EXERCISE 6.1-4

Normal Modes of Simple Polarization Systems.

(a) Show that the normal modes of the linear polarizer are linearly polarized waves.

(b) Show that the normal modes of the wave retarder are linearly polarized waves.

(c) Show that the normal modes of the polarization rotator are right and left circularly polarized waves.

What are the eigenvalues of the systems described above?

6.2 REFLECTION AND REFRACTION

In this section we examine the reflection and refraction of a monochromatic plane wave of arbitrary polarization incident at a planar boundary between two dielectric media. The media are assumed to be linear, homogeneous, and isotropic with impedances η_1 and η_2, and refractive indexes n_1 and n_2. The incident, refracted, and reflected waves are labeled with the subscripts 1, 2, and 3, respectively, as illustrated in Fig. 6.2-1.

As shown in Sec. 2.4A, the wavefronts of these waves are matched at the boundary if the angles of reflection and incidence are equal, $\theta_3 = \theta_1$, and if the angles of refraction and incidence satisfy Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{6.2-1}$$

To relate the amplitudes and polarizations of the three waves, we associate with each wave an x–y coordinate system in a plane normal to the direction of propagation (Fig. 6.2-1). The electric-field envelopes of these waves are described by the Jones vectors

$$\mathbf{J}_1 = \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}, \qquad \mathbf{J}_2 = \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix}, \qquad \mathbf{J}_3 = \begin{bmatrix} A_{3x} \\ A_{3y} \end{bmatrix}. \tag{6.2-2}$$

We proceed to determine the relations between $\mathbf{J}_2$ and $\mathbf{J}_1$ and between $\mathbf{J}_3$ and $\mathbf{J}_1$. These relations are written in the form of matrices $\mathbf{J}_2 = \mathbf{t}\mathbf{J}_1$ and $\mathbf{J}_3 = \mathbf{r}\mathbf{J}_1$, where $\mathbf{t}$ and $\mathbf{r}$ are 2×2 Jones matrices describing the transmission and reflection of the wave, respectively.

The elements of the transmission and reflection matrices may be determined by imposing the boundary conditions required by electromagnetic theory, namely the continuity at the boundary of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal components of $\mathbf{D}$ and $\mathbf{B}$. The electric field associated with each wave is orthogonal to the magnetic field; the ratio of their envelopes is the characteristic impedance, which is η_1 for the incident and reflected waves and η_2 for the transmitted wave. The result is a set of equations that are solved to obtain relations between the components of the electric fields of the three waves.

The algebra involved is reduced substantially if we observe that the two normal modes for this system are linearly polarized waves with polarizations along the x and y directions. This may be proved if we show that an incident, a reflected, and a refracted wave with their electric field vectors pointing in the x direction are self-consistent with the boundary conditions, and similarly for three waves linearly polarized in the y direction. This is indeed the case. The x and y polarized waves are therefore uncoupled.

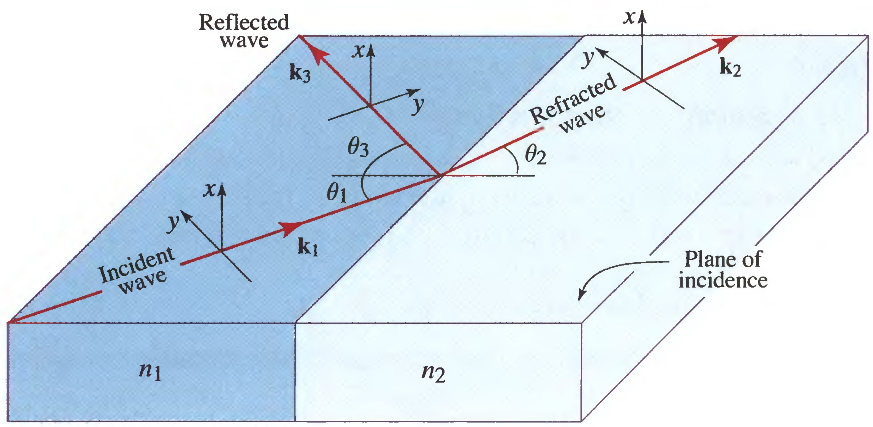


Figure 6.2-1 Reflection and refraction at the boundary between two dielectric media.

The x-polarized mode is called the **transverse electric (TE)** polarization or the **orthogonal** polarization, since the electric fields are orthogonal to the plane of incidence. The y-polarized mode is called the **transverse magnetic (TM)** polarization since the magnetic field is orthogonal to the plane of incidence, or the **parallel** polarization since the electric fields are parallel to the plane of incidence. The orthogonal and parallel polarizations are also called the s (for the German *senkrecht*, meaning "perpendicular") and p (for "parallel") polarizations, respectively. The y axes in Fig. 6.2-1 are arbitrarily defined such that their components parallel to the boundary between the dielectrics all point in the same direction.

The independence of the x and y polarizations implies that the Jones matrices $\mathbf{t}$ and $\mathbf{r}$ are diagonal,

$$\mathbf{t} = \begin{bmatrix} \mathrm{t}_x & 0 \\ 0 & \mathrm{t}_y \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} \mathrm{r}_x & 0 \\ 0 & \mathrm{r}_y \end{bmatrix} \tag{6.2-3}$$

so that

$$E_{2x} = \mathrm{t}_x E_{1x}, \qquad E_{2y} = \mathrm{t}_y E_{1y} \tag{6.2-4}$$

$$E_{3x} = \mathrm{r}_x E_{1x}, \qquad E_{3y} = \mathrm{r}_y E_{1y}. \tag{6.2-5}$$

The coefficients t_x and t_y are the complex amplitude transmittances for the TE and TM polarizations, respectively; r_x and r_y are the analogous complex amplitude reflectances.

Applying the boundary conditions (i.e., equating the tangential components of the electric fields and the tangential components of the magnetic fields at both sides of the boundary) in each of the TE and TM cases, we obtain the following expressions for the reflection and transmission coefficients:

$$\mathrm{r}_x = \frac{\eta_2 \sec\theta_2 - \eta_1 \sec\theta_1}{\eta_2 \sec\theta_2 + \eta_1 \sec\theta_1}, \qquad \mathrm{t}_x = 1 + \mathrm{r}_x, \tag{6.2-6}$$

TE Polarization

$$\mathrm{r}_y = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}, \qquad \mathrm{t}_y = (1 + \mathrm{r}_y)\frac{\cos\theta_1}{\cos\theta_2}. \tag{6.2-7}$$

TM Polarization
Reflection & Transmission

The characteristic impedance $\eta = \sqrt{\mu/\epsilon}$ is complex if ϵ and/or μ are complex, as is the case for lossy or conductive media. For nonlossy, nonmagnetic, dielectric media, $\eta = \eta_o/n$ is real, where $\eta_o = \sqrt{\mu_o/\epsilon_o}$ and n is the refractive index. In this case, the reflection and transmission coefficients in (6.2-6) and (6.2-7) yield the following equations, known as the **Fresnel equations**:

$$\mathfrak{r}_x = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}, \qquad \mathfrak{t}_x = 1 + \mathfrak{r}_x, \tag{6.2-8}$$

TE Polarization

$$\mathfrak{r}_y = \frac{n_1\sec\theta_1 - n_2\sec\theta_2}{n_1\sec\theta_1 + n_2\sec\theta_2}, \qquad \mathfrak{t}_y = (1 + \mathfrak{r}_y)\frac{\cos\theta_1}{\cos\theta_2}. \tag{6.2-9}$$

TM Polarization

Fresnel Equations

Given n_1, n_2, and θ_1, the reflection coefficients can be determined the Fresnel equations by first determining θ_2 using Snell's law, (6.2-1), from which

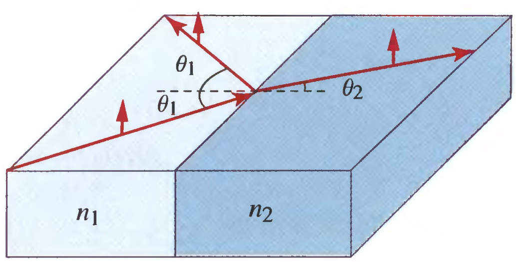
$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-(n_1/n_2)^2\sin^2\theta_1}\,. \tag{6.2-10}$$

Since the quantities under the square-root signs in (6.2-10) can be negative, the reflection and transmission coefficients are in general complex. The magnitudes $|\mathfrak{r}_x|$ and $|\mathfrak{r}_y|$, and the phase shifts $\varphi_x = \arg\{\mathfrak{r}_x\}$ and $\varphi_y = \arg\{\mathfrak{r}_y\}$, are plotted in Figs. 6.2-2 to 6.2-5 for the two polarizations, as functions of the angle of incidence θ_1. Plots are provided for external reflection ($n_1 < n_2$) as well as for internal reflection ($n_1 > n_2$).

TE Polarization

The dependence of the reflection coefficient $\mathfrak{r}_x$ on θ_1 for the TE-polarized wave is given by (6.2-8):

External reflection ($n_1 < n_2$). The reflection coefficient $\mathfrak{r}_x$ is always real and negative, corresponding to a phase shift $\varphi_x = \pi$. The magnitude $|\mathfrak{r}_x| = (n_2 - n_1)/(n_1 + n_2)$ at $\theta_1 = 0$ (normal incidence) and increases to unity at $\theta_1 = 90°$ (grazing incidence), as shown in Fig. 6.2-2.



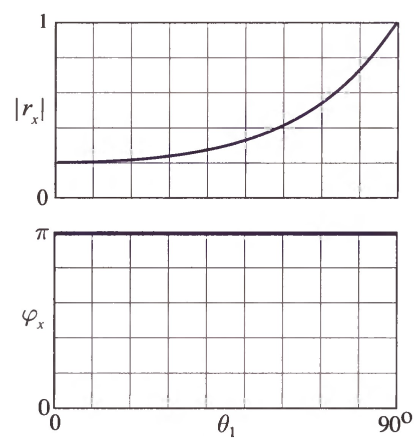


Figure 6.2-2 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *external reflection* of the *TE-polarized* wave ($n_2/n_1 = 1.5$).

Internal reflection ($n_1 > n_2$). For small θ_1 the reflection coefficient is real and positive. Its magnitude is $(n_1 - n_2)/(n_1 + n_2)$ when $\theta_1 = 0°$, and increases gradually to a value of unity, which is attained when θ_1 equals the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$. For $\theta_1 > \theta_c$, the magnitude of $\mathfrak{r}_x$ remains at unity, which corresponds to total internal reflection. This may be shown by using (6.2-10) to write† $\cos\theta_2 =$

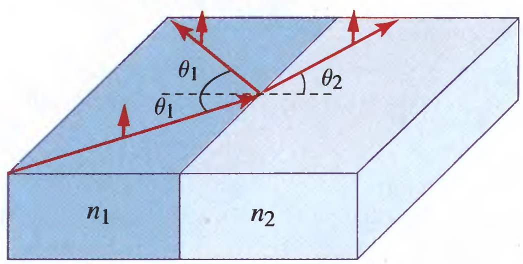
† The choice of the minus sign for the square root is consistent with the derivation that leads to the Fresnel equation.

$-\sqrt{1-\sin^2\theta_1/\sin^2\theta_c} = -j\sqrt{\sin^2\theta_1/\sin^2\theta_c - 1}$, and substituting into (6.2-8). Total internal reflection is accompanied by a phase shift $\varphi_x = \arg\{\mathfrak{r}_x\}$ given by

$$\tan\frac{\varphi_x}{2} = \sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_1} - 1} \qquad (6.2\text{-}11)$$

TE-Reflection Phase Shift

The phase shift φ_x increases from 0 at $\theta_1 = \theta_c$ to π at $\theta_1 = 90°$, as illustrated in Fig. 6.2-3. This phase plays an important role in dielectric waveguides (see Sec. 8.2).



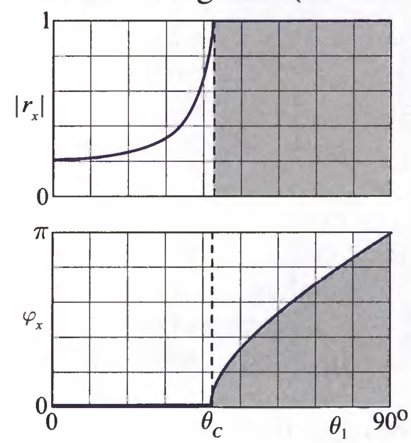


Figure 6.2-3 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *internal reflection* of the *TE-polarized* wave ($n_1/n_2 = 1.5$).

TM Polarization

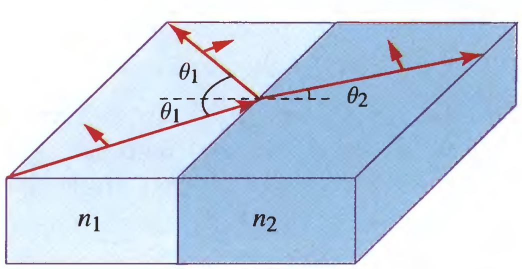
Similarly, the dependence of the reflection coefficient $\mathfrak{r}_y$ on θ_1 for the TM-polarized wave is provided by (6.2-9):

External reflection ($n_1 < n_2$). The reflection coefficient $\mathfrak{r}_y$ is always real. It assumes a negative value of $(n_1 - n_2)/(n_1 + n_2)$ at $\theta_1 = 0$ (normal incidence). Its magnitude then decreases until it vanishes when $n_1 \sec\theta_1 = n_2 \sec\theta_2$, at an angle $\theta_1 = \theta_B$, known as the **Brewster angle**:

$$\theta_B = \tan^{-1}(n_2/n_1) \qquad (6.2\text{-}12)$$

Brewster Angle

(see Prob. 6.2-5 for other properties of the Brewster angle). For $\theta_1 > \theta_B$, $\mathfrak{r}_y$ reverses sign (φ_y goes from π to 0) and its magnitude gradually increases until it approaches unity at $\theta_1 = 90°$. The absence of reflection of the TM wave at the Brewster angle is useful for making polarizers (see Sec. 6.6).



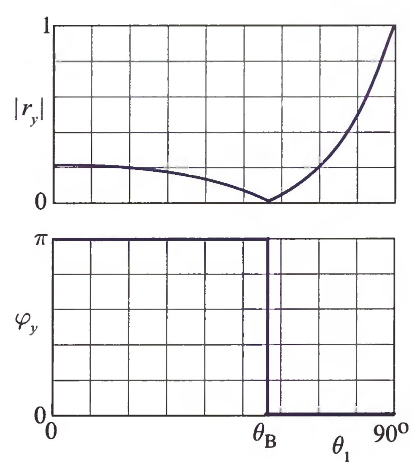


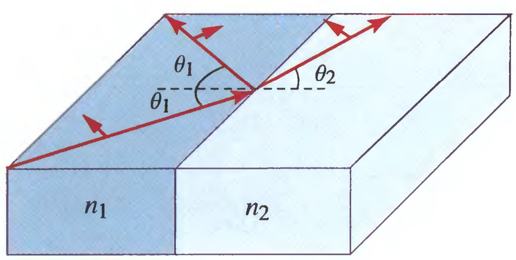
Figure 6.2-4 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *external reflection* of the *TM-polarized* wave ($n_2/n_1 = 1.5$).

Internal reflection ($n_1 > n_2$). At $\theta_1 = 0°$, r_y is positive and has magnitude $(n_1 - n_2)/(n_1 + n_2)$. As θ_1 increases, the magnitude decreases until it vanishes at the Brewster angle $\theta_B = \tan^{-1}(n_2/n_1)$. As θ_1 increases beyond θ_B, r_y becomes negative and its magnitude increases until it reaches unity at the critical angle θ_c. For $\theta_1 > \theta_c$ the wave undergoes total internal reflection accompanied by a phase shift $\varphi_y = \arg\{\mathrm{r}_y\}$ given by

$$\tan\frac{\varphi_y}{2} = \frac{-1}{\sin^2\theta_c}\sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_1} - 1}\,. \tag{6.2-13}$$

TM-Reflection Phase Shift

At normal incidence, evidently, the reflection coefficient is $\mathrm{r} = (n_1 - n_2)/(n_1 + n_2)$, whether the reflection is TE or TM, or external or internal.



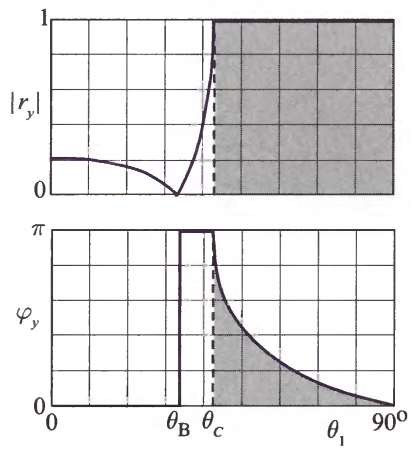


Figure 6.2-5 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *internal reflection* of the *TM-polarized wave* ($n_1/n_2 = 1.5$).

EXERCISE 6.2-1

Brewster Windows. At what angle is a TM-polarized beam of light transmitted through a glass plate of refractive index $n = 1.5$ placed in air ($n = 1$) without suffering reflection losses at either surface? Such plates, known as Brewster windows (Fig. 6.2-6), are used in lasers, as described in Sec. 15.2D.

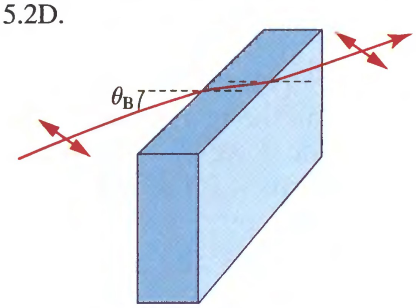


Figure 6.2-6 The Brewster window transmits TM-polarized light with no reflection loss.

Power Reflectance and Transmittance

The reflection and transmission coefficients r and t represent ratios of complex amplitudes. The power reflectance $\mathcal{R}$ and power transmittance $\mathcal{T}$ are defined as the ratios of power flow (along a direction normal to the boundary) of the reflected and transmitted waves to that of the incident wave. Because the reflected and incident waves propagate in the same medium and make the same angle with the normal to the surface, it follows that

$$\mathcal{R} = |\mathrm{r}|^2. \tag{6.2-14}$$

For both TE and TM polarizations, and for both external and internal reflection, the power reflectance at normal incidence is therefore

$$\boxed{\mathcal{R} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2.} \tag{6.2-15}$$

Power Reflectance at Normal Incidence

At the boundary between glass ($n = 1.5$) and air ($n = 1$), for example, $\mathcal{R} = 0.04$, so that 4% of the light is reflected at normal incidence. At the boundary between GaAs ($n = 3.6$) and air ($n = 1$), $\mathcal{R} \approx 0.32$, so that 32% of the light is reflected at normal incidence. However, at oblique angles the reflectance can be much greater or much smaller than 32%, as illustrated in Fig. 6.2-7.

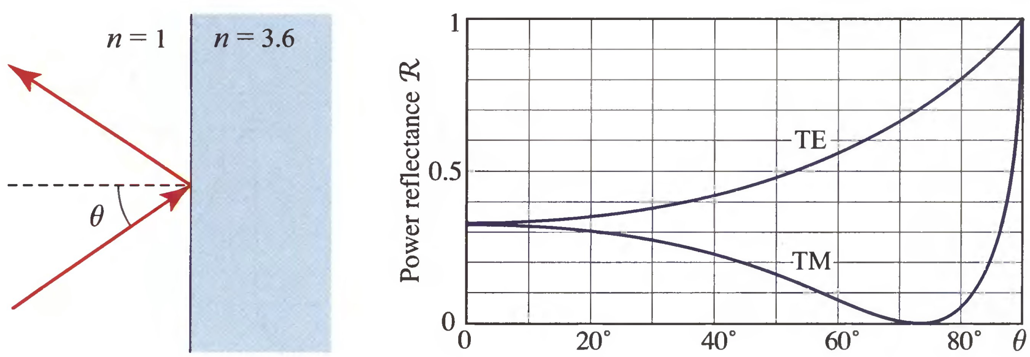


Figure 6.2-7 Power reflectance of TE- and TM-polarization plane waves at the boundary between air ($n = 1$) and GaAs ($n = 3.6$), as a function of the angle of incidence θ.

The power transmittance $\mathcal{T}$ is determined by invoking the conservation of power, so that in the absence of absorption loss the transmittance is simply

$$\mathcal{T} = 1 - \mathcal{R}. \tag{6.2-16}$$

It is important to note, however, that $\mathcal{T}$ is generally *not* equal to $|\mathrm{t}|^2$ since the power travels at different angles and with different impedances in the two media. For a wave traveling at an angle θ in a medium of refractive index n, the power flow in the direction normal to the boundary is $(|\mathcal{E}|^2/2\eta)\cos\theta = (|\mathcal{E}|^2/2\eta_o)\, n\cos\theta$. It follows that

$$\mathcal{T} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1}\,|\mathrm{t}|^2. \tag{6.2-17}$$

Reflectance from a plate. The power reflectance at normal incidence from a plate with two surfaces is described by $\mathcal{R}(1 + \mathcal{T}^2)$ since the power reflected from the far surface involves a double transmission through the near surface. For a glass plate in air, the overall reflectance is $\mathcal{R}(1 + \mathcal{T}^2) = 0.04[1 + (0.96)^2] \approx 0.077$, so that about 7.7% of the incident light power is reflected. However, this calculation does not include interference effects, which are washed out when the light is incoherent (see Sec.11.2), nor does it account for multiple reflections inside the plate. Optical transmission and reflectance from multiple boundaries in layered media are described in detail in Sec. 7.1.