

EXERCISE 6.2-2

Reflectance of a Conductive Medium. The equations for the reflection coefficients set forth in (6.2-6) and (6.2-7) can be used to determine the intensity reflectance \mathcal{R} at the boundary between a dielectric medium and a conductive medium.

- Show that $\mathcal{R} \approx 1$ if the conductivity of the conductive medium σ is infinite.
- Show that at normal incidence, and for $\sigma \gg \epsilon_o \omega$, the relation $\mathcal{R} = 1 - 2\sqrt{2\epsilon_o \omega / \sigma}$, known as the **Hagen–Rubens relation**, emerges. Use this relation to determine the reflectance of copper at the wavelengths $\lambda_o = 1.06 \mu\text{m}$ and $10.6 \mu\text{m}$. Assume that the conductivity of copper is $\sigma = 0.58 \times 10^8 (\Omega\text{-m})^{-1}$.
- Show that if the conductive medium is described by the Drude model, (5.5-39), then $\mathcal{R} = 1$ at frequencies below the plasma frequency.

6.3 OPTICS OF ANISOTROPIC MEDIA

A dielectric medium is said to be anisotropic if its macroscopic optical properties depend on direction. The macroscopic properties of a material are, of course, ultimately governed by its microscopic properties: the shape and orientation of the individual molecules and the organization of their centers in space. Optical materials have different kinds of positional and orientational types of order, which may be described as follows (see Fig. 6.3-1):

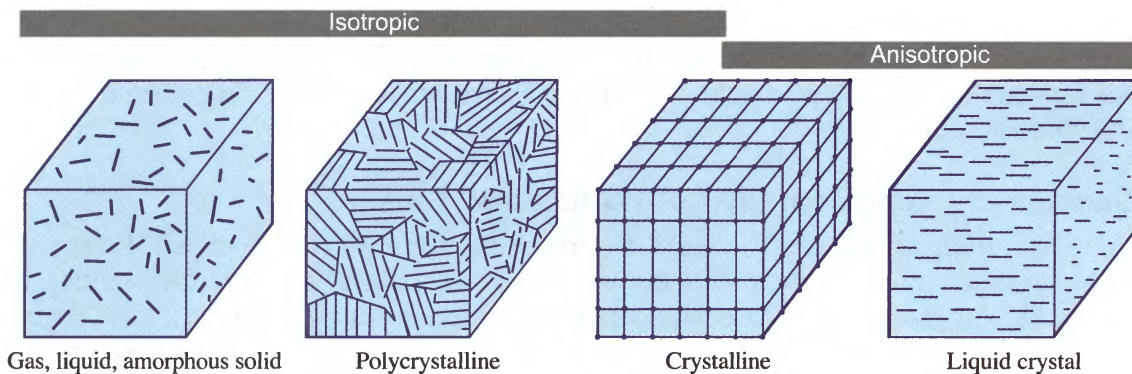


Figure 6.3-1 Positional and orientational order in different types of materials.

- If the molecules are located at totally random positions in space, and are themselves isotropic or oriented along random directions, the medium is isotropic. *Gases, liquids, and amorphous solids* follow this prescription.
- If the structure takes the form of disjointed crystalline grains that are randomly oriented with respect to each other, the material is said to be *polycrystalline*. The individual grains are, in general, anisotropic, but their averaged macroscopic behavior is isotropic.
- If the molecules are organized in space according to a regular periodic pattern and they are oriented in the same direction, as in *crystals*, the medium is, in general, anisotropic.

- If the molecules are anisotropic and their orientations are not totally random, the medium is anisotropic, even if their positions are totally random. This is the case for *liquid crystals*, which have orientational order but lack complete positional order.

A. Refractive Indexes

Permittivity Tensor

In a linear anisotropic dielectric medium (a crystal, for example), each component of the electric flux density \mathbf{D} is a linear combination of the three components of the electric field,

$$D_i = \sum_j \epsilon_{ij} E_j. \quad (6.3-1)$$

The indexes $i, j = 1, 2, 3$ refer to the $x, y,$ and z components, respectively, as described in Sec. 5.2B. The dielectric properties of the medium are therefore characterized by a 3×3 array of nine coefficients, $\{\epsilon_{ij}\}$, that form the **electric permittivity tensor** ϵ , which is a tensor of second rank. The **material equation** (6.3-1) is usually written in the symbolic form

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (6.3-2)$$

For most dielectric media, the electric permittivity tensor is symmetric, i.e., $\epsilon_{ij} = \epsilon_{ji}$. This means that the relation between the vectors \mathbf{D} and \mathbf{E} is reciprocal, i.e., their ratio remains the same if their directions are exchanged. This symmetry is obeyed for dielectric nonmagnetic materials that do not exhibit optical activity, and in the absence of an external magnetic field (see Sec. 6.4). With this symmetry, the medium is characterized by only six independent numbers in an arbitrary coordinate system. For crystals of certain symmetries, even fewer coefficients suffice since some vanish and some are related.

Geometrical Representation of Vectors and Tensors

A *vector*, such as the electric field \mathbf{E} , for example, describes a physical variable with magnitude and direction. It is represented *geometrically* by an arrow pointing in that particular direction, whose length is proportional to the magnitude of the vector [Fig. 6.3-2(a)]. A vector, which is a tensor of first rank, is represented *numerically* by three numbers: its projections on the three axes of a particular coordinate system. Though these components depend on the choice of the coordinate system, the magnitude and direction of the vector in physical space are independent of the choice of the coordinate system. A scalar, which is described by a single number, is a tensor of zero rank.

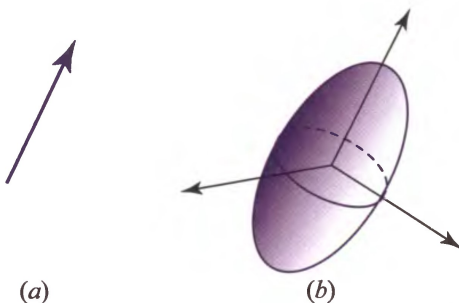


Figure 6.3-2 Geometrical representation of (a) a vector and (b) a symmetric second-rank tensor.

A second-rank *tensor* is a rule that relates two vectors. In a given coordinate system, it is represented *numerically* by nine numbers. Changing the coordinate system yields a different set of nine numbers, but the physical nature of the rule is unchanged. A useful *geometrical* representation [Fig. 6.3-2(b)] of a *symmetric* second-rank tensor (the dielectric tensor ϵ , for example), is a quadratic surface (an ellipsoid) defined by

$$\sum_{ij} \epsilon_{ij} x_i x_j = 1, \quad (6.3-3)$$

which is known as the **quadric representation**. This surface is invariant to the choice of the coordinate system; if the coordinate system is rotated, both x_i and ϵ_{ij} are altered but the ellipsoid remains intact in physical space. The ellipsoid has six degrees of freedom and carries all information about the symmetric second-rank tensor. In the principal coordinate system, ϵ_{ij} is diagonal and the ellipsoid assumes a particularly simple form:

$$\epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 = 1. \quad (6.3-4)$$

Its principal axes are those of the tensor, and its axes have half-lengths $1/\sqrt{\epsilon_1}$, $1/\sqrt{\epsilon_2}$, and $1/\sqrt{\epsilon_3}$.

Principal Axes and Principal Refractive Indexes

The elements of the permittivity tensor depend on how the coordinate system is chosen relative to the crystal structure. However, a coordinate system can always be found for which the off-diagonal elements of ϵ_{ij} vanish, so that

$$D_1 = \epsilon_1 E_1, \quad D_2 = \epsilon_2 E_2, \quad D_3 = \epsilon_3 E_3, \quad (6.3-5)$$

where $\epsilon_1 = \epsilon_{11}$, $\epsilon_2 = \epsilon_{22}$, and $\epsilon_3 = \epsilon_{33}$. According to (6.3-1), \mathbf{E} and \mathbf{D} are parallel along these particular directions so that if, for example, \mathbf{E} points in the x direction, then so too must \mathbf{D} . This coordinate system defines the **principal axes** and principal planes of the crystal. Throughout the remainder of this chapter, the coordinate system x, y, z , which is equivalently denoted x_1, x_2, x_3 , is assumed to lie along the principal axes of the crystal. This choice simplifies all analyses without loss of generality. The permittivities ϵ_1, ϵ_2 , and ϵ_3 correspond to refractive indexes

$$n_1 = \sqrt{\epsilon_1/\epsilon_o}, \quad n_2 = \sqrt{\epsilon_2/\epsilon_o}, \quad n_3 = \sqrt{\epsilon_3/\epsilon_o}, \quad (6.3-6)$$

respectively, where ϵ_o is the permittivity of free space; these are known as the **principal refractive indexes**.

Biaxial, Uniaxial, and Isotropic Crystals

Crystals in which the three principal refractive indexes are different are termed **biaxial**. For crystals with certain symmetries, namely a single axis of threefold, fourfold, or sixfold symmetry, two of the refractive indexes are equal ($n_1 = n_2$) and the crystal is called **uniaxial**. In this case, the indexes are usually denoted $n_1 = n_2 = n_o$ and $n_3 = n_e$, which are known as the **ordinary** and **extraordinary** indexes, respectively, for reasons that will become clear shortly. The crystal is said to be **positive uniaxial** if $n_e > n_o$, and **negative uniaxial** if $n_e < n_o$. The z axis of a uniaxial crystal is called the **optic axis**. In certain crystals with even greater symmetry (those with cubic unit cells, for example), all three indexes are equal and the medium is optically isotropic.

Impermeability Tensor

The relation $\mathbf{D} = \epsilon \mathbf{E}$ can be inverted and written in the form $\mathbf{E} = \epsilon^{-1} \mathbf{D}$, where ϵ^{-1} is the inverse of the tensor ϵ . It is also useful to define the **electric impermeability tensor** $\eta = \epsilon_o \epsilon^{-1}$ (not to be confused with the impedance of the medium η), so that $\epsilon_o \mathbf{E} = \eta \mathbf{D}$. Since ϵ is symmetric, so too is η . Both tensors, ϵ and η , share the same principal axes. In the principal coordinate system, η is diagonal with principal values $\epsilon_o/\epsilon_1 = 1/n_1^2$, $\epsilon_o/\epsilon_2 = 1/n_2^2$, and $\epsilon_o/\epsilon_3 = 1/n_3^2$. Either tensor, ϵ or η , fully describes the optical properties of the crystal.

Index Ellipsoid

The **index ellipsoid** (also called the **optical indicatrix**) is the quadric representation of the electric impermeability tensor $\eta = \epsilon_o \epsilon^{-1}$:

$$\sum_{ij} \eta_{ij} x_i x_j = 1, \quad i, j = 1, 2, 3. \quad (6.3-7)$$

If the principal axes were to be used as the coordinate system, we would obtain

$$\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1, \quad (6.3-8)$$

Index Ellipsoid

with principal values $1/n_1^2$, $1/n_2^2$, and $1/n_3^2$, and axes of half-lengths n_1 , n_2 , and n_3 .

The optical properties of the crystal (the directions of the principal axes and the values of the principal refractive indexes) are therefore completely described by the index ellipsoid (Fig. 6.3-3). For a uniaxial crystal, the index ellipsoid reduces to an ellipsoid of revolution; for an isotropic medium it becomes a sphere.

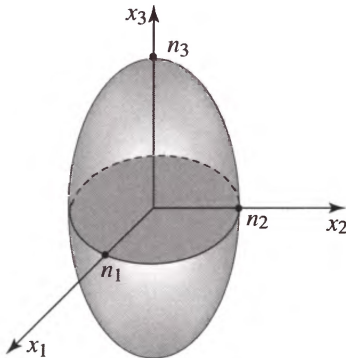


Figure 6.3-3 The index ellipsoid. The coordinates (x_1, x_2, x_3) are the principal axes while (n_1, n_2, n_3) are the principal refractive indexes of the crystal.

B. Propagation Along a Principal Axis

The rules that govern the propagation of light in crystals under general conditions are rather complex. However, they become relatively simple if the light is a plane wave traveling along one of the principal axes of the crystal. We begin with this case.

Normal Modes

Let x - y - z be a coordinate system that coincides with the principal axes of a crystal. A plane wave traveling in the z direction and linearly polarized along the x direction [Fig. 6.3-4(a)] travels with phase velocity c_o/n_1 (wavenumber $k = n_1 k_o$) without changing its polarization. The reason for this is that the electric field has only one

component, E_1 pointed along the x direction, so that \mathbf{D} is also in the x direction with $D_1 = \epsilon_1 E_1$; the wave equation derived from Maxwell's equations therefore provides a velocity of light given by $1/\sqrt{\mu_0 \epsilon_1} = c_0/n_1$. Similarly, a plane wave traveling in the z direction and linearly polarized along the y direction [Fig. 6.3-4(b)] travels with phase velocity c_0/n_2 , thereby experiencing a refractive index n_2 . Thus, the normal modes for propagation in the z direction are linearly polarized waves in the x and y directions. These waves are said to be normal modes because their velocities and polarizations are maintained as they propagate (see Appendix C). Other cases in which the wave propagates along one of the principal axes and is linearly polarized along another are treated similarly [Fig. 6.3-4(c)].

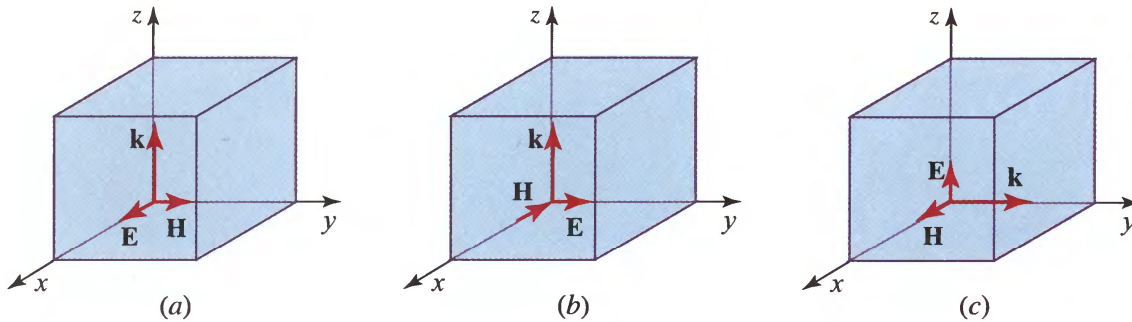


Figure 6.3-4 A wave traveling along a principal axis and polarized along another principal axis has phase velocity c_0/n_1 , c_0/n_2 , or c_0/n_3 , when the electric field vector points in the x , y , or z directions, respectively. (a) $k = n_1 k_0$; (b) $k = n_2 k_0$; (c) $k = n_3 k_0$.

Polarization Along an Arbitrary Direction

We now consider a wave traveling along one principal axis (the z axis, for example) that is linearly polarized along an arbitrary direction in the x - y plane. This case is addressed by analyzing the wave as a sum of the normal modes, namely the linearly polarized waves in the x and y directions. These two components travel with different phase velocities, c_0/n_1 and c_0/n_2 , respectively. They therefore undergo different phase shifts, $\varphi_x = n_1 k_0 d$ and $\varphi_y = n_2 k_0 d$, respectively, after propagating a distance d . Their phase retardation is thus $\varphi = \varphi_y - \varphi_x = (n_2 - n_1) k_0 d$. Recombination of the two components yields an elliptically polarized wave, as explained in Sec. 6.1 and illustrated in Fig. 6.3-5. Such a crystal can therefore serve as a **wave retarder**, a device in which two orthogonal polarizations travel at different phase velocities so that one is retarded with respect to the other (see Fig. 6.1-8).

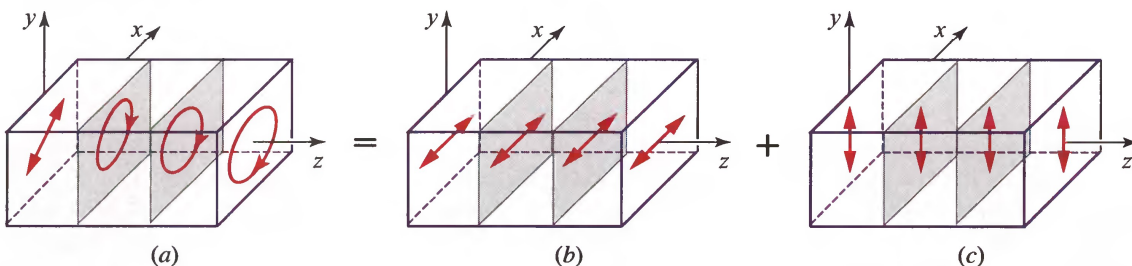


Figure 6.3-5 A linearly polarized wave at 45° in the $z = 0$ plane (a) is analyzed as a superposition of two linearly polarized components in the x and y directions (normal modes), which travel at velocities c_0/n_1 and c_0/n_2 [(b) and (c), respectively]. As a result of phase retardation, the wave is converted from plane polarization to elliptical polarization (a). It is therefore clear that the initial linearly polarized wave is not a normal mode of the system.

C. Propagation in an Arbitrary Direction

We now consider the general case of a plane wave traveling in an anisotropic crystal in an arbitrary direction defined by the unit vector $\hat{\mathbf{u}}$. We demonstrate that the two normal modes are linearly polarized waves. The refractive indexes n_a and n_b , and the directions of polarization of these modes, may be determined by use of a procedure based on the index ellipsoid:

Index-Ellipsoid Construction for Determining Normal Modes

Figure 6.3-6 illustrates a geometrical construction for determining the polarizations and refractive indexes n_a and n_b of the normal modes of a wave traveling in the direction of the unit vector $\hat{\mathbf{u}}$ in an anisotropic material characterized by the index ellipsoid:

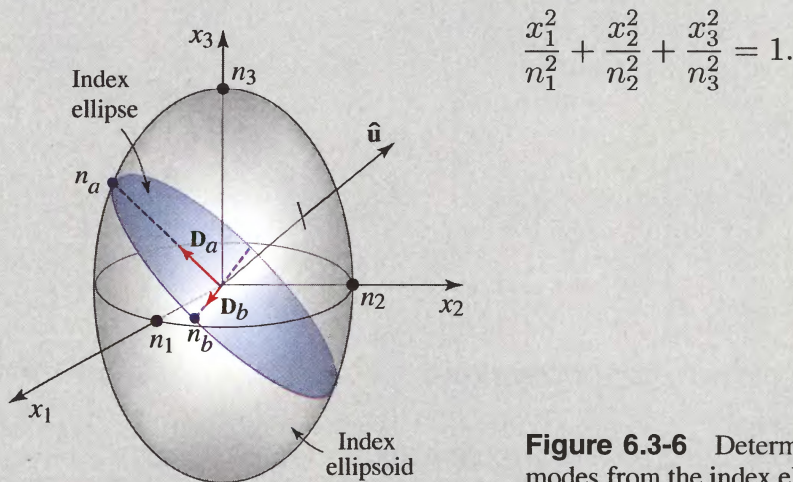


Figure 6.3-6 Determination of the normal modes from the index ellipsoid.

- Draw a plane passing through the origin of the index ellipsoid, normal to $\hat{\mathbf{u}}$. The intersection of the plane with the ellipsoid is an ellipse called the **index ellipse**.
- The half-lengths of the major and minor axes of the index ellipse are the refractive indexes n_a and n_b of the two normal modes.
- The directions of the major and minor axes of the index ellipse are the directions of the vectors \mathbf{D}_a and \mathbf{D}_b for the normal modes. These directions are orthogonal.
- The vectors \mathbf{E}_a and \mathbf{E}_b may be determined from \mathbf{D}_a and \mathbf{D}_b with the help of (6.3-5).

□ **Proof of the Index-Ellipsoid Construction for Determining the Normal Modes.** To determine the normal modes (see Sec. 6.1B) for a plane wave traveling in the direction $\hat{\mathbf{u}}$, we cast Maxwell's equations (5.3-2)–(5.3-5), and the material equation $\mathbf{D} = \epsilon\mathbf{E}$ given in (6.3-2), as an eigenvalue problem. Since all fields are assumed to vary with the position \mathbf{r} as $\exp(-j\mathbf{k} \cdot \mathbf{r})$, where $\mathbf{k} = k\hat{\mathbf{u}}$, Maxwell's equations (5.4-3) and (5.4-4) reduce to

$$\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D} \quad (6.3-9)$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu_o\mathbf{H} \quad (6.3-10)$$

Substituting (6.3-10) into (6.3-9) leads to

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = -\omega^2 \mu_o \mathbf{D}. \quad (6.3-11)$$

Using $\mathbf{E} = \epsilon^{-1} \mathbf{D}$, we obtain

$$\mathbf{k} \times (\mathbf{k} \times \epsilon^{-1} \mathbf{D}) = -\omega^2 \mu_o \mathbf{D}. \quad (6.3-12)$$

This is an eigenvalue equation that \mathbf{D} must satisfy. Working with \mathbf{D} is convenient since we know that it lies in a plane normal to the wave direction $\hat{\mathbf{u}}$.

We now simplify (6.3-12) by using $\boldsymbol{\eta} = \epsilon_o \epsilon^{-1}$, $\mathbf{k} = k \hat{\mathbf{u}}$, $n = k/k_o$, and $k_o^2 = \omega^2 \mu_o \epsilon_o$ to obtain

$$-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta} \mathbf{D}) = \frac{1}{n^2} \mathbf{D}. \quad (6.3-13)$$

The operation $-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta} \mathbf{D})$ may be interpreted as a projection of the vector $\boldsymbol{\eta} \mathbf{D}$ onto a plane normal to $\hat{\mathbf{u}}$. We may therefore rewrite (6.3-13) in the form

$$\mathbf{P}_u \boldsymbol{\eta} \mathbf{D} = \frac{1}{n^2} \mathbf{D}, \quad (6.3-14)$$

where \mathbf{P}_u is an operator representing projection. Equation (6.3-14) is an eigenvalue equation for the operator $\mathbf{P}_u \boldsymbol{\eta}$, with eigenvalue $1/n^2$ and eigenvector \mathbf{D} . The two eigenvalues, $1/n_a^2$ and $1/n_b^2$, and two corresponding eigenvectors, \mathbf{D}_a and \mathbf{D}_b , represent the two normal modes.

The eigenvalue problem (6.3-14) has a simple geometrical interpretation. The tensor $\boldsymbol{\eta}$ is represented geometrically by its quadric representation, the index ellipsoid. The operator $\mathbf{P}_u \boldsymbol{\eta}$ represents projection onto a plane normal to $\hat{\mathbf{u}}$. Solving the eigenvalue problem in (6.3-14) is thus equivalent to finding the principal axes of the ellipse formed by the intersection of the plane normal to $\hat{\mathbf{u}}$ with the index ellipsoid. This is precisely the construction set forth in Fig. 6.3-6 for determining the normal modes. ■

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$) the index ellipsoid of Fig. 6.3-6 is an ellipsoid of revolution. For a wave whose direction of travel $\hat{\mathbf{u}}$ forms an angle θ with the optic axis, the index ellipse has half-lengths n_o and $n(\theta)$, where

$$\frac{1}{n^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2}, \quad (6.3-15)$$

Refractive Index
of Extraordinary Wave

so that the normal modes have refractive indexes $n_b = n_o$ and $n_a = n(\theta)$. The first mode, called the **ordinary wave**, has a refractive index n_o regardless of θ . In accordance with the ellipse shown in Fig. 6.3-7, the second mode, called the **extraordinary wave**, has a refractive index $n(\theta)$ that varies from n_o when $\theta = 0^\circ$, to n_e when $\theta = 90^\circ$. The vector \mathbf{D} of the ordinary wave is normal to the plane defined by the optic axis (z axis) and the direction of wave propagation \mathbf{k} , and the vectors \mathbf{E} and \mathbf{D} are parallel. The extraordinary wave, on the other hand, has a vector \mathbf{D} that is normal to \mathbf{k} and lies in the k - z plane, and \mathbf{E} is not parallel to \mathbf{D} , as shown in Fig. 6.3-7.

D. Dispersion Relation, Rays, Wavefronts, and Energy Transport

We now examine other properties of waves in anisotropic media including the dispersion relation (the relation between ω and \mathbf{k}).

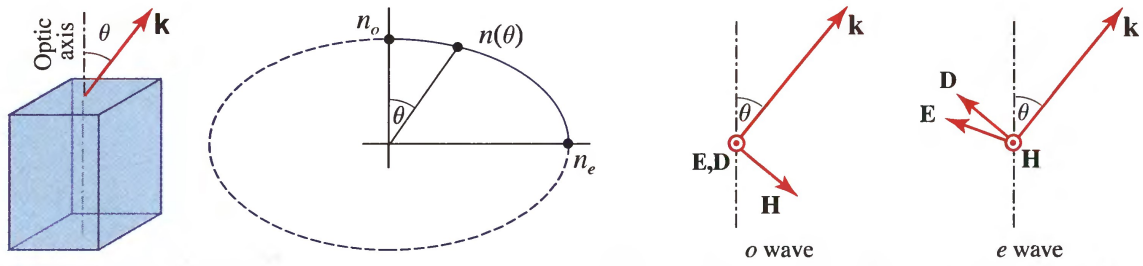


Figure 6.3-7 Variation of the refractive index $n(\theta)$ of the extraordinary wave with θ (the angle between the direction of propagation and the optic axis) in a uniaxial crystal, and directions of the electromagnetic fields of the ordinary (o) and extraordinary (e) waves. The circle with a dot at the center located at the origin signifies that the direction of the vector is out of the plane of the paper, toward the reader.

The optical wave is characterized by the wavevector \mathbf{k} , the field vectors \mathbf{E} , \mathbf{D} , \mathbf{H} , and \mathbf{B} , and the complex Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ (direction of power flow). These vectors are related by (6.3-9) and (6.3-10). It follows from (6.3-9) that \mathbf{D} is normal to both \mathbf{k} and \mathbf{H} . Equation (6.3-10) similarly indicates that \mathbf{H} is normal to both \mathbf{k} and \mathbf{E} . These geometrical conditions are illustrated in Fig. 6.3-8, which also shows the complex Poynting vector \mathbf{S} , which is orthogonal to both \mathbf{E} and \mathbf{H} . Thus, \mathbf{D} , \mathbf{E} , \mathbf{k} , and \mathbf{S} lie in one plane to which \mathbf{H} and \mathbf{B} are normal. In this plane $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{S} \perp \mathbf{E}$; but \mathbf{D} is not necessarily parallel to \mathbf{E} , and \mathbf{S} is not necessarily parallel to \mathbf{k} .

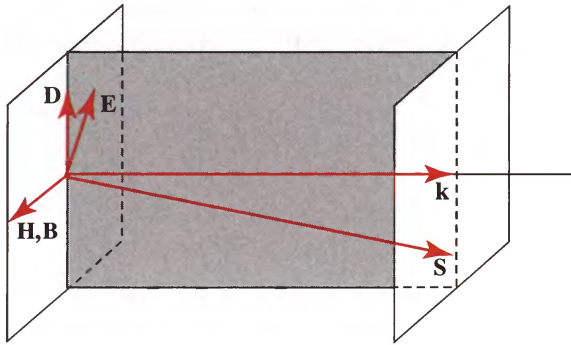


Figure 6.3-8 The vectors \mathbf{D} , \mathbf{E} , \mathbf{k} , and \mathbf{S} all lie in one plane to which \mathbf{H} and \mathbf{B} are normal. $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{E} \perp \mathbf{S}$.

Using the relation $\mathbf{D} = \epsilon\mathbf{E}$ in (6.3-11), we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu_0 \epsilon \mathbf{E} = 0. \quad (6.3-16)$$

This vector equation, which \mathbf{E} must satisfy, translates to three linear homogeneous equations for the components E_1 , E_2 , and E_3 along the principal axes, written in the matrix form

$$\begin{bmatrix} n_1^2 k_o^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_o^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_o^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (6.3-17)$$

where (k_1, k_2, k_3) are the components of \mathbf{k} , $k_o = \omega/c_o$, and (n_1, n_2, n_3) are the principal refractive indexes given by (6.3-6). The condition for these equations to have a nontrivial solution is obtained by setting the determinant of the matrix to zero. The result is an equation that relates ω to k_1 , k_2 , and k_3 and that takes the form $\omega = \omega(k_1, k_2, k_3)$, where $\omega(k_1, k_2, k_3)$ is a nonlinear function. This relation, known as the **dispersion relation**, is the equation of a surface in the k_1, k_2, k_3 space, known

as the **normal surface** or the **k surface**. The intersection of the direction $\hat{\mathbf{u}}$ with the k surface determines the vector \mathbf{k} whose magnitude $k = n\omega/c_o$ provides the refractive index n . There are two intersections corresponding to the two normal modes associated with each direction.

The k surface is a centrosymmetric surface comprising two sheets, each corresponding to a solution (a normal mode). It can be shown that the k surface intersects each of the principal planes in an ellipse and a circle, as illustrated in Fig. 6.3-9. For biaxial crystals ($n_1 < n_2 < n_3$), the two sheets meet at four points, defining two optic axes. In the uniaxial case ($n_1 = n_2 = n_o, n_3 = n_e$), the two sheets become a sphere and an ellipsoid of revolution that meet at only two points, thereby defining a single optic axis (the z axis). In the isotropic case ($n_1 = n_2 = n_3 = n$), the two sheets degenerate into a single sphere.

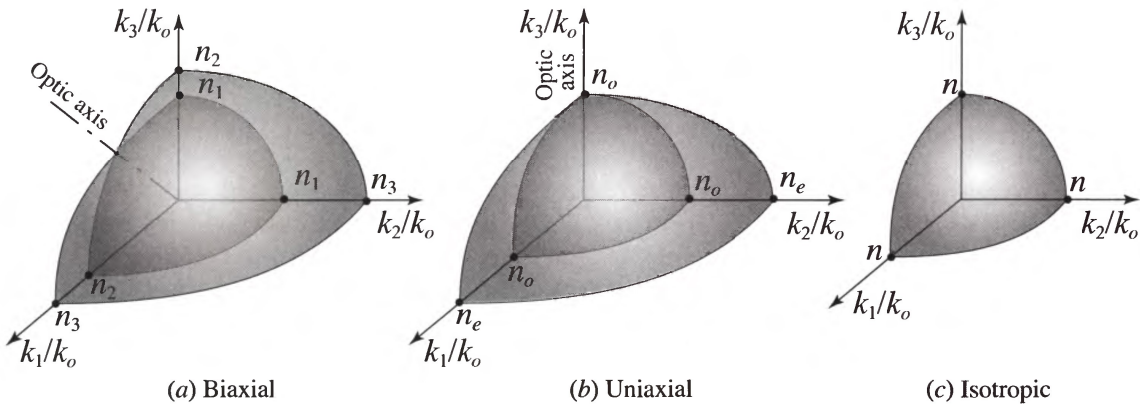


Figure 6.3-9 One octant of the k surface for (a) a biaxial crystal ($n_1 < n_2 < n_3$); (b) a uniaxial crystal ($n_1 = n_2 = n_o, n_3 = n_e$); and (c) an isotropic crystal ($n_1 = n_2 = n_3 = n$).

The intersection of the direction $\hat{\mathbf{u}} = (u_1, u_2, u_3)$ with the k surface corresponds to a wavenumber k that satisfies

$$\sum_{j=1,2,3} \frac{u_j^2 k^2}{k^2 - n_j^2 k_o^2} = 1. \quad (6.3-18)$$

This is a fourth-order equation in k (or second order in k^2). It has four solutions, $\pm k_a$ and $\pm k_b$, of which only the two positive values are meaningful, since the negative values represent a reversed direction of propagation. The problem is therefore solved: the wavenumbers of the normal modes are k_a and k_b and the refractive indexes are $n_a = k_a/k_o$ and $n_b = k_b/k_o$.

To determine the directions of polarization of the two normal modes, we determine the components $(k_1, k_2, k_3) = (ku_1, ku_2, ku_3)$ and the elements of the matrix in (6.3-17) for each of the two wavenumbers $k = k_a$ and $k = k_b$. We then solve two of the three equations in (6.3-17) to determine the ratios E_1/E_3 and E_2/E_3 , from which we determine the direction of the corresponding electric field \mathbf{E} .

The nature of waves in anisotropic media is best explained by examining the k surface $\omega = \omega(k_1, k_2, k_3)$ obtained by equating the determinant of the matrix in (6.3-17) to zero, as illustrated in Fig. 6.3-9. The variation of the phase velocity $c = \omega/k$ with the direction $\hat{\mathbf{u}}$ can be determined from the k surface: the distance from the origin to the k surface in the direction of $\hat{\mathbf{u}}$ is inversely proportional to the phase velocity.

The group velocity may also be determined from the k surface. In analogy with the group velocity $v = d\omega/dk$ that governs the propagation of light pulses (wavepackets), as discussed in Sec. 5.6, the group velocity for *rays* (localized beams or spatial

wavepackets) is the vector $\mathbf{v} = \nabla_{\mathbf{k}}\omega(\mathbf{k})$, the gradient of ω with respect to \mathbf{k} . Since the \mathbf{k} surface is the surface $\omega(k_1, k_2, k_3) = \text{constant}$, \mathbf{v} must be normal to the \mathbf{k} surface. Thus, rays travel along directions normal to the \mathbf{k} surface. The wavefronts are perpendicular to the wavevector \mathbf{k} since the phase of the wave is $\mathbf{k} \cdot \mathbf{r}$. The wavefront normals are therefore parallel to the wavevector \mathbf{k} .

The complex Poynting vector $\mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*$ is also normal to the \mathbf{k} surface. This can be demonstrated by choosing a value for ω and considering two vectors \mathbf{k} and $\mathbf{k} + \Delta\mathbf{k}$ that lie on the \mathbf{k} surface. By taking the differential of (6.3-9) and (6.3-10), and using certain vector identities, it can be shown that $\Delta\mathbf{k} \cdot \mathbf{S} = 0$, so that \mathbf{S} is normal to the \mathbf{k} surface. Consequently, \mathbf{S} is also parallel to the group velocity vector \mathbf{v} .

If the \mathbf{k} surface is a sphere, as it is for isotropic media, the vectors \mathbf{v} , \mathbf{S} , and \mathbf{k} are all parallel, indicating that rays are parallel to the wavevector \mathbf{k} and energy flows in the same direction, as illustrated in Fig. 6.3-10(a). On the other hand, if the \mathbf{k} surface is not normal to the wavevector \mathbf{k} , as illustrated in Fig. 6.3-10(b), the rays and the direction of energy transport are not orthogonal to the wavefronts. Rays then have the “extraordinary” property of traveling at an oblique angle to their wavefronts [Fig. 6.3-10(b)].

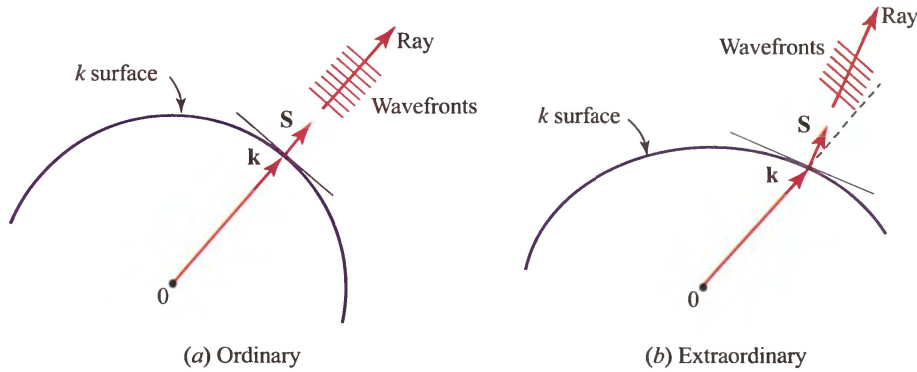


Figure 6.3-10 Rays and wavefronts for (a) a spherical \mathbf{k} surface, and (b) a nonspherical \mathbf{k} surface.

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$), the equation of the \mathbf{k} surface $\omega = \omega(k_1, k_2, k_3)$ simplifies to

$$(k^2 - n_o^2 k_o^2) \left(\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} - k_o^2 \right) = 0. \quad (6.3-19)$$

This equation has two solutions: a sphere, corresponding to the leftmost factor being zero:

$$k = n_o k_o, \quad (6.3-20)$$

and an ellipsoid of revolution, corresponding to the rightmost factor being zero:

$$\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} = k_o^2. \quad (6.3-21)$$

Because of symmetry about the z axis (optic axis), there is no loss of generality in assuming that the vector \mathbf{k} lies in the y - z plane. Its direction is then characterized by

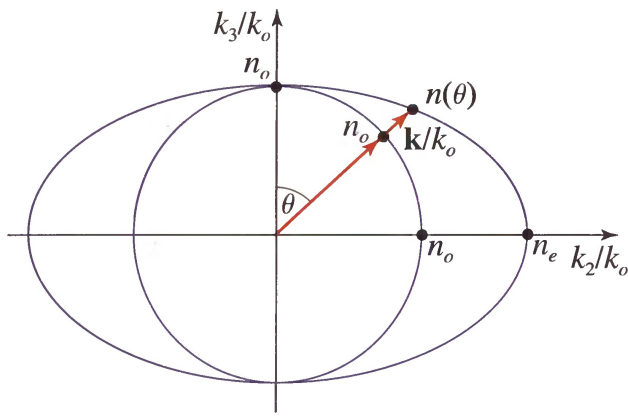


Figure 6.3-11 Intersection of the k surfaces with the y - z plane for a positive uniaxial crystal ($n_e > n_o$).

the angle θ it makes with the optic axis. It is thus convenient to draw the k -surfaces only in the y - z plane, as a circle and an ellipse, as shown in Fig. 6.3-11.

Given the direction $\hat{\mathbf{u}}$ of the vector \mathbf{k} , the wavenumber k is determined by finding the intersection with the k surfaces. The two solutions define the two normal modes, the ordinary and extraordinary waves. The ordinary wave has wavenumber $k = n_o k_o$ regardless of the direction of $\hat{\mathbf{u}}$, whereas the extraordinary wave has wavenumber $n(\theta)k_o$, where $n(\theta)$ is given by (6.3-15), thereby confirming earlier results obtained from the index-ellipsoid geometrical construction. The directions of the rays, wavefronts, energy flow, and field vectors \mathbf{E} and \mathbf{D} for the ordinary and extraordinary waves in a uniaxial crystal are illustrated in Fig. 6.3-12.

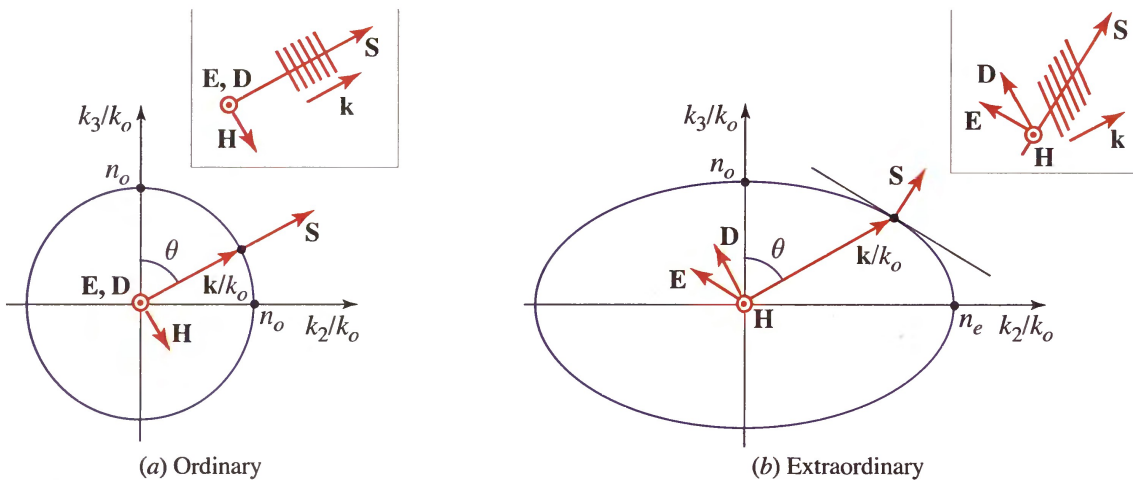


Figure 6.3-12 The normal modes for a plane wave traveling in a direction \mathbf{k} that makes an angle θ with the optic axis z of a uniaxial crystal are: (a) An ordinary wave of refractive index n_o polarized in a direction normal to the k - z plane. (b) An extraordinary wave of refractive index $n(\theta)$ [given by (6.3-15)] polarized in the k - z plane along a direction tangential to the ellipse (the k surface) at the point of its intersection with \mathbf{k} . This wave is “extraordinary” in the following ways: \mathbf{D} is not parallel to \mathbf{E} but both lie in the k - z plane and \mathbf{S} is not parallel to \mathbf{k} so that power does not flow along the direction of \mathbf{k} ; the rays are therefore not normal to the wavefronts so that the wave travels “sideways.”

E. Double Refraction

Refraction of Plane Waves

We now examine the refraction of a plane wave at the boundary between an isotropic medium (say air, $n = 1$) and an anisotropic medium (a crystal). The key principle

that governs the refraction of waves for this configuration is that the wavefronts of the incident and refracted waves must be matched at the boundary. Because the anisotropic medium supports two modes with distinctly different phase velocities, and therefore different indexes of refraction, an incident wave gives rise to two refracted waves with different directions and different polarizations. The effect is known as **double refraction** or **birefringence**.

The phase-matching condition requires that Snell's law be obeyed, i.e.,

$$k_o \sin \theta_1 = k \sin \theta, \quad (6.3-22)$$

where θ_1 and θ are the angles of incidence and refraction, respectively. In an anisotropic medium, however, the wavenumber $k = n(\theta)k_o$ is itself a function of θ , so that

$$\sin \theta_1 = n(\theta_a + \theta) \sin \theta, \quad (6.3-23)$$

where θ_a is the angle between the optic axis and the normal to the surface, so that $\theta_a + \theta$ is the angle the refracted ray makes with the optic axis. Equation (6.3-23) is a modified version of Snell's law. To solve (6.3-22), we draw the intersection of the k surface with the plane of incidence and search for an angle θ for which (6.3-22) is satisfied. Two solutions, corresponding to the two normal modes, are expected. The polarization state of the incident light governs the distribution of energy among the two refracted waves.

Take, for example, a uniaxial crystal and a plane of incidence parallel to the optic axis. The k surfaces intersect the plane of incidence in a circle and an ellipse (Fig. 6.3-13). The two refracted waves that satisfy the phase-matching condition are determined by satisfying (6.3-23):

- An ordinary wave of orthogonal polarization (TE) at an angle $\theta = \theta_o$ for which

$$\sin \theta_1 = n_o \sin \theta_o; \quad (6.3-24)$$

- An extraordinary wave of parallel polarization (TM) at an angle $\theta = \theta_e$, for which

$$\sin \theta_1 = n(\theta_a + \theta_e) \sin \theta_e, \quad (6.3-25)$$

where $n(\theta)$ is given by (6.3-15).

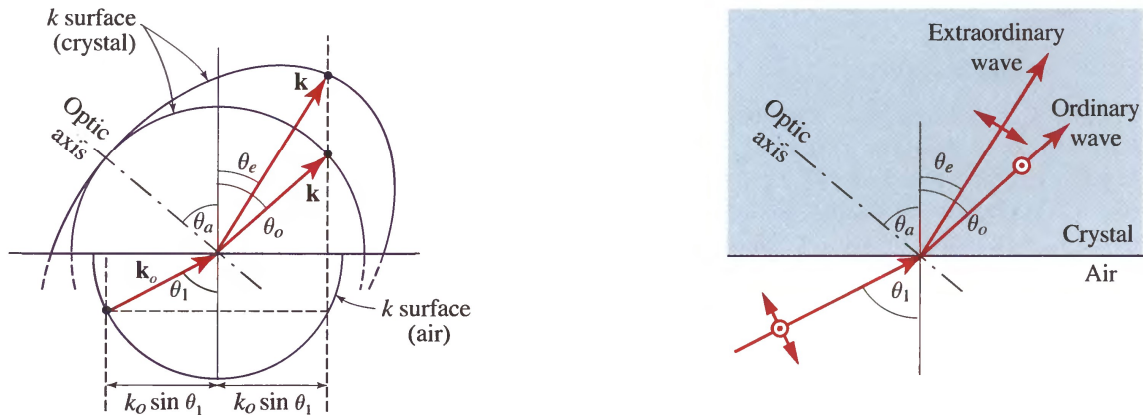


Figure 6.3-13 Determination of the angles of refraction by matching projections of the k vectors in air and in a uniaxial crystal.

If the incident wave carries the two polarizations, the two refracted waves will emerge, as shown in Fig. 6.3-13.

Refraction of Rays

The analysis immediately above dealt with the refraction of plane waves. The refraction of rays is different in an anisotropic medium, since rays do not necessarily travel in directions normal to the wavefronts. In air, before entering the crystal, the wavefronts are normal to the rays. The refracted wave must have a wavevector that satisfies the phase-matching condition, so that Snell's law (6.3-23) is applicable, with the angle of refraction θ determining the direction of \mathbf{k} . However, since the direction of \mathbf{k} is not the direction of the ray, Snell's law is not applicable to rays in anisotropic media.

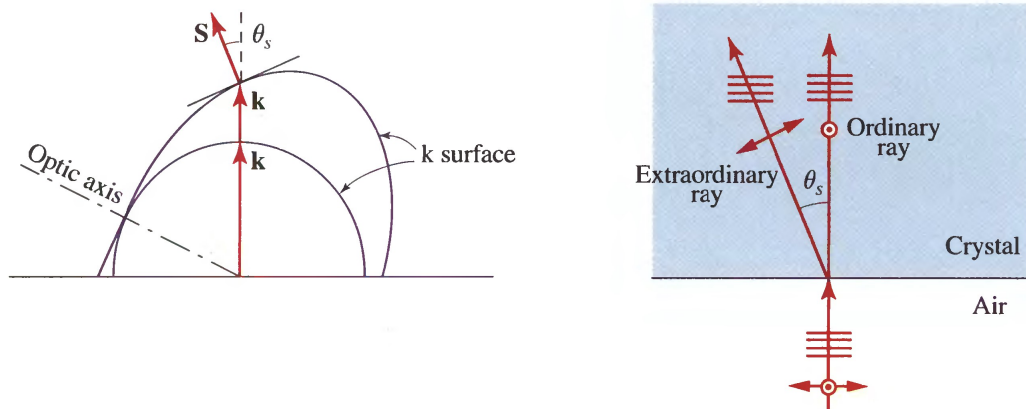


Figure 6.3-14 Double refraction at normal incidence.

An example that dramatizes the deviation from Snell's law is that of normal incidence into a uniaxial crystal whose optic axis is neither parallel nor perpendicular to the crystal boundary. The incident wave has a \mathbf{k} vector normal to the boundary. To ensure phase matching, the refracted waves must also have wavevectors in the same direction. Intersections with the \mathbf{k} surface yield two points corresponding to two waves. The ordinary ray is parallel to \mathbf{k} . But the extraordinary ray points in the direction of the normal to the \mathbf{k} surface, at an angle θ_s with the normal to the crystal boundary, as illustrated in Fig. 6.3-14. Thus, normal incidence creates oblique refraction. The principle of phase matching is maintained, however: wavefronts of both refracted rays are parallel to the crystal boundary and to the wavefront of the incident ray.

When light rays are transmitted through a plate of anisotropic material as described above, the two rays refracted at the first surface refract again at the second surface, creating two laterally separated rays with orthogonal polarizations, as illustrated in Fig. 6.3-15.

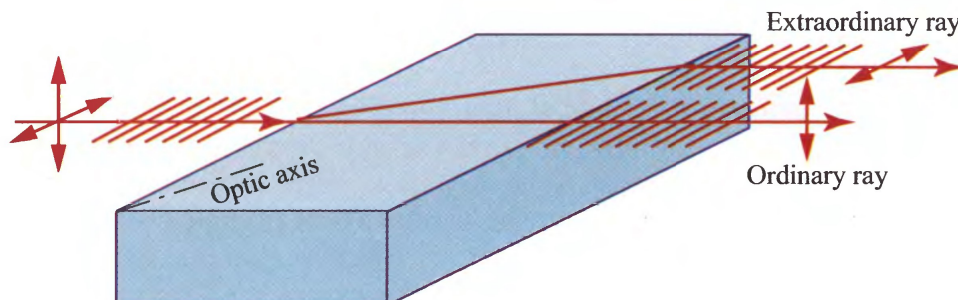


Figure 6.3-15 Double refraction through an anisotropic plate. The plate serves as a polarizing beamsplitter.

6.4 OPTICAL ACTIVITY AND MAGNETO-OPTICS

A. Optical Activity

Certain materials act as natural polarization rotators, a property known as **optical activity**. Their normal modes are waves that are circularly, rather than linearly polarized; waves with right- and left-circular polarizations travel at different phase velocities.

We demonstrate below that an optically active medium with right- and left-circular-polarization phase velocities c_o/n_+ and c_o/n_- acts as a polarization rotator with an angle of rotation $\pi(n_- - n_+)d/\lambda_o$ that is proportional to the thickness of the medium d . The rotatory power (rotation angle per unit length) of the optically active medium is therefore

$$\rho = \frac{\pi}{\lambda_o} (n_- - n_+). \quad (6.4-1)$$

Rotatory Power

The direction in which the polarization plane rotates is the same as that of the circularly polarized component with the greater phase velocity (smaller refractive index). If $n_+ < n_-$, ρ is positive and the rotation is in the same direction as the electric field vector of the right circularly polarized wave [clockwise when viewed from the direction toward which the wave is approaching, as illustrated in Fig. 6.4-1(a)]. Such materials are said to be **dextrorotatory**, whereas those for which $n_+ > n_-$ are termed **levorotatory**.

□ **Derivation of the Rotatory Power.** Equation (6.4-1) may be derived by decomposing the incident linearly polarized wave into a sum of right and left circularly polarized components of equal amplitudes (see Exercise 6.1B),

$$\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = \frac{1}{2} e^{-j\theta} \begin{bmatrix} 1 \\ j \end{bmatrix} + \frac{1}{2} e^{j\theta} \begin{bmatrix} 1 \\ -j \end{bmatrix}, \quad (6.4-2)$$

where θ is the initial angle of the plane of polarization. After propagating a distance d through the medium, the phase shifts encountered by the right and left circularly polarized waves are $\varphi_+ = 2\pi n_+ d/\lambda_o$ and $\varphi_- = 2\pi n_- d/\lambda_o$, respectively, resulting in a Jones vector

$$\frac{1}{2} e^{-j\theta} e^{-j\varphi_+} \begin{bmatrix} 1 \\ j \end{bmatrix} + \frac{1}{2} e^{j\theta} e^{-j\varphi_-} \begin{bmatrix} 1 \\ -j \end{bmatrix} = e^{-j\varphi_o} \begin{bmatrix} \cos(\theta - \varphi/2) \\ \sin(\theta - \varphi/2) \end{bmatrix}, \quad (6.4-3)$$

where $\varphi_o = \frac{1}{2}(\varphi_+ + \varphi_-)$ and $\varphi = \varphi_- - \varphi_+ = 2\pi(n_- - n_+)d/\lambda_o$. This Jones vector represents a linearly polarized wave with the plane of polarization rotated by an angle $\varphi/2 = \pi(n_- - n_+)d/\lambda_o$, as provided in (6.4-1). ■

Optical activity occurs in materials with an intrinsically helical structure. Examples include selenium, tellurium, tellurium oxide (TeO_2), quartz ($\alpha\text{-SiO}_2$), and cinnabar (HgS). Optically active liquids consist of so-called chiral molecules, which come in distinct left- and right-handed mirror-image forms. Many organic compounds, such as amino acids and sugars, exhibit optical activity. Almost all amino acids are levorotatory, whereas common sugars come in both forms: dextrose (d-glucose) and levulose (fructose) are dextrorotatory and levorotatory, respectively, as their names imply. The rotatory power and sense of rotation for solutions of such substances are therefore sensitive to both the concentration and structure of the solute. A saccharimeter is used to determine the optical activity of sugar solutions, from which the sugar concentration is calculated.

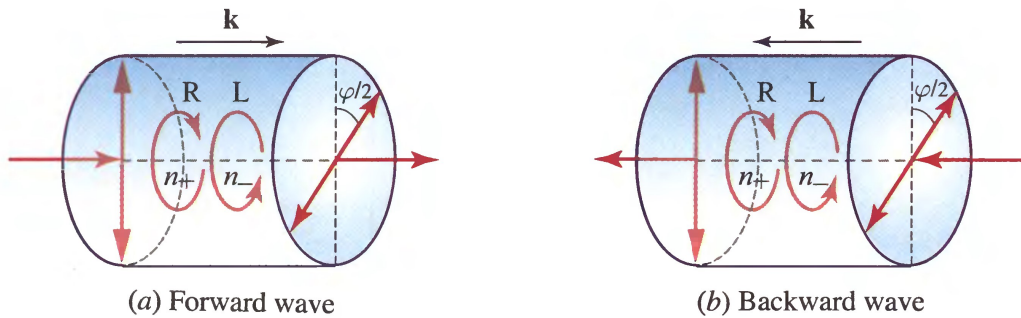


Figure 6.4-1 (a) The rotation of the plane of polarization by an optically active medium results from the difference in the velocities for the two circular polarizations. In this illustration, the right circularly polarized wave (R) is faster than the left circularly polarized wave (L), i.e., $n_+ < n_-$, so that ρ is positive and the material is dextrorotatory. (b) If the wave in (a) is reflected after traversing the medium, the plane of polarization rotates in the opposite direction so that the wave retraces itself.

Material Equations

A time-varying magnetic flux density \mathbf{B} applied to an optically active structure induces a circulating current, by virtue of its helical character, that sets up an electric dipole moment (and hence a polarization) proportional to $j\omega\mathbf{B} = -\nabla \times \mathbf{E}$. The optically active medium is therefore spatially dispersive; i.e., the relation between $\mathbf{D}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ is not local. $\mathbf{D}(\mathbf{r})$ at position \mathbf{r} is determined not only by $\mathbf{E}(\mathbf{r})$, but also by $\mathbf{E}(\mathbf{r}')$ at points \mathbf{r}' in the immediate vicinity of \mathbf{r} , since it is dependent on the spatial derivatives contained in $\nabla \times \mathbf{E}(\mathbf{r})$. For a plane wave, we have $\mathbf{E}(\mathbf{r}) = \mathbf{E} \exp(-j\mathbf{k} \cdot \mathbf{r})$ and $\nabla \times \mathbf{E} = -j\mathbf{k} \times \mathbf{E}$, so that the dielectric permittivity tensor is dependent on the wavevector \mathbf{k} . Spatial dispersiveness is analogous to temporal dispersiveness, which has its origin in the noninstantaneous response of the medium (see Sec. 5.2). While the permittivity of a medium exhibiting temporal dispersion depends on the frequency ω , that of a medium exhibiting spatial dispersion depends on the wavevector \mathbf{k} .

An optically active medium is described by the \mathbf{k} -dependent material equation

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_0 \xi \mathbf{k} \times \mathbf{E}, \quad (6.4-4)$$

where ξ is a quantity (called a pseudoscalar) that changes sign depending on the handedness of the coordinate system. This relation is a first-order approximation of the \mathbf{k} dependence of the permittivity tensor, under appropriate symmetry conditions.[†] The first term represents the response of an isotropic dielectric medium whereas the second term accounts for the optical activity, as will be shown subsequently. This \mathbf{D} - \mathbf{E} relation is often written in the form

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_0 \mathbf{G} \times \mathbf{E}, \quad (6.4-5)$$

where $\mathbf{G} = \xi \mathbf{k}$ is known as the **gyration vector**. In such media the vector \mathbf{D} is clearly not parallel to \mathbf{E} since the vector $\mathbf{G} \times \mathbf{E}$ in (6.4-5) is perpendicular to \mathbf{E} .

Normal Modes of the Optically Active Medium

We proceed to show that the two normal modes of the medium described by (6.4-5) are circularly polarized waves, and we determine the velocities c_0/n_+ and c_0/n_- in terms of the constant $G = \xi k$.

[†] See, for example, L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, Pergamon, 2nd revised ed. 1984, Chapter 12.

We assume that the wave propagates in the z direction, so that $\mathbf{k} = (0, 0, k)$ and thus $\mathbf{G} = (0, 0, G)$. Equation (6.4-5) may then be written in matrix form as

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \epsilon_o \begin{bmatrix} n^2 & -jG & 0 \\ jG & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (6.4-6)$$

where $n^2 = \epsilon/\epsilon_o$. The diagonal elements in (6.4-6) correspond to propagation in an isotropic medium with refractive index n , whereas the off-diagonal elements, proportional to G , represent the optical activity.

To prove that the normal modes are circularly polarized, consider the two circularly polarized waves with electric-field vectors $\mathbf{E} = (E_0, \pm jE_0, 0)$. The $+$ and $-$ signs correspond to right and left circularly polarized waves, respectively. Substitution in (6.4-6) yields $\mathbf{D} = (D_0, \pm jD_0, 0)$, where $D_0 = \epsilon_o(n^2 \pm G)E_0$. It follows that $\mathbf{D} = \epsilon_o n_{\pm}^2 \mathbf{E}$, where

$$n_{\pm} = \sqrt{n^2 \pm G}. \quad (6.4-7)$$

Hence, for either of the two circularly polarized waves the vector \mathbf{D} is parallel to the vector \mathbf{E} . Equation (6.3-11) is satisfied if the wavenumber $k = n_{\pm}k_o$. Thus, the right and left circularly polarized waves propagate without changing their state of polarization, with refractive indexes n_+ and n_- , respectively. They are therefore the normal modes for this medium.

EXERCISE 6.4-1

Rotatory Power of an Optically Active Medium. Show that if $G \ll n$, the rotatory power of an optically active medium (rotation of the polarization plane per unit length) is approximately given by

$$\rho \approx -\frac{\pi G}{\lambda_o n}. \quad (6.4-8)$$

The rotatory power is strongly dependent on the wavelength. Since G is proportional to k , as indicated by (6.4-5), it is inversely proportional to the wavelength λ_o . Thus, the rotatory power in (6.4-8) is inversely proportional to λ_o^2 . Moreover, the refractive index n is itself wavelength dependent. By way of example, the rotatory power ρ of quartz is ≈ 31 deg/mm at $\lambda_o = 500$ nm and ≈ 22 deg/mm at $\lambda_o = 600$ nm; for silver thiogallate (AgGaS_2), ρ is ≈ 700 deg/mm at 490 nm and ≈ 500 deg/mm at 500 nm.

B. Magneto-Optics: The Faraday Effect

Many materials act as polarization rotators in the presence of a static magnetic field, a property known as the **Faraday effect**. The angle of rotation is then proportional to the thickness of the material, and the rotatory power ρ (rotation angle per unit length) is proportional to the component of the magnetic flux density B in the direction of the wave propagation,

$$\rho = \mathfrak{V}B, \quad (6.4-9)$$

where \mathfrak{V} is called the **Verdet constant**.

The sense of rotation is governed by the direction of the magnetic field: for $\mathfrak{V} > 0$, the rotation is in the direction of a right-handed screw pointing in the direction of the magnetic field [Fig. 6.4-2(a)]. In contrast to optical activity, however, the sense of rotation does not reverse with the reversal of the direction of propagation of the wave. Thus, when a wave travels through a Faraday rotator and then reflects back onto itself, traveling once more through the rotator in the opposite direction, it undergoes twice the rotation [Fig. 6.4-2(b)]. Materials that exhibit the Faraday effect include glasses, yt-

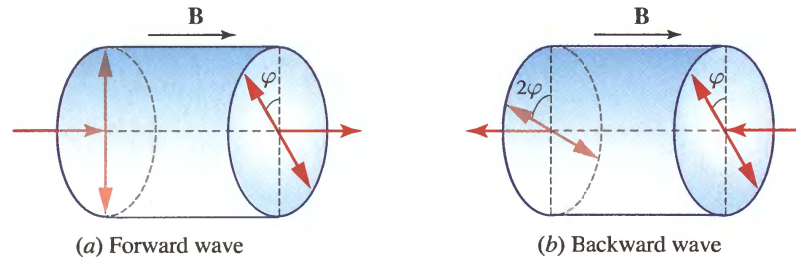


Figure 6.4-2 (a) Polarization rotation in a medium exhibiting the Faraday effect. (b) The sense of rotation is invariant to the direction of travel of the wave.

trium iron garnet (YIG), terbium gallium garnet (TGG), and terbium aluminum garnet (TbAlG). The Verdet constant of TbAlG is $\mathfrak{V} \approx -1.16 \text{ min/Oe-cm}$ at $\lambda_o = 500 \text{ nm}$. Thin films of these ferrimagnetic materials are used to make compact devices.

Material Equations

In magneto-optic materials, the electric permittivity tensor ϵ is altered by the application of a *static* magnetic field \mathbf{H} , so that $\epsilon = \epsilon(\mathbf{H})$. This effect originates from the interaction of the static magnetic field with the motion of the electrons in the material in response to an *optical* electric field \mathbf{E} . For the Faraday effect, in particular, the material equation is

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_o\mathbf{G} \times \mathbf{E} \quad (6.4-10)$$

with

$$\mathbf{G} = \gamma\mathbf{B}. \quad (6.4-11)$$

Here, $\mathbf{B} = \mu\mathbf{H}$ is the static magnetic flux density, and γ is a constant of the medium known as the **magnetogyration coefficient**.

Equation (6.4-10) is identical to (6.4-5) so that the vector $\mathbf{G} = \gamma\mathbf{B}$ in Faraday rotators plays the role of the gyration vector $\mathbf{G} = \xi\mathbf{k}$ in optically active media. For the Faraday effect, however, \mathbf{G} does not depend on \mathbf{k} , so that reversing the direction of propagation does not reverse the sense of rotation of the plane of polarization. This property is useful for constructing optical isolators, as explained in Sec. 6.6C.

With this analogy, and using (6.4-8), we conclude that the rotatory power of the Faraday medium is $\rho \approx -\pi G/\lambda_o n = -\pi\gamma B/\lambda_o n$, from which the Verdet constant (rotatory power per unit magnetic flux density) is seen to be

$$\mathfrak{V} \approx -\frac{\pi\gamma}{\lambda_o n}. \quad (6.4-12)$$

The Verdet constant is clearly a function of the wavelength λ_o .