PHYS 201 Mathematical Physics, Fall 2016, Homework 5

Due date: Tuesday, November 8th, 2016

1. Evaluate using rectangular contours (or otherwise):

   i. $\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{ikx} dx$ for real $\alpha > 0, k$.

   ii. $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ for $0 < a < 1$. By making an appropriate substitution, show that this integral reduces to the beta integral $B(a, 1-a)$.

2. Show that $\pi \cot \pi z$ has simple poles at integer values of z. Consider the integral

$$\frac{1}{2\pi i} \oint_{C_N} f(z) \pi \cot(\pi z) dz$$

with $f(z) = 1/z^2$ and C_N a square contour with diagonally opposite vertices $-(N + 1/2)(1+i)$ and $(N+1/2)(1+i)$. Argue why the integral goes to zero as $N \to \infty$ and consequently show that the sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ by computing the residue at 0.

3. Evaluate the integral

$$\int_0^{\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2} d\theta$$

for real $a > 1$ and integer $n > 0$. (**Hint:** Consider the real part of $\int \frac{-iz^{n-1} dz}{1 - a(z+z^{-1}) + a^2}$ where $z = e^{i\theta}$. Many integrals involving trigonometric integrands can be solved this way.)

4. Show that for $\alpha > 0$,

$$\int_0^{\infty} \frac{t \sin \alpha t}{1+t^2} dt = \frac{\pi}{2} e^{-\alpha}$$

5. The function $f(z) = (1-z^2)^{1/2}$ can be made single-valued using cuts running along the real axis for $|x| > 1$. Using these cuts and a suitable contour, evaluate the integral

$$\int_1^{\infty} \frac{dx}{x(x^2-1)^{1/2}}$$

Verify your answer by substituting $x = \sec\theta$.

6. Use a semicircular contour in the upper half plane (with a bump at the origin) to compute the integral

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} dx$$

and deduce, as a byproduct of your calculation, that

$$\int_0^\infty \frac{\ln x}{1+x^2}dx = 0$$

7. Use a keyhole contour to show that

$$\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}dx = -\sqrt{2}\pi^2$$

8. Evaluate the integral

$$\int_0^1 \frac{x^{1/2}(1-x)^{1/2}}{2-x}dx$$

(**Hint:** Apply Cauchy's Integral Formula to the entire complex plane excluding the branch cut between 0 and 1. Be careful about your choice of branch - verify that the branch you've chosen is continuous across the negative real line and the positive real line > 1. To compute the residue at infinity, use the formula $\text{Res}(f(z))$ at $z = \infty$ is equal to $\text{Res}(-z^{-2}f(1/z))$ at $z = 0$.)