PHYS 201 Mathematical Physics, Fall 2016, Homework 2

Due date: Tuesday, October 11th, 2016

1. Compute the integral around the unit circle $|z| = 1$ for the following functions using Cauchy's Integral Formula. Explain the choice of branch cuts if necessary.

   i. $\cot z$

   ii. $\frac{ze^z}{z^2+1}$

2. In class, we derived Cauchy's Integral Formula for a point z *on* the contour C as PV $\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \pi i f(z)$ by integrating on a semicircle around z with the semicircle lying inside the region enclosed by C (for an illustration, see Fig 2-2 of Carrier et al). Show that we obtain the same result when the semicircle is outside of the region enclosed by C.

3. In this exercise, we prove the converse of Cauchy's theorem, called *Morera's theorem*: If a function is continuous in a simple connected region and has the property that $\oint f(z)dz = 0$ on any closed contour lying in that region, then $f(z)$ must be analytic in that region. To prove this, first show that if $\oint f(z)dz = 0$ on any closed contour, then the integral $\int_{z_1}^{z_2} f(z)dz$ depends only on the end points z_1 and z_2. Next, suppose $F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)dz$. Show that the function F is analytic and explain why this implies that f is analytic.

4. In this exercise, we prove the complex analog of the theorem in electrostatics that if a charge-free domain has a constant potential on the boundary, then the potential is a constant inside the domain.

   i. Show that if a function $f(z)$ is analytic and has a constant modulus $|f(z)|$ inside a domain D, then $f(z)$ is constant in that domain. (**Hint**: Write $f(z) = u(z) + iv(z)$. Differentiate $|f(z)|^2$ w.r.t the real and imaginary components (x, y) of z, and show that if $|f(z)|$ is a constant, then the Jacobian of the mapping $(x, y) \to (u, v)$ is zero).

   ii. Using the above result, show that if $f(z)$ is analytic and non-vanishing inside D and is constant on the boundary of D, then f is constant everywhere in D. Assume that f is continuous on the closed domain $D \cup \partial D$ i.e., domain plus boundary. (**Hint**: Assume the contrary. By the max/min modulus theorem and continuity, the maximum and minimum of $|f|$ can only lie on the boundary.).

   iii. Show that if a function $u(x, y)$ is harmonic and non-constant in a domain D, then u has neither a maximum nor a minimum in D. Using ideas from the previous results, show that this implies that if u is constant on the boundary, then it is constant everywhere in D. Further, show that if two harmonic functions are equal on the boundary, then they are equal everywhere i.e., given the boundary conditions, the potential is unique. (**Hint**: For the first part, write $g(z) = e^{f(z)}$ where u is the real part of f and use max/min modulus theorem).