PHYS 201 Mathematical Physics, Fall 2016, Homework 4

Due date: Thursday, October 27th, 2016

1. In this exercise, we will use Rouché's theorem (a consequence of the residue theorem) and prove the existence of a single-valued and analytic local inverse function for any analytic function whose derivative is non-zero. *Rouché's theorem*: Let $f(z)$ and $g(z)$ be analytic inside and on C with $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

i. Prove that if $w = f(z)$ is analytic at z_0, with $w_0 = f(z_0)$ and $f'(z_0) \neq 0$, then there exists a contour C around z_0 such that $f(z)$ does not take the value w_0 on or within C except at z_0. This shows that $|f(z) - w_0|$ has a lower bound, say m, on C. (**Hint:** use the identity theorem, which states that if f is analytic, we have $f \equiv 0$ in a domain D if and only if you can construct a sequence $a_n \to a$ within D such that $f(a_n) = 0$ for all n and a lies in D.)

ii. Next, consider w such that $|w - w_0| < m$. Show that for any such w, say w_1, there is only one point inside C such that $w_1 = f(z_1)$. Conclude the proof. (**Hint:** use Rouché's theorem)

2. Let two regions R_1 and R_2 be adjacent to one another, with a portion Γ of their boundaries in common. Let $f_1(z)$ be analytic in R_1, $f_2(z)$ in R_2; let each function be continuous onto Γ, and let $f_1(z) = f_2(z)$ on Γ. Show that the combined function is analytic over the combined region. (**Hint:** use Morera's theorem from HW 2.)

3. Most of the special functions in mathematical physics can be generated by a generating function of the form

$$g(t, x) = \sum_n f_n(x) t^n$$

Show that we can give an integral representation of the special function $f_n(x)$ as

$$f_n(x) = \frac{1}{2\pi i} \oint g(t, x) t^{-n-1} dt$$

where the contour encloses the origin and no other singular points. The generating function for Bessel functions $J_n(x)$ is given by $g_B(t, x) = e^{(x/2)(t-1/t)}$. Expand $g_B(t, x)$ as a Laurent series near the origin and find the first three terms of $J_n(x)$ by computing the residue.