PHYS 201 Mathematical Physics, Fall 2016, Homework 1

Due date: Tuesday, October 4th, 2016

1. A series of mappings can be used to map a unit disk to an infinite horizontal strip and back to a unit disk.

i. Find a bilinear mapping that maps the unit circle $|z| = 1$ to the imaginary axis. Note that the mapping is not necessarily unique.

ii. Using this result, find a mapping from the unit disk $|z| \leq 1$ to the right half plane $\mathrm{Re}(z) \geq 0$.

iii. Find a mapping (not necessarily bilinear) from the right half plane to an infinite horizontal strip satisfying $|\mathrm{Im}(z)| \leq \pi/2$.

iv. Rotate and scale the infinite horizontal strip to an infinite vertical strip $|\mathrm{Re}(z)| \leq \pi/4$. Finally, show that the mapping $\tan(z)$ maps this infinite vertical strip to the unit disk $|z| \leq 1$.

2. Consider the function $g(z) = (z^2 + 1)^{1/2}$. One possible branch cut is the segment joining i and $-i$. A point z may be written relative to these two points in polar form as $z - i = r_1 e^{i\theta_1}$ and $z + i = r_2 e^{i\theta_2}$ which yields $g(z) = \sqrt{r_1 r_2} e^{i\frac{\theta_1+\theta_2}{2}}$. Choose careful definitions for the arguments θ_1 and θ_2 and argue that the function $g(z)$ is indeed continuous by following the function through values of z around this branch cut. Show that the function is discontinuous (as it should) across the branch cut.

3. Enumerate the branch points and discuss the possible branch cuts for the following functions:

i. $\log(z^2 - 1)$

ii. $\log \frac{z-1}{z+1}$

iii. $(z - 1)^{1/3}(z + 1)^{1/2}$

iv. $\log(1+\sqrt{1 + z^2})$ (Consider both the branches of $\sqrt{1 + z^2}$ and enumerate the branch points and possible cuts in each case).