmark$

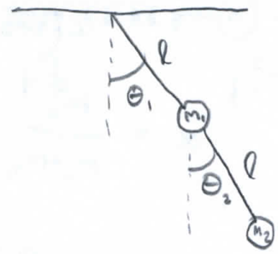
No, cannot write L_x, L_y, L_z as canonical momenta because $[L_i, L_j] \neq 0$.

$x_1 = l\sin\theta_1$ $\qquad$ $\dot{x}_1 = l\dot{\theta}_1\cos\theta_1$

$y_1 = -l\cos\theta_1$ $\qquad$ $\dot{y}_1 = l\dot{\theta}_1\sin\theta_1$

$x_2 = l\sin\theta_1 + l\sin\theta_2$ $\qquad$ $\dot{x}_2 = l\dot{\theta}_1\cos\theta_1 + l\dot{\theta}_2\cos\theta_2$

$y_2 = -l\cos\theta_1 - l\cos\theta_2$ $\qquad$ $\dot{y}_2 = l\dot{\theta}_1\sin\theta_1 + l\dot{\theta}_2\sin\theta_2$



$$L = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y_1 - m_2 g y_2$$

$$= \frac{1}{2}m_1 l^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + l^2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right]$$

$$+ m_1 g l\cos\theta_1 + m_2 g l\left(\cos\theta_1 + \cos\theta_2\right)$$

$$= m_1 l^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l^2\left(\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right) + gl\left[m_1\cos\theta_1 + m_2\cos\theta_1 + m_2\cos\theta_2\right]$$

Now make small angle approximation and let $\eta_i = l\theta_i$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$L = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m_2\left(\dot{\eta}_1 + \dot{\eta}_2\right)^2 - \frac{g}{2l}\left[\left(m_1 + m_2\right)\eta_1^2 + m_2\eta_2^2\right]$$

LEOM :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_1} = \frac{\partial L}{\partial \eta_1} \Rightarrow m_1\ddot{\eta}_1 + m_2\left(\ddot{\eta}_1 + \ddot{\eta}_2\right) = -\frac{g}{l}\left(m_1 + m_2\right)\eta_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_2} = \frac{\partial L}{\partial \eta_2} \Rightarrow m_2\left(\ddot{\eta}_1 + \ddot{\eta}_2\right) = -\frac{g}{l}m_2\eta_2$$

To find frequency of small osc. let $\ddot{\eta}_i = -\omega^2\eta_i$, Then,

$$\omega^2\eta_1 + \frac{m_2}{m_1 + m_2}\omega^2\eta_2 = \frac{g}{l}\eta_1$$

$$\omega^2\eta_1 + \omega^2\eta_2 = \frac{g}{l}\eta_2$$

Let $\alpha \equiv \frac{m_2}{m_1 + m_2}$, then $\begin{pmatrix} \omega^2 - \frac{g}{l} & \alpha\omega^2 \\ \omega^2 & \omega^2 - \frac{g}{l} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\omega^4 - \frac{2g}{l}\omega^2 + \frac{g^2}{l^2} - \alpha\omega^4 = (1 - \alpha)\omega^4 - \frac{2g}{l}\omega^2 + \frac{g^2}{l^2} = 0$$

$$\omega^2 = \frac{2g/\ell \pm \sqrt{\frac{4g^2}{\ell^2} - \frac{4g^2}{\ell^2}(1-\alpha)}}{2(1-\alpha)} = \frac{g}{\ell}\left[\frac{1 \pm \sqrt{\alpha}}{1-\alpha}\right]$$

$$\alpha = \frac{m_2}{m_1+m_2} \qquad \gamma = \sqrt{\frac{m_2}{m_2(m_1+m_2)}} \Rightarrow \gamma = \sqrt{\alpha}$$

$$\omega^2 = \frac{g}{\ell}\left[\frac{1 \pm \gamma}{1-\gamma^2}\right] = \frac{g}{\ell}\left[\frac{1 \pm \gamma}{(1+\gamma)(1-\gamma)}\right] = \boxed{\frac{g}{\ell}(1 \pm \gamma)^{-1} = \omega^2}$$

c. First find η_+:

$$\left(\frac{g}{\ell}(1+\gamma)^{-1} - \frac{g}{\ell}\right)\eta_1 + \left(\gamma^2 \frac{g}{\ell}(1+\gamma)^{-1}\right)\eta_2 = 0$$

$$(1 - (1+\gamma))\eta_1 + \gamma^2 \eta_2 = 0$$

$$\eta_1 = \gamma \eta_2 \Rightarrow \eta_+ = \frac{1}{\sqrt{1+\gamma^2}}\begin{pmatrix} \gamma \\ 1 \end{pmatrix}$$] Symmetric Mode

Now find η_-:

$$\left(\frac{g}{\ell}(1-\gamma)^{-1} - \frac{g}{\ell}\right)\eta_1 + \left(\gamma^2 \frac{g}{\ell}(1-\gamma)^{-1}\right)\eta_2 = 0$$

$$(1 - (1-\gamma))\eta_1 + \gamma^2 \eta_2 = 0$$

$$\eta_1 = -\gamma \eta_2 \Rightarrow \eta_1 = \frac{1}{\sqrt{1+\gamma^2}}\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}$$] Antisymmetric Mode

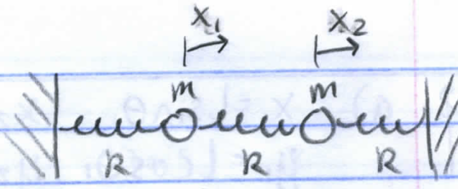
For $m_1 \gg m_2$

$\gamma \to 0$: recover single pendulum frequency $\sqrt{\frac{g}{\ell}}$

For $m_2 \gg m_1$

$\gamma \to 1$: recover single pendulum frequency $\sqrt{\frac{g}{2\ell}}$

9. a)



$$L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}k(x_2 - x_1)^2$$

EOM's

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = kx_2 - 2kx_1$$
$$m\ddot{x}_2 = -kx_2 + k(x_2 - x_1) = kx_1 - 2kx_2$$

b) $\ddot{x}_i = -\omega^2 x_i \qquad x_i = A_i\cos(\omega t + \phi)$

$$-m\omega^2 x_1 = kx_2 - 2kx_1$$
$$-m\omega^2 x_2 = kx_1 - 2kx_2$$
$$\Rightarrow (-m\omega^2 + 2k)A_1 - kA_2 = 0$$
$$-kA_1 + (-m\omega^2 + 2k)A_2 = 0$$

Put in matrix form

$$\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set determinant $= 0$

$$\Rightarrow (-m\omega^2 + 2k)^2 - k^2 = 0$$
$$m^2\omega^4 + 4mk\omega^2 + 4k^2 - k^2 = 0$$
$$\omega^4 - 4k/m\,\omega^2 + 3k^2/m^2 = 0$$
$$\omega^2 = \begin{cases} k/m \\ 3k/m \end{cases}$$

$\omega^2 = k/m$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rho_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\omega^2 = 3k/m$

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rho_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Describe the motions

ω_1^2: The masses oscillate in sync/phase

ω_2^2: The masses oscillate in opposite phase

c) Modal Matrix

$$A = [\rho_1 \ \rho_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Generalized Coordinates: η

$$x(t) = A\,\eta(t) \qquad * \ A^T \vec{m} A = 0$$

To find $\vec{m}$ matrix, we know that the EOMs can be written as,

$$(\vec{V} - \vec{m}\omega^2)\vec{x} = 0 \quad \text{where } \vec{V} \text{ is the potential matrix}$$

So we can deduce that

$$\vec{m} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now to find $\eta(t)$

$$\eta(t) = A^T \vec{m}\, x(t)$$

$$\vec{\eta} = \frac{m}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{m}{\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$\eta_1 = m/\sqrt{2}\,(x_1 + x_2)$$

$$\eta_2 = m/\sqrt{2}\,(x_1 - x_2)$$

Lagrangian in diagonal form

$$L = \frac{1}{2}\sum_{i=1}^{2}\left[\dot{\eta}_i^2 - \omega_i^2 \eta_i^2\right]$$

were ω_i^2 is the ith eigenfrequency