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Chapter 4

Continuous Translations and Rotations

4.1 Translations and their Representations

4.1.1 Discrete translations and their continuum limit

Consider an infinite one-dimensional lattice of points $x_n = na$, where $n \in \mathbb{Z}$ and a is the lattice spacing. If all sites are equivalent, the relevant symmetry group is \mathbb{Z} , *i.e.* the group of the integers under addition. The unitary representations are labeled by a *wavevector* k , with dimensions of inverse length, such that $D^k(n) = \exp(-ikna)$. Then $D^k(n)D^k(n') = D^k(n+n')$ yadda yadda yadda. Clearly the representation matrices are periodic under $k \rightarrow k + \frac{2\pi}{a}$, hence k may be restricted to the interval $k \in [-\frac{\pi}{a}, +\frac{\pi}{a}]$, with the endpoints identified. Alternatively, we could define $z \equiv \exp(-ika)$ and say that the unitary representations are labeled by a unimodular complex number $z \in S^1$. This interval over which k may be restricted (without loss of generality) is called the *first Brillouin zone* in condensed matter physics.

In d space dimensions, the unitary IRREPS of the group \mathbb{Z}^d are labeled by a d -component wavevector¹ $\mathbf{k} = \{k_1, \dots, k_d\}$, with $D^{\mathbf{k}}(\mathbf{n}) = \exp(-i\mathbf{k} \cdot \mathbf{n}a)$, where $\mathbf{n} = \{n_1, \dots, n_d\} \in \mathbb{Z}^d$. The Brillouin zone becomes a d -dimensional product of intervals of the $d = 1$ type, or, equivalently, a d -dimensional torus T^d , with $\mathbf{z} = \{z_1, \dots, z_d\} \in T^d$. The set of points $\mathbf{R} = a \sum_{j=1}^d n_j \hat{\mathbf{e}}_j$ in d -dimensional space describes a d -dimensional cubic lattice.

In general, we can choose a set of linearly independent vectors $\{\mathbf{a}_j\}$ with $j \in \{1, \dots, d\}$ and define the lattice position $\mathbf{R} = \sum_j n_j \mathbf{a}_j$. This is the construction for a d -dimensional *Bravais lattice*. The representations are given by $D^{\mathbf{k}}(\mathbf{R}) = e^{-i\mathbf{k} \cdot \mathbf{R}}$. The unit cell volume is given by

$$\Omega = \epsilon_{\mu_1 \dots \mu_d} a_1^{\mu_1} \dots a_d^{\mu_d} \quad , \quad (4.1)$$

and is by definition positive². The $\{\mathbf{a}_j\}$ are called the *elementary direct lattice vectors* and by convention one chooses them to have the shortest possible lengths. One can then define the *elementary reciprocal*

¹For $d = 1$, the *wavevector* is in fact a scalar. Ain't that a kick in the head?

²If the expression in Eqn. 4.1 is negative, swap the labels of two of the elementary direct lattice vectors \mathbf{a}_j .

lattice vectors,

$$b_k^\nu \equiv \frac{2\pi}{\Omega} \epsilon_{\mu_1 \dots \mu_{k-1} \nu \mu_{k+1} \dots \mu_d} a_1^{\mu_1} \dots a_{k-1}^{\mu_{k-1}} a_{k+1}^{\mu_{k+1}} \dots a_d^{\mu_d} \quad , \quad (4.2)$$

which satisfy

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij} \quad . \quad (4.3)$$

For example, with $d = 3$ we have $\Omega = \mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$ and

$$\mathbf{b}_1 = \frac{2\pi}{\Omega} \mathbf{a}_2 \times \mathbf{a}_3 \quad , \quad \mathbf{b}_2 = \frac{2\pi}{\Omega} \mathbf{a}_3 \times \mathbf{a}_1 \quad , \quad \mathbf{b}_3 = \frac{2\pi}{\Omega} \mathbf{a}_1 \times \mathbf{a}_2 \quad . \quad (4.4)$$

The Brillouin zone volume is $\hat{\Omega} = (2\pi)^d / \Omega$. The first Brillouin zone is the set of wavevectors \mathbf{k} such that

$$\mathbf{k} = \sum_{j=1}^d \frac{\theta_j}{2\pi} \mathbf{b}_j \quad , \quad (4.5)$$

with $\theta_j \in [-\pi, \pi]$ for all $j \in \{1, \dots, d\}$.

For example, the triangular lattice is described by

$$\mathbf{a}_1 = a \left(\frac{1}{2} \hat{\mathbf{x}} - \frac{\sqrt{3}}{2} \hat{\mathbf{y}} \right) \quad , \quad \mathbf{a}_2 = a \left(\frac{1}{2} \hat{\mathbf{x}} + \frac{\sqrt{3}}{2} \hat{\mathbf{y}} \right) \quad (4.6)$$

with $\Omega = \frac{\sqrt{3}}{2} a^2$. The elementary reciprocal lattice vectors are then

$$\mathbf{b}_1 = \frac{4\pi}{a\sqrt{3}} \left(\frac{\sqrt{3}}{2} \hat{\mathbf{x}} - \frac{1}{2} \hat{\mathbf{y}} \right) \quad , \quad \mathbf{b}_2 = \frac{4\pi}{a\sqrt{3}} \left(\frac{\sqrt{3}}{2} \hat{\mathbf{x}} + \frac{1}{2} \hat{\mathbf{y}} \right) \quad . \quad (4.7)$$

In the continuum limit, $\mathbf{R} \rightarrow \mathbf{r}$ becomes continuous and $D^{\mathbf{k}}(\mathbf{r}) = e^{-i\mathbf{k} \cdot \mathbf{r}}$.

4.1.2 The cyclic group and its continuum limit

Recall how the cyclic group C_N , describing a single n -fold axis, is isomorphic to \mathbb{Z}_N , the group of integer clock arithmetic modulo N . The group elements are $\{1, \omega, \dots, \omega^{N-1}\}$, with $\omega \equiv \exp(-2\pi i/N)$, and the representations Γ_j are labeled by integers $j \in \{0, \dots, N-1\}$, with the 1×1 representation matrices $D^j(\omega^l) = \omega^{jl}$. In the continuum limit $N \rightarrow \infty$, we define $\theta \equiv 2\pi l/N$, and $D^j(\theta) = e^{-ij\theta}$ with $j \in \mathbb{Z}$. Note the periodicity under $\theta \rightarrow \theta + 2\pi$. These are the IRREPS of the group $C_\infty \cong \text{SO}(2)$. Note that there are an infinite number of IRREPS, as there must be upon consideration of the $N \rightarrow \infty$ limit of the formula $N_G = \sum_{\Gamma} d_\Gamma^2$.

Note that $\text{SO}(2)$ is also the group of continuous *translations* in one dimension when *periodic boundary conditions* (PBCs) are imposed. PBCs mean that x is equivalent to $x + L$, where L is the length of the system. This is equivalent to placing our one-dimensional system on a circle. One must then have $D^k(x) = e^{-ikx} = e^{-ik(x+L)}$, i.e. $e^{-ikL} = 1$ for all k , which requires $k = 2\pi j/L$ with $j \in \mathbb{Z}$. Equivalently, define $\theta \equiv 2\pi x/L$, and label the IRREPS by j , in which case we recover $D^j(\theta) = e^{-ij\theta}$.

For discrete translations, we may still implement PBCs, equivalencing sites n and $n + N$, where $L = Na$ is the size of the system. We then have $\exp(ikNa) = 1$, in which case k is among a discrete set,

$$k \in \frac{2\pi}{Na} \left\{ -\frac{1}{2}N, \dots, \frac{1}{2}N \right\} \quad (4.8)$$

where we have assumed N even for simplicity. In the $N \rightarrow \infty$ limit, with a remaining finite, this discrete set becomes the interval $ka \in [-\pi, \pi]$.

4.1.3 Consequences of parity symmetry on the eigenspectrum

One might ask, as I did during a moment of intellectual weakness at 2 am, how it is then that the dispersion relation $\varepsilon(k)$ for a free particle in one dimension satisfies $\varepsilon(k) = \varepsilon(-k) = \hbar^2 k^2 / 2m$, *i.e.* the states other than $k = 0$ are all doubly degenerate. Where are the two-dimensional IRREPs? Now if you are going to ask this question, you should make sure that you ask a very smart person, like my colleague John McGreevy, who will tell you that the symmetries of the free particle are not just continuous translations along the real line, *i.e.* the group \mathbb{R} under addition (or \mathbb{Z} for a discrete infinite chain), but rather $\mathbb{R} \rtimes \mathbb{Z}_2$, where \mathbb{Z}_2 is due to parity, and where, as we discussed in chapter 1, the symbol \rtimes denotes the *semidirect product*. If we denote the parity operation σ and translation through u by $t(u)$ (rather than u itself, which is slightly awkward notation in this context), then the elements of $\mathbb{R} \times \mathbb{Z}_2$ are $t(u)$ and $s(u)$, where $s(u) \equiv \sigma t(u) = t(-u) \sigma$. The multiplication table for $\mathbb{R} \times \mathbb{Z}_2$ is as follows:

$$\begin{aligned} t(u_1) t(u_2) &= t(u_1 + u_2) \\ t(u_1) s(u_2) &= s(u_2 - u_1) \\ s(u_1) t(u_2) &= s(u_1 + u_2) \\ s(u_1) s(u_2) &= t(u_2 - u_1) \quad . \end{aligned} \quad (4.9)$$

Now for the IRREPs, which are still labeled by a real scalar k . The one-dimensional IRREPs are given by

$$D^{(k)}(t(u)) = e^{-iku} \quad , \quad D^{(k)}(s(u)) = 1 \quad . \quad (4.10)$$

However, now there are also two-dimensional IRREPs, given by

$$D^{(k)}(t(u)) = \begin{pmatrix} e^{-iku} & 0 \\ 0 & e^{iku} \end{pmatrix} \quad , \quad D^{(k)}(s(u)) = \begin{pmatrix} 0 & e^{iku} \\ e^{-iku} & 0 \end{pmatrix} \quad . \quad (4.11)$$

The student should verify that the group multiplication law is satisfied by each of these IRREPs. The above results directly generalize to the case where \mathbb{R} is replaced in d dimensions by \mathbb{R}^d (continuous and infinite), \mathbb{T}^d (continuous and periodic) or \mathbb{Z}^d (discrete and infinite) or \mathbb{Z}_n^d (discrete and periodic). Note that in §1.5.3 we discussed how $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$. For all $n \geq 2$, the dihedral groups D_n have two-dimensional IRREPs. And as John notes, \mathbb{R} is essentially indistinguishable from a very large circle \mathbb{S}^1 , which is the continuum limit of a periodic chain.

4.1.4 Invariant measure for $\mathbf{SO}(2)$

Recall the orthogonality and completeness theorems for finite groups:

$$\begin{aligned} \frac{1}{N_G} \sum_{g \in G} D_{ik}^\Gamma(g)^* D_{i'k'}^{\Gamma'}(g) &= \frac{1}{d_\Gamma} \delta_{\Gamma\Gamma'} \delta_{ii'} \delta_{kk'} \\ \sum_{\Gamma, i, k} d_\Gamma D_{ik}^\Gamma(g)^* D_{ik}^\Gamma(g') &= N_G \delta_{gg'} \quad . \end{aligned} \quad (4.12)$$

How do these generalize for continuous (Lie) groups? A natural guess might be

$$\begin{aligned} \int_G d\mu(g) D_{ik}^\Gamma(g)^* D_{i'k'}^{\Gamma'}(g) &\sim \frac{1}{d_\Gamma} \delta_{\Gamma\Gamma'} \delta_{ii'} \delta_{kk'} \\ \sum_{\Gamma, i, k} d_\Gamma D_{ik}^\Gamma(g)^* D_{ik}^\Gamma(g') &\sim \delta(g - g') \quad . \end{aligned} \quad (4.13)$$

Here we assume that while the number of IRREPs is infinite, the dimension d_Γ of each IRREPs is finite. The expression $d\mu(g)$ is an integration measure on the group manifold G , and $\delta(g - g')$ is a generalization of the Dirac delta function³. An important feature of the measure is that it should be invariant under replacement of g by gh , where h is any element in G . I.e. $d\mu(g) = d\mu(gh)$ for all $g, h \in G$. This is because the group integration is the continuum limit of a sum over all the elements of a finite group, and by rearrangement we know that $\sum_{g \in G} F(g)$ is the same as $\sum_{g \in G} F(gh^{-1})$ for all h . For $G = \mathbf{SO}(2)$, the group elements $g = g(\phi)$ are parameterized by the angle variable $\phi \in [0, 2\pi]$. We then have

$$d\mu(g) = \rho(\phi) d\phi \quad , \quad (4.14)$$

where $\rho(\phi)$ is a weighting function. The condition that $d\mu(g)$ be an invariant measure means that it is unchanged under group multiplication $g \rightarrow gh$ for all fixed h . This means $\rho(\phi) = \rho(\phi + \alpha)$ for all $\alpha \in [0, 2\pi)$, hence $\rho(\phi) = C$, a constant. Normalizing $\int_G d\mu(g) \equiv 1$, we have $\rho(\phi) = \frac{1}{2\pi}$. The completeness and orthonormality relations then become

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi}{2\pi} [D^j(\phi)]^* D^{j'}(\phi) &= \delta_{jj'} \\ \sum_{j=-\infty}^{\infty} [D^j(\phi)]^* D^j(\phi') &= 2\pi \delta(\phi - \phi') \quad . \end{aligned} \quad (4.15)$$

with $D^j(\phi) = e^{-ij\phi}$.

³The formal statement of completeness is known as the *Peter-Weyl theorem*, which we shall discuss in §4.3.5. Since we do not add elements of the group itself, the notation $\delta(g - g')$ is problematic, and we shall see further below how to make proper sense of all this.

4.2 Remarks about $SO(N)$ and $SU(N)$

4.2.1 How $SO(N)$ acts on vectors, matrices, and tensors

$SO(2)$ is abelian and while important in physics, it is rather trivial from the point of view of group theory. We've just classified all its unitary representations. On to $SO(N)$, which is nonabelian for $N > 2$.

$SO(N) \subset GL(N, \mathbb{R})$ is the group of proper rotations in N Euclidean dimensions, *i.e.* the group of $N \times N$ real matrices R satisfying $RR^T = R^T R = \mathbf{1}$. Acting on the vector space \mathbb{R}^N , rotation matrices act on basis vectors in the following manner:

$$\hat{e}^a \rightarrow \hat{e}'^a = \hat{e}^b R_{ba} \quad . \quad (4.16)$$

Thus with $\mathbf{x} = x_a \hat{e}^a$, we have

$$\mathbf{x}' = x_a \hat{e}'^a = x_a \hat{e}^b R_{ba} \Rightarrow x'_a = R_{ab} x_b \quad . \quad (4.17)$$

The group also acts on $N \times N$ matrices, with $M' = RMR^T$, *i.e.*

$$M'_{ij} = R_{ia} R_{jb} M_{ab} = (RMR^T)_{ij} \quad . \quad (4.18)$$

A matrix M_{ij} may be generalized to an object with more indices, called a *tensor*. A p -tensor carries p indices $\{i_1, \dots, i_p\}$, with each $i_p \in \{1, \dots, N\}$, and is denoted $M_{i_1 i_2 \dots i_p}$. Under elements of $SO(N)$, an n -tensor transforms thusly:

$$M'_{i_1 \dots i_p} = R_{i_1 a_1} \cdots R_{i_p a_p} M_{a_1 \dots a_p} \quad . \quad (4.19)$$

The transformation coefficients $C_{a_1 \dots a_p}^{i_1 \dots i_p}(R) \equiv R_{i_1 a_1} \cdots R_{i_p a_p}$ form a representation of $SO(N)$ because

$$C_{a_1 \dots a_p}^{i_1 \dots i_p}(S) C_{j_1 \dots j_p}^{a_1 \dots a_p}(R) = C_{j_1 \dots j_p}^{i_1 \dots i_p}(SR) \quad . \quad (4.20)$$

What is the dimension of this representation? It is given by the dimension of the space on which the transformation coefficients act, *i.e.* the space of p -tensors M , where each index runs over N possible values. Thus, we have obtained a representation of dimension N^p .

But is this representation reducible? To address this question, let's first think about the case $n = 2$, *i.e.* good old matrices. Our representation is then of dimension N^2 . But any matrix M^{ik} can be written as

$$M^{ik} = \overbrace{\frac{1}{N} \text{Tr } M \delta^{ik}}^c \delta^{ik} + \overbrace{\frac{1}{2} (M^{ik} - M^{ki})}^A + \overbrace{\frac{1}{2} (M^{ik} + M^{ki}) - \frac{1}{N} \text{Tr } M \delta^{ik}}^S \quad (4.21)$$

where $c = \frac{1}{N} \text{Tr } M$, $A = -A^T$ is an antisymmetric matrix, and $S = S^T$ is a traceless symmetric matrix. Orthogonal transformations preserve all these forms: RR^T is the identity, RAR^T is itself antisymmetric, and RSR^T is itself traceless and symmetric. Note that A has $\frac{1}{2}N(N-1)$ independent components, and S has $\frac{1}{2}N(N+1) - 1$ independent components. Thus, our representation, which is N^2 -dimensional, reduces as

$$N^2 = 1 \oplus \frac{1}{2}N(N-1) \oplus [\frac{1}{2}N(N+1) - 1] \quad . \quad (4.22)$$

For $N = 3$, this says $9 = 1 \oplus 3 \oplus 5$. Note that the dimension of the antisymmetric tensor representation is the same as that of the vector representation. This is because a three component vector is *dual* to a 3×3 antisymmetric 2-tensor (*i.e.* matrix).

4.2.2 Invariant symbols, dual tensors, and up/down index notation

Any matrix $R \in \text{SO}(3)$ preserves the Kronecker delta symbol δ^{ab} as well as the totally antisymmetric symbol ϵ^{abc} . Recall that

$$\epsilon_{i_1 \dots i_N} = \text{sgn} \begin{pmatrix} 1 & 2 & \dots & N \\ i_1 & i_2 & \dots & i_N \end{pmatrix} . \quad (4.23)$$

and that $\epsilon_{i_1 \dots i_N} = 0$ if any of the indices i_p are repeated. Note that for any matrix R , one has

$$\begin{aligned} R^i_a R^j_b \delta^{ab} &= (RR^T)_{ij} \\ R^{i_1}_{a_1} \dots R^{i_N}_{a_N} \epsilon^{a_1 \dots a_N} &= \det(R) \epsilon^{i_1 \dots i_N} . \end{aligned} \quad (4.24)$$

Hence if $R \in \text{SO}(N)$, the Kronecker and epsilon symbols remain invariant under an orthogonal transformation. The raised and lowered indices don't do anything but aid us in identifying which pairs are to be contracted, *i.e.* $R^i_j = R_i^j = R_{ij}$. We always contract an upper index with a lower index⁴.

True facts about the epsilon symbol:

$$\begin{aligned} \epsilon^{i_1 \dots i_N} \epsilon_{i_1 \dots i_N} &= N! \\ \epsilon^{i_1 \dots i_N} \epsilon_{j_1 i_2 \dots i_N} &= (N-1)! \delta^{i_1}_{j_1} \\ \epsilon^{i_1 \dots i_N} \epsilon_{j_1 j_2 i_3 \dots i_N} &= (N-2)! \left(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right) . \end{aligned} \quad (4.25)$$

The general case:

$$\epsilon^{i_1 \dots i_N} \epsilon_{j_1 \dots j_K i_{K+1} \dots i_N} = (N-K)! \sum_{\sigma \in S_K} \text{sgn}(\sigma) \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_K}_{j_{\sigma(K)}} . \quad (4.26)$$

Given a totally antisymmetric K -tensor $A^{a_1 \dots a_K}$, we may use the ϵ -symbol to construct its *dual*, which is a totally antisymmetric $(N-K)$ -tensor $\tilde{A}^{i_1 \dots i_{N-K}}$, *viz.*

$$\tilde{A}^{i_1 \dots i_{N-K}} \equiv \frac{1}{K!} \epsilon_{i_1 \dots i_{N-K} a_1 \dots a_K} A^{a_1 \dots a_K} . \quad (4.27)$$

What is the dual of the dual? We have

$$\begin{aligned} \tilde{\tilde{A}}^{b_1 \dots b_K} &= \frac{1}{(N-K)!} \epsilon^{b_1 \dots b_K i_1 \dots i_{N-K}} \tilde{A}_{i_1 \dots i_{N-K}} \\ &= \frac{1}{K!(N-K)!} \epsilon^{b_1 \dots b_K i_1 \dots i_{N-K}} \epsilon_{i_1 \dots i_{N-K} a_1 \dots a_K} A^{a_1 \dots a_K} \\ &= \frac{(-1)^{K(N-K)}}{K!} \sum_{\sigma \in S_K} \text{sgn}(\sigma) \delta^{b_1}_{a_{\sigma(1)}} \dots \delta^{b_K}_{a_{\sigma(K)}} A^{a_1 \dots a_K} = (-1)^{K(N-K)} A^{b_1 \dots b_K} . \end{aligned} \quad (4.28)$$

⁴Since R is not necessarily a symmetric matrix, we offset the upper and lower indices to indicate which is the row and which is the column index, *i.e.* $R^i_j = R_{ij}$ but $R_j^i = R_{ji}$. For diagonal matrices like the δ -symbol, we don't need to do this, and we write $\delta^a_b = \delta_{ab} = \delta_{ba}$. We can use the δ -symbol to raise and lower indices, *viz.* $\delta^{ij} M_{jk} = M^i_k$. In relativistic theories, the metric tensor $g_{\mu\nu} = \text{diag}(+, -, -, -)$ is used to raise and lower indices, which introduces sign changes.

Thus, up to a sign, the dual of the dual tensor is the original tensor. We see that for $N = 3$, the dual of a vector V^a is the antisymmetric tensor

$$\tilde{V}_{ab} = \epsilon_{abc} V^c = \begin{pmatrix} 0 & +V^3 & -V^2 \\ -V^3 & 0 & +V^1 \\ +V^2 & -V^1 & 0 \end{pmatrix} . \quad (4.29)$$

This establishes the equivalence between vector and antisymmetric matrix representations of $SO(3)$. For $N = 4$, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \Rightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix} . \quad (4.30)$$

One can also readily establish that if

$$B^{b_1 \dots b_K} = R^{b_1}_{a_1} \dots R^{b_K}_{a_K} A^{a_1 \dots a_K} , \quad (4.31)$$

then

$$\tilde{B}_{j_1 \dots j_{N-K}} = R_{j_1}^{i_1} \dots R_{j_{N-K}}^{i_{N-K}} \tilde{A}_{i_1 \dots i_{N-K}} , \quad (4.32)$$

i.e. the dual of the orthogonal transform is the orthogonal transform of the dual.

4.2.3 Tensor representations of $SU(N)$

Let $U \in SU(N)$. Thus, $U_{ia} U_{aj}^\dagger = U_{ia} U_{ja}^* = \delta_{ij}$. Let Q be a rank $(p + q)$ tensor, with components $Q_{a_1 \dots a_p, b_1 \dots b_q}$. We define

$$Q'_{a_1 \dots a_p, b_1 \dots b_q} = C_{a_1 \dots a_p, b_1 \dots b_q}^{a'_1 \dots a'_p, b'_1 \dots b'_q}(U) Q_{a'_1 \dots a'_p, b'_1 \dots b'_q} \quad (4.33)$$

where

$$C_{a'_1 \dots a'_p, b'_1 \dots b'_q}^{a_1 \dots a_p, b_1 \dots b_q}(U) = U_{a_1 a'_1} \dots U_{a_p a'_p} U_{b_1 b'_1}^* \dots U_{b_q b'_q}^* . \quad (4.34)$$

This forms a representation of $SU(N)$ because

$$C_{a'_1 \dots a'_p, b'_1 \dots b'_q}^{a_1 \dots a_p, b_1 \dots b_q}(U) C_{a''_1 \dots a''_p, b''_1 \dots b''_q}^{a'_1 \dots a'_p, b'_1 \dots b'_q}(V) = C_{a''_1 \dots a''_p, b''_1 \dots b''_q}^{a_1 \dots a_p, b_1 \dots b_q}(UV) . \quad (4.35)$$

As with the special orthogonal group, the Levi-Civita ϵ symbol is an invariant tensor:

$$U_{a_1 b_1} U_{a_2 b_2} \dots U_{a_N b_N} \epsilon^{b_1 \dots b_N} = \overbrace{\det(U)}^{=1} \epsilon^{a_1 \dots a_N} . \quad (4.36)$$

Consider the tensor representation of $SU(N)$ with $C_{a'_1 a'_2}^{a_1 a_2}(U) = U_{a_1 a'_1} U_{a_2 a'_2}$. For $SO(N)$, there is always a one-dimensional IRREP where the tensor $M_{a_1 a_2} = A \delta_{a_1 a_2}$, because $R_{a_1 a'_1} R_{a_2 a'_2} \delta_{a'_1 a'_2} = (RR^\top)_{a_1 a_2} = \delta_{a_1 a_2}$.

Not so for $SU(N)$, because $UU^\top \neq 1$ in general. Still, symmetric and antisymmetric tensors transform among their respective kinds, hence rather than eqn. 4.22, we have

$$N^2 = \frac{1}{2}N(N-1) \oplus \frac{1}{2}N(N+1) \quad . \quad (4.37)$$

Thus, the trivial IRREP in $SO(N)$ adjoins in $SU(N)$ to the symmetric matrix IRREP to form a larger symmetric matrix IRREP of dimension $\frac{1}{2}N(N+1)$. The general classification scheme for IRREPs of $SO(N)$ and $SU(N)$ is facilitated by the use of Young tableaux similar to those encountered in our study of the symmetric group. The rules for counting IRREPs and their dimensions are different, however. In the language of Young diagrams,

$$\begin{array}{|c|} \hline \square \\ \hline N \end{array} \times \begin{array}{|c|} \hline \square \\ \hline N \end{array} = \begin{array}{|c|} \hline \square \\ \hline \frac{1}{2}N(N-1) \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \frac{1}{2}N(N+1) \end{array} \quad . \quad (4.38)$$

Next, consider the tensor representation of $SU(N)$ with $C_{a'_1 a'_2}^{a_1 a_2}(U) = U_{a_1 a'_1} U_{a_2 a'_2}^*$. Now we find that the Kronecker matrix $\delta_{a_1 a_2}$ does indeed remain invariant, hence there is a trivial one-dimensional representation in the decomposition of this representation. However, one can also see that symmetric and antisymmetric matrices will in general mix under this transformation, hence the symmetric and antisymmetric IRREPs of $SO(N)$ adjoin in $SU(N)$ to yield an IRREP of dimension $N^2 - 1$, which is the adjoint representation:

$$N \times \overline{N} = \text{id} \oplus \text{adj} \quad . \quad (4.39)$$

Here we denote by N the fundamental IRREP of $SU(N)$, and by \overline{N} the *antifundamental* (i.e. complex conjugate) IRREP.

4.3 $SO(3)$ and $SU(2)$

Recall that $SO(3)$ is the matrix Lie group of rotations in Euclidean 3-space. Its elements can be represented as $R(\xi, \hat{n})$, meaning a (right-handed) rotation by ξ about \hat{n} , with $\xi \in [0, \pi]$. As discussed in chapter 1, topologically this means that each element of $SO(3)$ can be associated with a point $\xi \hat{n}$ in a filled sphere of radius π . Since $R(\pi, \hat{n}) = R(\pi, -\hat{n})$, points on the surface of this sphere are identified with their antipodes, resulting in $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

The Lie algebra $\mathfrak{so}(3)$ consists of real antisymmetric 3×3 matrices. We can define a basis for this algebra,

$$\Sigma^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \quad \Sigma^y = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Sigma^z = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \quad (4.40)$$

Note that $\Sigma_{ij}^a = -\epsilon_{aij}$, from which one easily establishes the commutation relations $[\Sigma^a, \Sigma^b] = \epsilon_{abc} \Sigma^c$. Then with $\xi = \xi \hat{n}$, we have $R(\xi, \hat{n}) = \exp(\xi \cdot \Sigma) = \exp(\xi \hat{n} \cdot \Sigma)$. Note that

$$(\hat{n} \cdot \Sigma)_{ab} = \begin{pmatrix} 0 & -n^z & +n^y \\ +n^z & 0 & -n^x \\ -n^y & +n^x & 0 \end{pmatrix}_{ab} = -\epsilon_{abc} n^c \quad . \quad (4.41)$$

Squaring this, we obtain

$$[(\hat{\mathbf{n}} \cdot \boldsymbol{\Sigma})^2]_{ac} = \epsilon_{abd} n^d \epsilon_{bce} n^e = n^a n^c - \delta^{ac} \equiv -Q_{\hat{\mathbf{n}}}^{ac} \quad , \quad (4.42)$$

where $Q_{\hat{\mathbf{n}}}^{ab} = \delta^{ab} - n^a n^b$ is the orthogonal projector with respect to $\hat{\mathbf{n}}$. The projector onto $\hat{\mathbf{n}}$ is of course $P_{\hat{\mathbf{n}}}^{ab} = n^a n^b$. Defining $\Sigma_{\hat{\mathbf{n}}} \equiv \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma}$, it is now easy to show that

$$P_{\hat{\mathbf{n}}} Q_{\hat{\mathbf{n}}} = Q_{\hat{\mathbf{n}}} P_{\hat{\mathbf{n}}} = 0 \quad , \quad P_{\hat{\mathbf{n}}} \Sigma_{\hat{\mathbf{n}}} = \Sigma_{\hat{\mathbf{n}}} P_{\hat{\mathbf{n}}} = 0 \quad , \quad Q_{\hat{\mathbf{n}}} \Sigma_{\hat{\mathbf{n}}} = \Sigma_{\hat{\mathbf{n}}} Q_{\hat{\mathbf{n}}} = \Sigma_{\hat{\mathbf{n}}} \quad , \quad (4.43)$$

as well as the projector relations $P_{\hat{\mathbf{n}}}^2 = P_{\hat{\mathbf{n}}}$ and $Q_{\hat{\mathbf{n}}}^2 = Q_{\hat{\mathbf{n}}}$ and our previous result $\Sigma_{\hat{\mathbf{n}}}^2 = -Q_{\hat{\mathbf{n}}}$.

From these relations, we may sum the exponential series for $R(\xi, \hat{\mathbf{n}})$ to obtain

$$R_{ab}(\xi, \hat{\mathbf{n}}) = \exp(\xi \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma}) = n^a n^b + (\delta^{ab} - n^a n^b) \cos \xi - \epsilon_{abc} n^c \sin \xi \quad . \quad (4.44)$$

It is also a simple matter to show that if $S \in \text{SO}(3)$, then

$$S R(\xi, \hat{\mathbf{n}}) S^{-1} = R(\xi, \hat{\mathbf{n}}') \quad (4.45)$$

where $\hat{\mathbf{n}}' = S\hat{\mathbf{n}}$. This means that rotations through a fixed angle ξ form an equivalence class. Recall from chapter 1 how $\text{SO}(3)$ is topologically equivalent to a three-dimensional sphere of radius π , with radial coordinate ξ and angular coordinates given by the unit vector $\hat{\mathbf{n}}$. The condition $R(\pi, \hat{\mathbf{n}}) = R(\pi, -\hat{\mathbf{n}})$ means that $\text{SO}(3)$ is multiply connected, with $\pi_1(\text{SO}(3)) \simeq \mathbb{Z}_2$. Thus the equivalence classes of $\text{SO}(3)$ correspond to concentric two-dimensional spheres, with antipodes identified on the surface $\xi = \pi$.

In the physics literature, the $\text{so}(3)$ generators are Hermitian, and we write $J^a = i\Sigma^a = D(\hat{J}^a)$ is a 3×3 matrix representation of the operator \hat{J}^a , where the familiar commutation relations $[\hat{J}^a, \hat{J}^b] = i\epsilon_{abc} \hat{J}^c$ hold for both the angular momentum operator \hat{J}^a as well as its representation matrices J^a . Thus,

$$R(\xi, \hat{\mathbf{n}}) = \exp(-i\xi \hat{\mathbf{n}} \cdot \mathbf{J}) = D[\exp(-i\xi \hat{\mathbf{n}} \cdot \hat{\mathbf{J}})] \quad , \quad (4.46)$$

is the matrix representation of the rotation operator $\hat{R}(\xi, \hat{\mathbf{n}}) = \exp(-i\xi \hat{\mathbf{n}} \cdot \hat{\mathbf{J}})$. Rather than the $(\xi, \hat{\mathbf{n}})$ parameterization, we could also choose to parameterize a general $R \in \text{SO}(3)$ by the Euler angles (α, β, γ) familiar from the classical mechanics of rotating bodies⁵, where $\alpha, \gamma \in [0, 2\pi)$ and $\beta \in [0, \pi]$. The general rotation operation in terms of the Euler angles is depicted in Fig. 4.1 and is given by

$$R(\alpha, \beta, \gamma) = \exp(-i\gamma J^{z''}) \exp(-i\beta J^{y'}) \exp(-i\alpha J^z) \quad . \quad (4.47)$$

Here $\exp(-i\alpha J^z)$ rotates by α about the original $\hat{\mathbf{z}} = \hat{\mathbf{e}}_3^0$ axis, $\exp(-i\beta J^{y'})$ by β about the new $\hat{\mathbf{y}}' = \hat{\mathbf{e}}_2'$ axis, and $\exp(-i\gamma J^{z''})$ by γ about the new $\hat{\mathbf{z}}'' = \hat{\mathbf{e}}_3''$ axis. Then

$$\begin{aligned} \exp(-i\gamma J^{z''}) &= \exp(-i\beta J^{y'}) \exp(-i\gamma J^z) \exp(i\beta J^{y'}) \\ \exp(-i\beta J^{y'}) &= \exp(-i\alpha J^z) \exp(-i\beta J^y) \exp(i\alpha J^z) \end{aligned} \quad (4.48)$$

and so we find

$$R(\alpha, \beta, \gamma) = \exp(-i\alpha J^z) \exp(-i\beta J^y) \exp(-i\gamma J^z) \quad . \quad (4.49)$$

⁵Since we reflexively parameterize the unit vector $\hat{\mathbf{n}}$ in terms of its polar angle θ and azimuthal angle ψ , we'll call the Euler angles (α, β, γ) rather than the also common (ϕ, θ, ψ) to obviate any confusion.

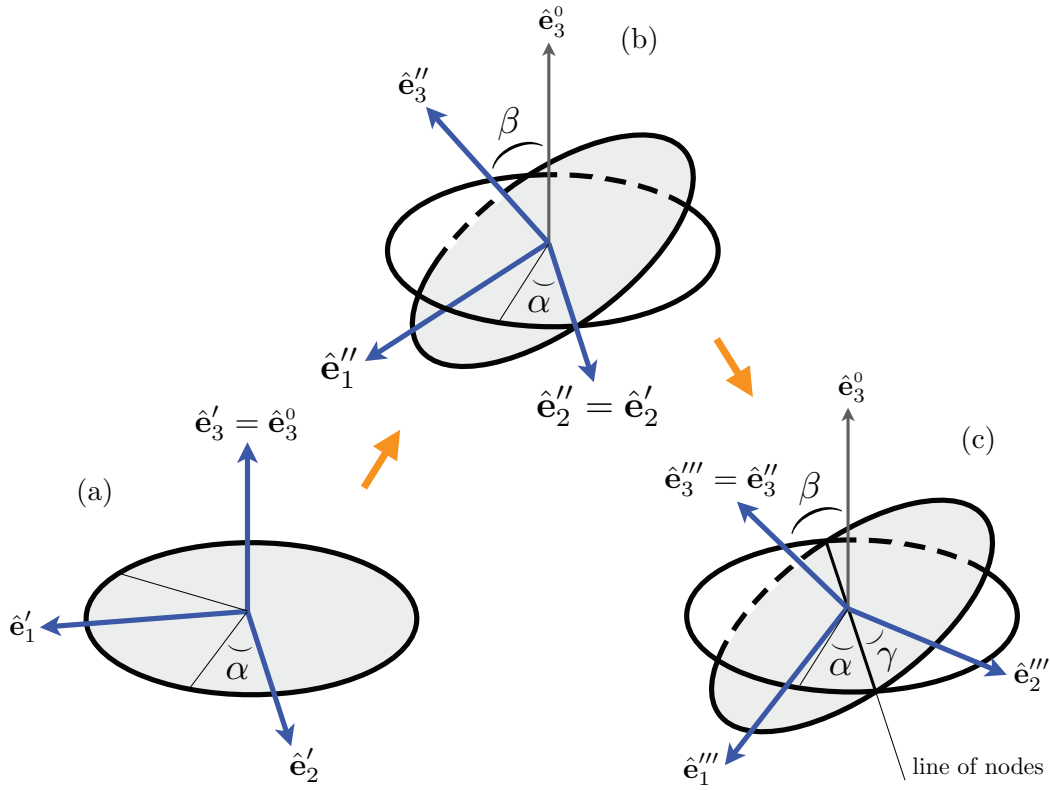


Figure 4.1: Definition of Euler angles and their order of operations in Eqn. 4.47.

Thus, we obtain an expression which looks very much like that in Eqn. 4.47, except (i) the rotations are now about lab-fixed axes and (ii) the order of operations is reversed. Identifying $R(\alpha, \beta, \gamma) \equiv R(\xi, \hat{n})$, one obtains a relation⁶

$$\phi = \frac{1}{2}(\pi + \alpha - \gamma) \quad , \quad \tan \theta = \frac{\tan(\frac{1}{2}\beta)}{\sin(\frac{1}{2}\alpha + \frac{1}{2}\gamma)} \quad , \quad \cos^2(\frac{1}{2}\xi) = \cos^2(\frac{1}{2}\beta) \cos^2(\frac{1}{2}\alpha + \frac{1}{2}\gamma) \quad . \quad (4.50)$$

Note that

$$[R(\alpha, \beta, \gamma)]^{-1} = \exp(i\gamma J^z) \exp(i\beta J^y) \exp(i\alpha J^z) \quad . \quad (4.51)$$

Explicitly, we may write

$$\begin{aligned} R(\alpha, \beta, \gamma) &= \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix} \quad . \end{aligned} \quad (4.52)$$

For future reference, we note that

$$\text{Tr } R(\alpha, \beta, \gamma) = \cos(\alpha + \gamma) \cos \beta + \cos(\alpha - \gamma) + \cos \beta \quad . \quad (4.53)$$

⁶See Wu-Ki Tung, *Group Theory in Physics*, p. 99.

4.3.1 Irreducible representations of SO(3)

We now promote the generators and group elements to operators acting on Hilbert space, writing $\hat{R}(\alpha, \beta, \gamma)$, \hat{J}^a , etc., and we seek representations of SO(3) which can be used to classify the eigenstates of a rotationally invariant Hamiltonian, *i.e.* one for which $[\hat{H}, \hat{R}(\xi, \hat{n})] = 0$ for all rotations $\hat{R}(\xi, \hat{n})$. Clearly it is enough that $[\hat{H}, J^a] = 0$ for all the generators J^a , so we will seek a representation of the Lie algebra so(3). This will yield a representation of the group SO(3) itself, provided global conditions such as $\hat{R}(\pi, \hat{n}) = \hat{R}(\pi, -\hat{n})$ are met. As we shall soon see, we will be led naturally to a set of projective representations, which you already know correspond to half-odd integer total angular momentum, as well as proper representations, corresponding to integer angular momentum.

We will seek finite-dimensional representations in which the generators \hat{J}^a are all represented by Hermitian matrices. In general we can pull this off for compact Lie groups like SO(3). We'll follow a general point of attack outlined by Élie Cartan, whose work in the first half of the 20th century laid the foundations for the theory of Lie groups. Cartan's approach was to start with a standard vector $|\psi_0\rangle \in \mathcal{H}$ and to generate the remaining vectors in an irreducible basis by iteratively applying various generators. One important entity which helps us label the IRREPS is the existence of a *Casimir operators*. A Casimir is an operator \hat{C} which commutes with all elements of the group, and hence with all operators in the Lie algebra. For so(3), this is $\hat{C} = \hat{\mathbf{J}}^2$. By Schur's first lemma, this means \hat{C} is a multiple of the identity, hence the individual IRREPS may be labeled by the eigenvalues $\{C_1, \dots, C_K\}$ of all the Casimirs, where K is the total number of Casimirs⁷.

For so(3), following Cartan's method, it is useful to define the ladder operators $\hat{J}^\pm = \hat{J}^x \pm i\hat{J}^y$. Then

$$[\hat{J}^z, \hat{J}^\pm] = \pm \hat{J}^\pm \quad , \quad [\hat{J}^+, \hat{J}^-] = 2\hat{J}^z \quad (4.54)$$

as well as $\hat{\mathbf{J}}^2 = (\hat{J}^z)^2 + \hat{J}^z + \hat{J}^- \hat{J}^+$ and $(\hat{J}^\pm)^\dagger = \hat{J}^\mp$. In the vector space \mathcal{V} of our representation, we label the basis vectors by the eigenvalues of the Hermitian operator \hat{J}^z as write them as $|m\rangle$, with $\hat{J}^z |m\rangle = m |m\rangle$. We then apply \hat{J}^+ , and from the commutation relations we have

$$\hat{J}^z \hat{J}^+ |m\rangle = (m+1) \hat{J}^+ |m\rangle \quad . \quad (4.55)$$

We can keep applying \hat{J}^+ , but eventually, if the representation is finite, we must reach a state $|j\rangle$ for which $\hat{J}^z |j\rangle = j |j\rangle$ but $\hat{J}^+ |j\rangle = 0$. We then have $\hat{\mathbf{J}}^2 |j\rangle = j(j+1) |j\rangle$. The eigenvalue of our Casimir is thus $j(j+1)$, and $|j\rangle$, our "standard vector", is called the *highest weight state*.

We now work downward from $|j\rangle$, successively applying \hat{J}^- . Note that for any normalized state $|m\rangle$ with $m \leq j$,

$$\begin{aligned} \langle m | (\hat{J}^-)^\dagger \hat{J}^- |m\rangle &= \langle m | \hat{J}^+ \hat{J}^- |m\rangle = \langle m | (\hat{\mathbf{J}}^2 - (\hat{J}^z)^2 + \hat{J}^z) |m\rangle \\ &= j(j+1) - m(m-1) \quad . \end{aligned} \quad (4.56)$$

We also have $\hat{J}^z \hat{J}^- |j, m\rangle = (m-1) \hat{J}^- |j, m\rangle$, hence we may take

$$\hat{J}^- |m\rangle = \sqrt{j(j+1) - m(m-1)} |m-1\rangle \quad . \quad (4.57)$$

⁷A Lie group can have several Casimirs. For example, SU(3) has two and in general SU(N) has $N-1$ Casimirs.

If the representation is finite, eventually one must reach a state where $\hat{J}^-|m\rangle = 0$, which requires $m = -j$. As this state is achieved by an integer number of applications of the lowering operator \hat{J}^- , we must have $j - (-j) = 2j \in \mathbb{N}$, where \mathbb{N} denotes the natural numbers $\{0, 1, 2, \dots\}$. Thus, $j \in \frac{1}{2}\mathbb{N}$, which is to say j may be a positive integer or a half odd positive integer. We now include the label j on all states, in order to label the representation, and we have

$$\begin{aligned}\hat{\mathbf{J}}^2 |j, m\rangle &= j(j+1) |j, m\rangle \\ \hat{J}^z |j, m\rangle &= m |j, m\rangle \\ \hat{J}^\pm |j, m\rangle &= \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle \quad ,\end{aligned}\tag{4.58}$$

where $m \in \{-j, \dots, j\}$. When we refer to the matrix elements of \hat{J}^a , we will respectfully remove the hats from the operators, *i.e.* J^a is the matrix whose elements are $J_{mm'}^a = \langle j, m | \hat{J}^a | j, m' \rangle$.

4.3.2 Rotation matrices

Rotation matrices are the matrices corresponding to a particular group element, and are specific to each representation. By definition,

$$\hat{R}(\alpha, \beta, \gamma) |j, m'\rangle = |j, m\rangle D_{mm'}^{(j)}(\alpha, \beta, \gamma) \quad .\tag{4.59}$$

Since $\hat{R}(\alpha, \beta, \gamma) = \exp(-i\alpha\hat{J}^z) \exp(-i\beta\hat{J}^y) \exp(-i\gamma\hat{J}^z)$, we have

$$D_{mm'}^{(j)}(\alpha, \beta, \gamma) = e^{-i\alpha m} e^{-i\gamma m'} d_{mm'}^{(j)}(\beta) \quad ,\tag{4.60}$$

with

$$d_{mm'}^{(j)}(\beta) = \langle j, m | \exp(-i\beta\hat{J}^y) | j, m' \rangle \quad .\tag{4.61}$$

As the matrices of \hat{J}^\pm are real⁸, iJ^y is real, and we conclude $d_{mm'}^{(j)}(\beta)$ is a real-valued matrix of rank $2j + 1$. For all j we have $[d^{(j)}(\beta)]^T = d^{(j)}(-\beta) = [d^{(j)}(\beta)]^{-1}$, as well as

$$d_{-m, -m'}^{(j)}(\beta) = (-1)^{2j-m-m'} d_{m, m'}^{(j)}(\beta) = (-1)^{m-m'} d_{m, m'}^{(j)}(\beta) \quad .\tag{4.62}$$

Note that we could have stuck with the $(\xi, \hat{\mathbf{n}})$ parameterization, and written

$$\hat{R}(\xi, \hat{\mathbf{n}}) |j, m'\rangle = |j, m\rangle D_{mm'}^{(j)}(\xi, \hat{\mathbf{n}}) \quad ,\tag{4.63}$$

but clearly the Euler angle parameterization is advantageous due to the particularly simple way in which the α and γ angles appear in the rotation matrices.

⁸This was actually a convention that we chose, by taking the prefactor on the RHS of the last of Eqn. 4.58 to be real, and is originally due to Condon and Shortley.

Characters

Recall that rotations through a fixed angle ξ form a class within SO(3). What is the character of this class? Since the axis doesn't matter, we choose $\hat{n} = \hat{z}$, in which case

$$\chi^{(j)}(\xi) = \text{Tr} \exp(-i\xi J^z) = \sum_{m=-j}^j e^{-im\xi} = \frac{\sin[(j + \frac{1}{2})\xi]}{\sin(\frac{1}{2}\xi)} . \quad (4.64)$$

Using the Euler angle parameterization,

$$\chi^{(j)}(\alpha, \beta, \gamma) = \sum_{m=-j}^j e^{-im(\alpha+\gamma)} d_{mm}^{(j)}(\beta) . \quad (4.65)$$

Examples

The simplest example of course is $j = \frac{1}{2}$, where $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ are the Pauli matrices. Then

$$d^{(1/2)}(\beta) = \exp(-i\beta\sigma^y/2) = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} . \quad (4.66)$$

For $j = 1$, we need to exponentiate the 3×3 matrix $i\beta J^y$. Let's first find the normalized eigenvalues and eigenvectors of J^y :

$$\psi^+ = \frac{1}{2} \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix} , \quad \psi^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} , \quad \psi^- = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix} , \quad (4.67)$$

with corresponding eigenvalues $+1, 0$, and -1 , respectively. From these we construct the projectors

$$P_{ij}^{\pm} = \psi_i^{\pm} \psi_j^{\pm*} = \frac{1}{4} \begin{pmatrix} 1 & \mp i\sqrt{2} & -1 \\ \pm i\sqrt{2} & 2 & \mp i\sqrt{2} \\ -1 & \pm i\sqrt{2} & 1 \end{pmatrix} , \quad P_{ij}^0 = \psi_i^0 \psi_j^{0*} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} . \quad (4.68)$$

The projectors are mutually orthogonal and complete: $P^a P^b = \delta^{ab} P^a$ (no sum) and $\sum_a P^a = \mathbf{1}$. We can decompose J^y into its projectors, writing $J^y = P^+ - P^-$, in which case

$$\begin{aligned} \exp(-i\beta J^y) &= \mathbf{1} - i\beta (P^+ - P^-) - \frac{1}{2}\beta^2 (P^+ - P^-)^2 + \dots \\ &= P^0 + \cos \beta (P^+ + P^-) - i \sin \beta (P^+ - P^-) . \end{aligned} \quad (4.69)$$

since $(P^+ - P^-)^{2n} = P^+ + P^-$ and $(P^+ - P^-)^{2n+1} = P^+ - P^-$, allowing us to sum the Taylor series. Thus, we have

$$d^{(1)}(\beta) = \exp(-i\beta J^y) = \frac{1}{2} \begin{pmatrix} 1 + \cos \beta & -\sqrt{2} \sin \beta & 1 - \cos \beta \\ \sqrt{2} \sin \beta & 2 \cos \beta & -\sqrt{2} \sin \beta \\ 1 - \cos \beta & \sqrt{2} \sin \beta & 1 + \cos \beta \end{pmatrix} . \quad (4.70)$$

For a general Hermitian matrix M , we can always decompose M into its orthogonal projectors, *viz.*

$$M = \sum_l \lambda_k P^{(k)} \quad , \quad (4.71)$$

where $P^{(k)}$ projects onto the k^{th} eigenspace and λ_k is the associated eigenvalue. We can always orthogonalize projectors associated with degenerate eigenspaces, and so

$$\exp(-i\theta M) = \sum_k \exp(-i\theta \lambda_k) P^{(k)} \quad , \quad (4.72)$$

where λ_k is the k^{th} eigenvalue of M . Therefore

$$d^{(j)}(2\pi) = \sum_{m=-j}^j e^{-2\pi i m} P^{(m)} = (-1)^{2j} \mathbb{1} \quad . \quad (4.73)$$

This immediately tells us that the IRREPS we have found with $j \in \mathbb{Z} + \frac{1}{2}$ are not proper IRREPS, but rather are *projective* IRREPS.

The general expression for the $d^{(j)}(\beta)$ matrices is

$$d_{mm'}^{(j)}(\beta) = \sum_{k=0}^{2j} (-1)^k \frac{[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}}{k!(j+m-k)!(j-m'+k)!(k-m+m')!} [\cos(\frac{1}{2}\beta)]^{2j+m-m'-2k} [\sin(\frac{1}{2}\beta)]^{2k-m+m'} \quad , \quad (4.74)$$

where it is to be understood that values of k which make the arguments of any of the factorials negative are excluded from the sum.

Parameterizations of SU(2)

$SU(2) \in GL(2, \mathbb{C})$ is the group of unitary 2×2 complex matrices with determinant 1. We have met up with $SU(2)$ along the way several times already. Let's recall some of its parameterizations. Any matrix $U \in SU(2)$ may be written as

$$U(w, x) = \begin{pmatrix} w & x \\ -x^* & w^* \end{pmatrix} \quad (4.75)$$

where $w, x \in \mathbb{C}$ and $\det U = |w|^2 + |x|^2 = 1$. Thus, $SU(2) \cong S^3$, the three dimensional sphere. We may also write

$$U(\xi, \hat{n}) = \exp\left(-\frac{i}{2}\xi \hat{n} \cdot \sigma\right) = \cos\left(\frac{1}{2}\xi\right) \mathbb{1} - i \sin\left(\frac{1}{2}\xi\right) \hat{n} \cdot \sigma \quad , \quad (4.76)$$

where σ are the Pauli matrices and $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector. The ranges of the parameters (ξ, θ, ϕ) are $\xi \in [0, 2\pi)$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$. This parameterization carries the interpretation of a rotation by an angle ξ about the axis \hat{n} . We've seen how $SU(2)$ is a double cover of $SO(3)$, for if $U = \exp\left(-\frac{i}{2}\xi \hat{n} \cdot \sigma\right)$, then

$$R_{ab} = \frac{1}{2} \text{Tr}(U \sigma^a U^\dagger \sigma^b) = n^a n^b + (\delta^{ab} - n^a n^b) \cos \xi - \epsilon_{abc} n^c \sin \xi \quad (4.77)$$

where $R \in \text{SO}(3)$. Note that $R(2\pi - \xi, \hat{\mathbf{n}}) = R(\xi, -\hat{\mathbf{n}})$, so the coordinates (ξ, θ, ϕ) cover $\text{SO}(3)$ twice, and for a unique expression of $\text{SO}(3)$ matrices we restrict $\xi \in [0, \pi]$ and identify $R(\pi, \hat{\mathbf{n}}) \equiv R(\pi, -\hat{\mathbf{n}})$, as we have discussed before. Finally, we have the Euler angle parameterization,

$$U(\alpha, \beta, \gamma) = \exp\left(-\frac{i}{2}\alpha\sigma^z\right) \exp\left(-\frac{i}{2}\beta\sigma^y\right) \exp\left(-\frac{i}{2}\gamma\sigma^z\right) \quad (4.78)$$

where $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, and $\gamma \in [0, 4\pi)$.

4.3.3 Guide for the perplexed

The action of rotation operators on wavefunctions can be confusing when it comes to active *versus* passive rotations. To set the record straight, consider the action of a rotation operator $\hat{U}(R)$ where $R(\xi, \hat{\mathbf{n}}) \in \text{SO}(3)$. When there is no *intrinsic* spin, we have

$$\hat{U}(R) \psi(\mathbf{r}) \equiv \langle \mathbf{r} | \hat{U}(R) | \psi \rangle = \langle \mathbf{r}R | \psi \rangle = \psi(\mathbf{r}R) = \psi(R^T \mathbf{r}) \quad , \quad (4.79)$$

where $R \equiv D^{\text{def}}(R)$ is the 3×3 matrix representation of the rotation R in the defining representation of $\text{SO}(3)$ ⁹. Now suppose there is intrinsic spin j . We assume the structure of Hilbert space is such that spatial and spin degrees of freedom enter as a direct product, *i.e.* that the wavefunction can be written

$$| \Psi \rangle = | \psi_{m'} \rangle \otimes | m' \rangle \quad . \quad (4.80)$$

with an implied sum on m' from over the range $\{-j, \dots, j\}$. We then have

$$\hat{U}(R) | \Psi \rangle = \left[\hat{U}_{\text{rot}}(R) | \psi_{m'} \rangle \right] \otimes | m \rangle D_{mm'}^{(j)}(R) \quad (4.81)$$

so that

$$\begin{aligned} \langle \mathbf{r} | \hat{U}(R) | \Psi \rangle &= \langle \mathbf{r} | \hat{U}_{\text{rot}}(R) | \psi_{m'} \rangle | m \rangle D_{mm'}^{(j)}(R) \\ &= \psi_{m'}(\mathbf{r}R) | m \rangle D_{mm'}^{(j)}(R) = [\hat{U}(R) \Psi(\mathbf{r})]_m | m \rangle \quad , \end{aligned} \quad (4.82)$$

where $\hat{U}_{\text{rot}}(R) = \exp(-i\xi \mathbf{L} \cdot \hat{\mathbf{n}}/\hbar)$ is the spatial rotation part of $\hat{U}(R)$. We can also write this as

$$\hat{U}(R) \begin{pmatrix} \psi_{+j}(\mathbf{r}) \\ \vdots \\ \psi_{-j}(\mathbf{r}) \end{pmatrix} = D^{(j)}(R) \begin{pmatrix} \psi_{+j}(\mathbf{r}R) \\ \vdots \\ \psi_{-j}(\mathbf{r}R) \end{pmatrix} \quad . \quad (4.83)$$

Attend to the order of operations here or you may lead an unhappy life: $R = D^{\text{def}}(R)$ multiplies the row vector \mathbf{r} on the *right*, while $D^{(j)}(R)$ multiplies the column vector Ψ on the *left*.

⁹Recall that in the defining representation of any matrix Lie group $G \subset \text{GL}(n, \mathbb{F})$, each element g is represented by itself.

4.3.4 Invariant measure for Lie groups

How does one sum over all the elements of a continuous group? Since the group space G is a manifold, we may integrate over G if we have an appropriate *measure* $d\mu(g)$. Integrating with respect to this measure should be the equivalent of summing over all elements of a discrete group. But then, by rearrangement, we must have

$$\int_G d\mu(g) \phi(gh^{-1}) = \int_G d\mu(g) \phi(g) \quad (4.84)$$

for any function $\phi(g)$ and any fixed $h \in G$. Thus, we require $d\mu(gh) = d\mu(g)$ in order that the integral remain invariant under rearrangement. A measure which satisfies this desideratum is called an *invariant* (or *Haar*) *measure*.

Let each group element $g \in G$ be parameterized by a set of coordinates $\mathbf{x} = \{x_1, \dots, x_{\dim(G)}\}$. We define \mathbf{x}_g to be the coordinates corresponding to the group element g . The coordinates \mathbf{x}_{gh} for the product gh must depend on those of the components g and h , and accordingly we write

$$\mathbf{x}_{gh} = \mathbf{f}(\mathbf{x}_g, \mathbf{x}_h) \quad , \quad (4.85)$$

where $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is the *group composition function*. Any group composition function must satisfy the following consistency relations:

$$\begin{aligned} \mathbf{f}(\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{z}) &= \mathbf{f}(\mathbf{x}, \mathbf{f}(\mathbf{y}, \mathbf{z})) \\ \mathbf{f}(\mathbf{x}_E, \mathbf{y}) &= \mathbf{f}(\mathbf{y}, \mathbf{x}_E) = \mathbf{y} \\ \mathbf{f}(\mathbf{x}, \mathbf{x}^{-1}) &= \mathbf{f}(\mathbf{x}^{-1}, \mathbf{x}) = \mathbf{x}_E \quad , \end{aligned} \quad (4.86)$$

where \mathbf{x}_E are the coordinates of the identity E , i.e. $g(\mathbf{x}_E) = E$, and \mathbf{x}^{-1} are the coordinates of the inverse of $g(\mathbf{x})$, i.e. $g(\mathbf{x}^{-1}) = [g(\mathbf{x})]^{-1}$. We can use the composition functions to construct an invariant measure, by writing

$$d\mu(g(\mathbf{x})) = \rho(\mathbf{x}) \prod_{j=1}^{\dim(G)} dx_j \quad (4.87)$$

with

$$\rho_0 = \rho(\mathbf{x}) \left| \det \left(\frac{\partial f_j(\boldsymbol{\epsilon}, \mathbf{x})}{\partial \epsilon_k} \right)_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_E} \right| \quad , \quad (4.88)$$

where $\rho_0 = \rho(\mathbf{x}_E)$. An equivalent and somewhat more convenient definition is the following. For any $g(\mathbf{x})$, express $g^{-1} \frac{\partial g}{\partial x_i}$ in terms of the Lie algebra generators T^a , i.e.

$$g^{-1}(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_i} = \sum_{a=1}^{\dim(G)} M_{ia}(\mathbf{x}) T^a \quad , \quad (4.89)$$

where $\{T^a\}$ are the generators of the Lie algebra \mathfrak{g} . Then¹⁰

$$\rho(\mathbf{x}) = \rho_0 |\det M(\mathbf{x})| \quad . \quad (4.90)$$

¹⁰Since we are taking the absolute value of the determinant, it doesn't matter whether we use the math or physics convention for the generators, since the difference is only a power of i , which is unimodular.

Note that

$$[hg(\mathbf{x})]^{-1} \frac{\partial [hg(\mathbf{x})]}{\partial x_i} = g^{-1}(\mathbf{x}) h^{-1} h \frac{\partial g(\mathbf{x})}{\partial x_i} = M_{ia}(\mathbf{x}) T^a \quad (4.91)$$

and therefore the measure is invariant under left multiplication of g . The student should check that it is also right-invariant. For *compact, semisimple* Lie groups, we will always be able to choose a normalization of the generators $\text{Tr}(T^a T^b) = c \delta^{ab}$, in which case we may write

$$M_{ia}(\mathbf{x}) = c^{-1} \text{Tr} \left(T^a g^{-1}(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_i} \right) . \quad (4.92)$$

It is conventional to normalize the invariant measure according to

$$\int_G d\mu(g) \equiv 1 . \quad (4.93)$$

Let's implement these formulae for the cases of SO(2) and SU(2) (and SO(3)). For SO(2),

$$U(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} , \quad U^{-1}(\phi) = g(-\phi) , \quad \frac{\partial U}{\partial \phi} = \begin{pmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \end{pmatrix} , \quad (4.94)$$

and thus

$$U^{-1}(\phi) \frac{\partial U}{\partial \phi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \quad (4.95)$$

The RHS is $-i\sigma^y$, and σ^y is the generator of SO(2) (physics convention). Thus, $\rho(\phi)$ is a constant, and normalizing over the group manifold, we have $\rho(\phi) = \frac{1}{2\pi}$.

The analysis for SU(2) is not quite so trivial, but still straightforward. Let's choose the parameterization $U(\xi, \hat{\mathbf{n}}) = \exp(-\frac{i}{2}\xi \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})$. Then $U^{-1}(\xi, \hat{\mathbf{n}}) = U(\xi, \hat{\mathbf{n}})$ and

$$dU(\xi, \hat{\mathbf{n}}) = -\frac{1}{2} \sin(\frac{1}{2}\xi) d\xi - \frac{1}{2} \cos(\frac{1}{2}\xi) \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} d\xi - i \sin(\frac{1}{2}\xi) d\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \quad (4.96)$$

and one readily obtains

$$U^{-1}(\xi, \hat{\mathbf{n}}) dU(\xi, \hat{\mathbf{n}}) = -\frac{i}{2} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} d\xi - \frac{i}{2} \sin \xi d\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} + \frac{i}{2} (1 - \cos \xi) \hat{\mathbf{n}} \times d\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} . \quad (4.97)$$

It is convenient to define vectors $\hat{\mathbf{e}}_{1,2}$ such that $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{n}}\}$ forms an orthonormal triad for all (θ, ϕ) . Explicitly,

$$\begin{aligned} \hat{\mathbf{e}}_1 &= (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\ \hat{\mathbf{e}}_2 &= (-\sin \phi, \cos \phi, 0) \\ \hat{\mathbf{n}} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) . \end{aligned} \quad (4.98)$$

One then finds

$$\begin{aligned} d\hat{\mathbf{n}} &= \hat{\mathbf{e}}_1 d\theta + \hat{\mathbf{e}}_2 \sin \theta d\phi \\ \hat{\mathbf{n}} \times d\hat{\mathbf{n}} &= -\hat{\mathbf{e}}_1 \sin \theta d\phi + \hat{\mathbf{e}}_2 d\theta , \end{aligned} \quad (4.99)$$

and we have

$$U^{-1}(\xi, \hat{\mathbf{n}}) dU(\xi, \hat{\mathbf{n}}) = -\frac{i}{2} (d\xi \quad d\theta \quad d\phi) \begin{pmatrix} \hat{\mathbf{n}} \\ \sin \xi \hat{\mathbf{e}}_1 + (1 - \cos \xi) \hat{\mathbf{e}}_2 \\ (1 - \cos \xi) \sin \theta \hat{\mathbf{e}}_1 - \sin \xi \sin \theta \hat{\mathbf{e}}_2 \end{pmatrix} \cdot \boldsymbol{\sigma} . \quad (4.100)$$

Thus, we have

$$\rho(\xi, \theta, \phi) \propto \hat{\mathbf{n}} \cdot [\sin \xi \hat{\mathbf{e}}_1 + (1 - \cos \xi) \hat{\mathbf{e}}_2] \times [(1 - \cos \xi) \hat{\mathbf{e}}_1 - \sin \xi \hat{\mathbf{e}}_2] \sin \theta = -4 \sin^2(\frac{1}{2}\xi) , \quad (4.101)$$

where $\xi \in [0, 2\pi]$. Normalizing, we have the invariant measure

$$d\mu(\xi, \theta, \phi) = \overbrace{\frac{1}{8\pi^2} (1 - \cos \xi) \sin \theta \, d\xi \, d\theta \, d\phi}^{\rho(\xi, \theta, \phi)} = \frac{(1 - \cos \xi) \, d\xi}{2\pi} \frac{d\hat{\mathbf{n}}}{4\pi} . \quad (4.102)$$

We can also compute the invariant measure using the Euler angle parameterization,

$$U(\alpha, \beta, \gamma) = \exp(-\frac{i}{2} \alpha \sigma^z) \exp(-\frac{i}{2} \beta \sigma^y) \exp(-\frac{i}{2} \gamma \sigma^z) . \quad (4.103)$$

One finds

$$\begin{aligned} U^{-1} \frac{\partial U}{\partial \alpha} &= -\frac{i}{2} [-\sin \beta \cos \gamma \sigma^x + \sin \beta \sin \gamma \sigma^y + \cos \beta \sigma^z] \\ U^{-1} \frac{\partial U}{\partial \beta} &= -\frac{i}{2} [\sin \gamma \sigma^x + \cos \gamma \sigma^y] \\ U^{-1} \frac{\partial U}{\partial \gamma} &= -\frac{i}{2} \sigma^z , \end{aligned} \quad (4.104)$$

and so

$$M = -\frac{i}{2} \begin{pmatrix} -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \\ -\sin \gamma & -\cos \gamma & 0 \\ 0 & 0 & -1 \end{pmatrix} . \quad (4.105)$$

Thus, $\det(M) = \frac{i}{8} \sin \beta$ and the normalized invariant measure in the Euler angle representation is

$$d\mu(\alpha, \beta, \gamma) = \frac{\sin \beta \, d\alpha \, d\beta \, d\gamma}{\text{vol}(G)} , \quad (4.106)$$

where $\text{vol}(G)$ is the group volume, *i.e.* the integral of the numerator over the allowed range of the angles (α, β, γ) . Remember that $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, and $\gamma \in [0, 4\pi)$ for $\text{SU}(2)$, hence $\text{vol}(\text{SU}(2)) = 16\pi^2$, but for $\text{SO}(3)$,

$$R(\alpha, \beta, \gamma) = \exp(-i\alpha J^z) \exp(-i\beta J^y) \exp(-i\gamma J^z) , \quad (4.107)$$

we have $\gamma \in [0, 2\pi)$, and accordingly $\text{vol}(\text{SO}(3)) = 8\pi^2$.

4.3.5 Peter-Weyl theorem

Armed with the invariant measure, we can now express the Great Orthogonality Theorem for Lie groups. It goes by the name of the *Peter-Weyl theorem*, and says that the functions

$$V_{\Gamma ik}(g) = \sqrt{d_{\Gamma}} D_{ik}^{\Gamma}(g) \quad (4.108)$$

form a complete and orthonormal basis in the space of square-integrable functions on the group manifold. This means

$$\int_G d\mu(g) V_{\Gamma ik}^*(g) V_{\Gamma' i' k'}(g) = \sqrt{d_{\Gamma} d_{\Gamma'}} \int_G d\mu(g) D_{ik}^{\Gamma*}(g) D_{i' k'}^{\Gamma'}(g) = \delta_{\Gamma \Gamma'} \delta_{ii'} \delta_{kk'} \quad (4.109)$$

$$\sum_{\Gamma, i, k} V_{\Gamma ik}(g) V_{\Gamma ik}^*(g') = \sum_{\Gamma ik} d_{\Gamma} D_{ik}^{\Gamma}(g) D_{ik}^{\Gamma*}(g') = \delta(g, g')$$

where the symbol $\delta(g, g')$ satisfies

$$\int_G d\mu(g') \delta(g, g') F(g') = \sum_{\Gamma, i, k} \langle V_{\Gamma ik} | F \rangle V_{\Gamma ik}(g) = F(g) \quad , \quad (4.110)$$

where $\langle V_{\Gamma ik} | F \rangle = \int_G d\mu(g) V_{\Gamma ik}^*(g) F(g)$, and where the last inequality must be understood in terms of "convergence in the L^2 norm". In other words, the convergence is in the norm, and not necessarily pointwise, just like in the analogous case of the Fourier transform. For any function $F(g)$ which can be expanded in terms of the basis functions $V_{\Gamma ik}(g)$, one has

$$\int_G d\mu(g') \delta(g, g') \overbrace{\sum_{\Gamma, i, k} C_{\Gamma ik} V_{\Gamma ik}(g')}^{F(g')} = \overbrace{\sum_{\Gamma ik} C_{\Gamma ik} V_{\Gamma ik}(g)}^{F(g)} \quad . \quad (4.111)$$

4.3.6 Projection operators

In analogy with the case for discrete groups, we can construct projectors onto the μ row of the Γ IRREP for any compact Lie group G , viz.

$$\hat{\Pi}_{\mu\nu}^{\Gamma} = d_{\Gamma} \int_G d\mu(g) D_{\mu\nu}^{\Gamma*}(g) \hat{U}(g) \quad . \quad (4.112)$$

Again, these satisfy

$$\hat{\Pi}_{\mu\nu}^{\Gamma} \hat{\Pi}_{\mu'\nu'}^{\Gamma'} = \delta_{\Gamma \Gamma'} \delta_{\nu \nu'} \hat{\Pi}_{\mu\nu}^{\Gamma} \quad , \quad (4.113)$$

and

$$\hat{U}(g) \hat{\Pi}_{\mu\nu}^{\Gamma} = \hat{\Pi}_{\rho\nu}^{\Gamma} D_{\rho\mu}^{\Gamma}(g) \quad . \quad (4.114)$$

Projection matrices are defined in analogous fashion, with

$$\Pi_{\mu\nu}^\Gamma = d_\Gamma \int_G d\mu(g) D_{\mu\nu}^{\Gamma*}(g) \Delta(g) \quad (4.115)$$

satisfying $\Pi_{\mu\nu}^\Gamma \Pi_{\mu'\nu'}^{\Gamma'} = \delta_{\Gamma\Gamma'} \delta_{\nu\nu'} \Pi_{\mu\mu'}^\Gamma$ and $\Delta(g) \Pi_{\mu\nu}^\Gamma = \Pi_{\rho\nu}^\Gamma D_{\rho\mu}^\Gamma(g)$. Here $\Delta(G)$ is a matrix representation of the Lie group.

As an example, consider the group $\text{SO}(2)$. To project an arbitrary periodic function $f(\phi)$ onto the j IRREP, we use $D^{(j)}(\alpha) = \exp(-ij\alpha)$ and $\hat{U}(\alpha) = \exp(-i\alpha L^z) = \exp(-\alpha \partial_\phi)$. The IRREPs are all one-dimensional. We then have

$$\hat{\Pi}^{(j)} f(\phi) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{ij\alpha} e^{-\alpha \frac{\partial}{\partial \phi}} f(\phi) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{ij\alpha} f(\phi - \alpha) = \hat{f}_j e^{ij\phi} \quad , \quad (4.116)$$

where

$$\hat{f}_j = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-ij\alpha} f(\alpha) \quad . \quad (4.117)$$

is the discrete Fourier transform of the function $f(\alpha)$. Note that $\hat{U}(\alpha)$ has eigenvalue $e^{-ij\alpha} = D^{(j)}(\alpha)$ when acting on the projected function $\hat{f}_j e^{ij\phi}$.

4.3.7 Product representations for $\text{SU}(2)$

In the product basis, we have

$$\hat{U}(g) \left[|j_1, m_1\rangle \otimes |j_2, m_2\rangle \right] = \sum_{m'_1, m'_2} |j_1, m'_1\rangle \otimes |j_2, m'_2\rangle \overbrace{D_{m'_1, m_1}^{j_1}(g) D_{m'_2, m_2}^{j_2}(g)}^{D_{m'_1, m'_2, m_1, m_2}^{j_1 \times j_2}(g)} \quad . \quad (4.118)$$

Taking traces, we have

$$\chi^{j_1 \times j_2}(g) = \chi^{(j_1)}(g) \chi^{(j_2)}(g) \quad . \quad (4.119)$$

Generalizing the decomposition formula to the case of continuous groups,

$$n_\Gamma(\Psi) = \int_G d\mu(g) \chi^{\Gamma*}(g) \chi^\Psi(g) \quad . \quad (4.120)$$

For $\text{SU}(2)$, the invariant measure is $d\mu(\xi, \hat{n}) = \frac{1}{\pi} \sin^2(\frac{1}{2}\xi) d\xi \cdot \frac{d\hat{n}}{4\pi}$.

Recall that $\chi^{(j)}(\xi) = \sin[(j + \frac{1}{2})\xi] / \sin(\frac{1}{2}\xi)$. Thus, according to the decomposition rule,

$$\begin{aligned}
n_j^{j_1 \times j_2} &= \frac{2}{\pi} \int_0^\pi d\xi \sin^2(\frac{1}{2}\xi) \chi^{(j)}(\xi) \chi^{j_1}(\xi) \chi^{j_2}(\xi) \\
&= \frac{2}{\pi} \int_0^\pi d\xi \sum_{m=-j}^j e^{-im\xi} \sin((j_1 + \frac{1}{2})\xi) \sin((j_2 + \frac{1}{2})\xi) \\
&= \frac{1}{\pi} \int_0^\pi d\xi \sum_{m=-j}^j e^{-im\xi} \left\{ \cos[(j_1 - j_2)\xi] - \cos[(j_1 + j_2 + 1)\xi] \right\} = \begin{cases} 1 & \text{if } |j_1 - j_2| \leq j \leq (j_1 + j_2) \\ 0 & \text{otherwise} \end{cases} .
\end{aligned} \tag{4.121}$$

Thus, for each j with $|j_1 - j_2| \leq j \leq (j_1 + j_2)$, there is one representation within the direct product $j_1 \times j_2$. Note that

$$\begin{aligned}
\sum_{|j_1 - j_2|}^{j_1 + j_2} (2j + 1) &= \frac{1}{2}(j_1 + j_2)(j_1 + j_2 + 1) - \frac{1}{2}(|j_1 - j_2| - 1)|j_1 - j_2| + (j_1 + j_2) - |j_1 - j_2| \\
&= (2j_1 + 1)(2j_2 + 1) \quad ,
\end{aligned} \tag{4.122}$$

which says that the dimension of the product representation is the product of the dimensions of its factors.

The direct product of two representations j_1 and j_2 is expanded as

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j, m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} |j, m\rangle \quad . \tag{4.123}$$

The CGCs are nonzero only if $|j_1 - j_2| \leq j \leq (j_1 + j_2)$ and $m = m_1 + m_2$. They are tabulated in various publications (*e.g.*, see *Wikipedia*). To derive the CGCs, one starts with the state with $m_1 = j_1$ and $m_2 = j_2$, which corresponds to $j = j_1 + j_2$ and $m = m_1 + m_2$. Since

$$|j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle \quad , \tag{4.124}$$

we have, trivially, that

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & j_1 + j_2 \end{pmatrix} = \delta_{m_1, j_1} \delta_{m_2, j_2} \quad . \tag{4.125}$$

Now apply the lowering operator \hat{J}^- to get

$$\begin{aligned}
\hat{J}^- |j_1 + j_2, j_1 + j_2\rangle &= [2(j_1 + j_2)]^{1/2} |j_1 + j_2, j_1 + j_2 - 1\rangle \\
&= [\hat{J}^- |j_1, j_1\rangle] \otimes |j_2, j_2\rangle + |j_1, j_1\rangle \otimes [\hat{J}^- |j_2, j_2\rangle] \\
&= \sqrt{2j_1} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle + \sqrt{2j_2} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \quad .
\end{aligned} \tag{4.126}$$

Thus,

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \quad . \tag{4.127}$$

Taking the inner product with eqn. 4.123 then gives

$$\left(\begin{array}{cc} j_1 & j_2 \\ m_1 & m_2 \end{array} \middle| \begin{array}{c} j_1 + j_2 \\ j_1 + j_2 - 1 \end{array} \right) = \sqrt{\frac{j_1}{j_1 + j_2}} \delta_{m_1, j_1 - 1} \delta_{m_2, j_2} + \sqrt{\frac{j_2}{j_1 + j_2}} \delta_{m_1, j_1} \delta_{m_2, j_2 - 1} \quad . \quad (4.128)$$

Continue to apply \hat{J}^- to generate all the states $|j_1 + j_2, m\rangle$ for $m \in \{-j_1 - j_2, \dots, j_1 + j_2\}$.

Next, consider the state

$$|j, m\rangle = \sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle - \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \quad , \quad (4.129)$$

which clearly has $m = j_1 + j_2 - 1$ but which is orthogonal to the state in Eqn. 4.127. We conclude that the above state is none other than $|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$, and consequently

$$\left(\begin{array}{cc} j_1 & j_2 \\ m_1 & m_2 \end{array} \middle| \begin{array}{c} j_1 + j_2 - 1 \\ j_1 + j_2 - 1 \end{array} \right) = \sqrt{\frac{j_2}{j_1 + j_2}} \delta_{m_1, j_1 - 1} \delta_{m_2, j_2} - \sqrt{\frac{j_1}{j_1 + j_2}} \delta_{m_1, j_1} \delta_{m_2, j_2 - 1} \quad . \quad (4.130)$$

Applying the lowering operator to this state, one creates a state with $j = j_1 + j_2 - 1$ and $m = j_1 + j_2 - 2$, and one may continue to apply \hat{J}^- to generate the entire family of basis states for the $j = j_1 + j_2 - 1$ representation. One then constructs a new state ψ with $m = j_1 + j_2 - 2$ which is normalized and orthogonal to both $|j_1 + j_2, j_1 + j_2 - 2\rangle$ and $|j_1 + j_2 - 1, j_1 + j_2 - 2\rangle$. This must be $|j_1 + j_2 - 2, j_1 + j_2 - 2\rangle$. Continuing in this manner, one eventually constructs all the basis states $|j, m\rangle$ in terms of the product states, from which one can read off the CGCs.

4.3.8 Spherical harmonics

The angular momentum operators,

$$\hat{L}^x = i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) \quad , \quad \hat{L}^y = i \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \quad , \quad \hat{L}^z = i \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \quad , \quad (4.131)$$

satisfy the SO(3) algebra $[\hat{L}^a, \hat{L}^b] = i\epsilon_{abc} \hat{L}^c$. Clearly $L^a f(r) = 0$ when acting on a spherically symmetric function. Therefore we may express the \hat{L}^a in terms of derivatives with respect to θ and ϕ , viz.

$$L^\pm = e^{\pm i\phi} \left(i \cot\theta \frac{\partial}{\partial \phi} \pm \frac{\partial}{\partial \theta} \right) \quad , \quad L^z = -i \frac{\partial}{\partial \phi} \quad , \quad \hat{\mathbf{L}}^2 = -\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \quad . \quad (4.132)$$

We then have

$$\hat{\mathbf{L}}^2 Y_{lm}(\hat{\mathbf{n}}) = l(l+1) Y_{lm}(\hat{\mathbf{n}}) \quad , \quad \hat{L}^z Y_{lm}(\hat{\mathbf{n}}) = m Y_{lm}(\hat{\mathbf{n}}) \quad , \quad (4.133)$$

with $l \in \mathbb{N}$, where $Y_{lm}(\hat{\mathbf{n}})$ is the *spherical harmonic*. The spherical harmonics are related to the rotation matrices. If we define

$$|\hat{\mathbf{n}}\rangle \equiv \hat{R}(\phi, \theta, 0) |\hat{\mathbf{z}}\rangle \quad , \quad (4.134)$$

where $|\hat{z}\rangle$ is the ket vector corresponding to an orientation along \hat{z} ¹¹, then

$$\langle l, m | \hat{\mathbf{n}} \rangle = \sum_{m'=-l}^l \langle l, m | \hat{R}(\phi, \theta, 0) | l, m' \rangle \langle l, m' | \hat{z} \rangle . \quad (4.135)$$

We then have $Y_{lm}(\hat{\mathbf{n}}) = \langle \hat{\mathbf{n}} | l, m \rangle$. Now

$$\langle l, m | \hat{z} \rangle = Y_{lm}(\hat{z}) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0} \quad (4.136)$$

and therefore

$$Y_{lm}(\hat{\mathbf{n}}) = \sqrt{\frac{2l+1}{4\pi}} [D_{m0}^{(l)}(\phi, \theta, 0)]^* = \sqrt{\frac{2l+1}{4\pi}} d_{m0}^{(l)}(\theta) e^{im\phi} . \quad (4.137)$$

Note that $Y_{l,-m}(\hat{\mathbf{n}}) = (-1)^m Y_{lm}^*(\hat{\mathbf{n}})$ and that $D_{m0}^{(l)}(\phi, \theta, \psi) = D_{m0}^{(l)}(\phi, \theta, 0)$ because $e^{-im'\gamma} = 1$ for $m' = 0$. The spherical harmonics are normalized according to

$$\int d\hat{\mathbf{n}} Y_{lm}^*(\hat{\mathbf{n}}) Y_{l'm'}(\hat{\mathbf{n}}) = \delta_{ll'} \delta_{mm'} . \quad (4.138)$$

Note that there is no factor of 4π in the denominator of the measure, which is $d\hat{\mathbf{n}} = \sin\theta d\theta d\phi$. The associated Legendre polynomials¹² $P_{lm}(\cos\theta)$ are related to the $d^{(l)}$ matrices by

$$P_{lm}(\cos\theta) = (-1)^m \sqrt{\frac{(l+m)!}{(l-m)!}} d_{m0}^{(l)}(\theta) , \quad (4.139)$$

and therefore we have

$$Y_{lm}(\hat{\mathbf{n}}) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) e^{im\phi} . \quad (4.140)$$

See https://en.wikipedia.org/wiki/Table_of_spherical_harmonics for explicit expressions of $Y_{lm}(\hat{\mathbf{n}})$ for low orders of the angular momentum l . Finally, note that

$$\langle \hat{\mathbf{n}} | \hat{R}(\xi, \hat{\xi}) | l, m \rangle = \sum_{m'=-l}^l Y_{lm}(\hat{\mathbf{n}}) D_{m'm}^{(l)}(\phi_\xi, \theta_\xi, \psi_\xi) , \quad (4.141)$$

where $(\phi_\xi, \theta_\xi, \psi_\xi)$ are the Euler angles corresponding to the rotation $\hat{R}(\xi, \hat{\xi})$. Writing $|\hat{\mathbf{n}}'\rangle = \hat{R}(-\xi, \hat{\xi}) |\hat{\mathbf{n}}\rangle$ as the ket vector $|\hat{\mathbf{n}}\rangle$ rotated by $-\xi$ about the direction $\hat{\xi}$, and taking $m = 0$, we obtain the *spherical harmonic addition formula*,

$$Y_{l0}(\hat{\mathbf{n}}') = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{n}}) Y_{lm}^*(\hat{\mathbf{n}}_\xi) , \quad (4.142)$$

¹¹That is, we suppress the radial coordinate in $|\mathbf{r}\rangle \equiv |r, \hat{\mathbf{n}}\rangle$.

¹²Wisconsin Senator Joseph McCarthy was famous for his aggressive questioning of witnesses before the U.S. Senate Subcommittee on Investigations in 1954, theatrically haranguing them by demanding, "Are you now or have you ever been associated with the Legendre polynomials?" Those who answered in the affirmative or refused to answer were blacklisted and forbidden from working on special functions. A similar fate befell those who associated with Laguerre, Jacobi, or Gegenbauer polynomials.

where \hat{n}_ξ is the unit vector whose polar and azimuthal angles are (θ_ξ, ϕ_ξ) . Note that $\hat{n}_\xi \neq \hat{\xi}$ in general! Since $P_{l0}(\cos \theta) = P_l(\cos \theta)$, we have

$$P_l(\cos \vartheta) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{n}) Y_{lm}^*(\hat{n}_\xi) \quad , \quad (4.143)$$

where, as the student should confirm, ϑ is the angle between \hat{n} and \hat{n}_ξ .

4.3.9 Tensor operators and the Wigner-Eckart theorem for SU(2)

An irreducible tensor operator \hat{Q}_M^J is one which transforms according to the J IRREP of SU(2), meaning

$$\hat{R} \hat{Q}_M^J \hat{R}^\dagger = \sum_{M'=-J}^J \hat{Q}_{M'}^J D_{M'M}^{(J)}(R) \quad , \quad (4.144)$$

where \hat{R} is the unitary operator corresponding to the group element R . Writing $\hat{R} = \exp(i\epsilon \cdot \hat{J})$ where \hat{J} are the operator generators of $\mathfrak{su}(2)$, and expanding for infinitesimal ϵ , one finds

$$[\hat{J}^a, \hat{Q}_M^J] = \sum_{M'=-J}^J \hat{Q}_{M'}^J \langle J, M' | J^a | J, M \rangle \quad . \quad (4.145)$$

Mercifully, as we have seen, SU(2) is simply reducible, meaning that the product representation $j_1 \times j_2$ contains representations with $j \in \{|j_1 - j_2|, \dots, j_1 + j_2\}$ where each j IRREP occurs only once. This means that we can decompose the state $\hat{Q}_m^J |j', m'; \lambda'\rangle$ as

$$\hat{Q}_M^J |j', m'; \lambda'\rangle = \sum_{j,m} \left(\begin{matrix} J & j' & j \\ M & m' & m \end{matrix} \middle| \begin{matrix} j \\ m \end{matrix} \right) |\Psi_m^{j,\lambda'}\rangle \quad . \quad (4.146)$$

Here and below, λ and λ' are extra indices corresponding to quantum numbers not associated with the group symmetry. The state $|\Psi_m^{j,\lambda'}\rangle$ transforms as

$$\hat{R} |\Psi_m^{j,\lambda'}\rangle = \sum_{m'=-j}^j |\Psi_{m'}^{j,\lambda'}\rangle D_{m'm}^{(j)}(R) \quad . \quad (4.147)$$

It follows that

$$\langle j, m; \lambda | \hat{Q}_M^J |j', m'; \lambda'\rangle = \sum_{j'', m''} \left(\begin{matrix} J & j' & j'' \\ M & m' & m'' \end{matrix} \middle| \begin{matrix} j'' \\ m'' \end{matrix} \right) \langle j, m; \lambda | \Psi_{m''}^{j'',\lambda'} \rangle \quad (4.148)$$

where

$$\langle j, m; \lambda | \Psi_{m''}^{j'',\lambda'} \rangle = \delta_{jj''} \delta_{mm''} \times \frac{1}{2j+1} \sum_{\tilde{m}=-j}^j \langle j, \tilde{m}; \lambda | \Psi_{\tilde{m}}^{j,\lambda'} \rangle \quad (4.149)$$

We now define¹³

$$\langle j; \lambda \parallel \hat{Q}^J \parallel j'; \lambda' \rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{\tilde{m}=-j}^j \langle j, \tilde{m}; \lambda \mid \Psi_{\tilde{m}}^{j, \lambda'} \rangle, \quad (4.150)$$

in which case we have the Wigner-Eckart theorem for SU(2):¹⁴

$$\langle j, m; \lambda \mid \hat{Q}_M^J \mid j', m'; \lambda' \rangle = \begin{pmatrix} J & j' & j \\ M & m' & m \end{pmatrix} \frac{\langle j; \lambda \parallel \hat{Q}^J \parallel j'; \lambda' \rangle}{\sqrt{2j+1}}. \quad (4.151)$$

All the M , m , and m' dependence is in the CGC.

4.4 Joke for Chapter Four

A rabbit one day managed to break free from the laboratory where he had been born and raised. As he scurried away from the fencing of the compound, he felt grass under his little feet and saw the dawn breaking for the first time in his life. 'Wow, this is great,' he thought. It wasn't long before he came to a hedge and, after squeezing under it he saw a wonderful sight: lots of other bunny rabbits, all free and nibbling on the lush green grass.

'Hey,' he called. 'I'm a rabbit from the laboratory and I've just escaped. Are you wild rabbits?'

'Yes. Come and join us!' they cried.

He hopped over to them and started eating the grass. It was delicious. 'What else do you wild rabbits do?' he asked.

'Well,' one of them said. 'You see that field there? It's got carrots growing in it. We dig them up and eat them.'

This, he couldn't resist and he spent the next hour eating the most succulent carrots. They were scrumptious – out of this world.

Later, he asked them again, 'What else do you do?'

'You see that field there? It's got lettuce growing in it. We eat that as well.'

The lettuce was as yummy as the grass and the carrots, and he returned a while later completely full. 'Is there anything else you guys do?' he asked.

One of the other rabbits came a bit closer to him and spoke softly. 'There's one other thing you must try. You see those rabbits there?' he said, pointing to the far corner of the field. 'They're lady rabbits. We shag them. Check it out.'

The rabbit spent the rest of the morning screwing his little heart out until, completely exhausted, he staggered back to the group.

'That was awesome,' he panted.

¹³Where does j' come from in the reduced matrix element on the LHS when it doesn't appear on the RHS? Well, you see, the RHS *does* know about j' , as a check of Eqn. 4.146 should make clear. I've suppressed this label in the state $|\Psi_{\tilde{m}}^{j, \lambda'}\rangle$ just to keep you on your toes.

¹⁴Look, I'm very sorry about the awkward $\sqrt{2j+1}$ factors. In my defense, it's a convention which was established long before I was even born.

'So are you going to live with us then?' one of them asked.

'I'm sorry, I had a great time – but I just can't.'

The wild rabbits all stared at him, a bit surprised. 'Why? We thought you liked it here.'

'I do,' he said. 'But I've got to get back to the lab. I'm dying for a cigarette.'